CHAPTER 7

SUMMARY

This chapter summarizes new methods and theory for circular random fields (CRF) and circular-spatial data: the circular dataimage for visualization, the empirical cosineogram for extraction of spatial correlation, the fitted cosineogram model to provide a positive definite estimate of the circular-spatial correlation, a circular kriging solution with variance estimate, and possibly the first method to simulate circular random fields.

Chapter 1, which is the foundation for subsequent chapters, introduced the circular random variable (CRV) and the CRF. The CRF was defined as a space containing spatially correlated CRVs. With Θ the circular RV and $\boldsymbol{x}$ the location in 2 dimensional space, the CRF is the set $\{\Theta(\boldsymbol{x}), \boldsymbol{x} \in \mathrm{R}^2\}$. A CRV takes random directions with the total probability of all possible directions distributed on the circular support $[0, 2\pi)$ or $[-\pi, \pi)$. Spatial correlation increases as distance between measurement locations decreases, i.e., the random components of direction tend to be more similar. An isotropic CRF was defined as a CRF in which spatial correlation is the same in all directions in space. Circular-spatial methods were summarized in a flow chart.

Chapter 2 introduced the circular dataimage. Traditional plots of circular-spatial data become less intelligible as random variation, missing data, and data density increase. These issues were resolved by the circular dataimage. The circular dataimage was defined by coding direction as the color at the same angle on a color wheel, with the color wheel defined as a sequence of three or more two-color gradients with the same color between connecting gradients. This eliminated color discontinuity at the cross over point 0 and 2π (or $-\pi$ and π) resulting in a high resolution continuous image of circular-spatial data in which fine detail on a small scale and large-scale structure on a global scale can be simultaneously recognized. Various suitable color wheels were shown and compared to motivate experimentation, the objective being to effectively contrast and highlight interesting circular-spatial structure. The discrete color wheel was constructed from a continuous color wheel by holding color in an angular interval to the start color of the interval of the continuous color wheel. The advantages of various color schemes were summarized. Circular dataimage examples included:

1) Global and zoomed views of average wind direction
2) Internal flow of the Space Shuttle solid rocket motor nozzle
3) Families of circular time series of rocket nozzle vectoring direction angle vs. time
4) Direction of the Earth main magnetic horizontal (H) field
5) Deuteranopic (red-green color impairment) simulations
6) Highlighting a narrow band of directions (focus plot)
7) Overlay of magnitude as contour curves on circular dataimages
8) 3D polar plots of Earth main magnetic H field with magnitude as radius, and direction coded as color in a color wheel, and magnitude and direction depending on longitude and latitude.

Chapter 3 defined the empirical cosineogram. The cosineogram expresses the spatial correlation in circular-spatial data in a form consistent with the circular kriging solution of Chapter 4. The circular kriging solution requires the mean cosine of the angles between the random components of direction as a function of the distance between observation locations *d*. In the presence of a spatial trend, the random component equals the observed direction minus the mean direction at the observation location. In the absence of a spatial trend, the random component equals the observed direction. With $\hat{\varsigma}(d)$ the mean cosine, $\|\boldsymbol{x}_j - \boldsymbol{x}_i\|$ the linear distance between observations *i* and *j*, and $N(d)$ the number of pairs of observations of direction separated by a distance within a tolerance ε of *d*, the cosineogram is the plot of

$$\hat{\varsigma}(d) = \frac{1}{N(d)} \sum_{\left| \| \boldsymbol{x}_j - \boldsymbol{x}_i \| - d \right| < \varepsilon} \cos(\theta_j - \theta_i) \tag{7.1}$$

vs. *d*. For a example, a cosineogram was computed from homogeneous ocean wind data in a south polar region.

The cosine model fitted to the cosineogram characterizes the spatial correlation as a smooth, continuous, and positive definite function with

- The mean cosine equals 1 at zero distance
- A reduction in the mean cosine at distance close to 0, which is called the nugget effect
- The range (scale parameter, which is also the distance CRV are uncorrelated when the input spatial covariance function is spherical)
- The sill (mean cosine at distances where CRV are uncorrelated).

The theoretical sill was derived as the square of the resultant vector mean length parameter of the circular probability distribution underlying the circular-spatial data. For the circular probability distributions uniform $(\rho = 0)$, cardioid, triangular, von Mises, and wrapped Cauchy, it was determined that the resultant vector mean length equals the parameter ρ of circular probability distributions. The theoretical sill was verified by simulation.

Introductory cosine models for fitting to the empirical cosineogram were adapted from covariance functions for linear kriging by shifting and scaling. With $c(d)$ the mean cosine of the angle between random components of direction a distance *d* apart, ρ the resultant vector mean length of the circular probability distribution, $0 \le \rho < 1$, n_g the nugget, $0 \le n_g \le 1 - \rho^2$, and $c(d)$ the covariance function with a maximum of 1, the general form of the cosine model is

$$\varsigma(d) = \begin{cases} 1, d = 0 \\ \rho^2 + \left(1 - n_g - \rho^2\right) c(d), d > 0. \end{cases} \quad (7.2)$$

The general cosine model was proved to be positive definite for optimum circular kriging.

Chapter 4 developed a circular kriging solution. With $\boldsymbol{w}$ a computed vector of weights based on the circular-spatial correlation, the estimated direction is the matrix of observed directions $\boldsymbol{U}$ (each column is an observation of direction as a unit vector) post multiplied by $\boldsymbol{w}$. The approach avoided the first order Taylor series approximation of McNeill (1993), which results in a nonunit vector estimator. The solution was derived in full detail, and verified to produce a unit vector of maximum fit. With $\boldsymbol{K}$ the positive definite matrix of cosines equal to the cosine model of the matrix of pairwise distances, and $\boldsymbol{c}$ the vector of cosines between the estimation location and sample locations, the weight vector $\boldsymbol{w}$ is

$$\boldsymbol{w} = \boldsymbol{K}^{-1}\boldsymbol{c} \Big/ \sqrt{\boldsymbol{c}^T \boldsymbol{K}^{-1} \boldsymbol{U}^T \boldsymbol{U} \boldsymbol{K}^{-1} \boldsymbol{c}}. \quad (7.3)$$

A computationally efficient form of the estimator of direction was derived by omitting the denominator of (7.3). With h and v being the horizontal and vertical components of the vector $\boldsymbol{UK}^{-1}\boldsymbol{c}$, respectively, the estimated direction in $[0, 2\pi)$ radians at location $\boldsymbol{x}_0$ is

$$\hat{\theta}_0 = \begin{cases} \tan^{-1}(v/h), & h > 0, v \geq 0 \\ \pi/2, & h = 0, v > 0 \\ \tan^{-1}(v/h) + \pi, & h < 0 \\ \frac{3}{2}\pi, & h = 0, v < 0 \\ \tan^{-1}(v/h) + 2\pi, & h > 0, v < 0 \\ \text{undefined}, & h = v = 0. \end{cases} \quad (7.4)$$

The estimated direction at a sampled location was proven to be the observed direction.

An estimate of the circular kriging variance $\hat{\sigma}_{CK}^2$ was defined as the mean squared length of the error vector between the estimator and the unobserved direction. $0 \le \hat{\sigma}_{CK}^2 < 4$. It was approximated by a first order Taylor's series. The circular kriging variance approximation is

$$\hat{\sigma}_{CK}^2 = 2 - 2\sqrt{\boldsymbol{c}^T \boldsymbol{K}^{-1} \boldsymbol{c}}. \tag{7.5}$$

McNeill's (1993) estimate $\hat{\sigma}_{CK}^2 = \sqrt{\boldsymbol{c}^T \boldsymbol{K}^{-1} \boldsymbol{c}}$ is actually proportional to concentration, which is in a sense opposite to variance, i.e., as concentration about the mean direction increases, variance about the mean direction decreases. The estimate at a sampled location is exact and has zero variance.

In Chapter 5, the CRF was defined as a set of $(\theta, \boldsymbol{x})$ of where θ denotes direction and $\boldsymbol{x}$ denotes the location of observation. In a CRF with spatial correlation, the mean cosine of the angle between random components of directions (nonrandom component removed) increases as the distance between observation locations decreases. The nonrandom component is removed so spatial correlation is not confused with a global or first order trend The well known inverse cumulative distribution function (CDF) method was extended to the simulation of a CRF by applying the inverse CDF of a circular probability distribution to the cumulative probabilities of observations of the Gaussian random variables (GRV) of a Gaussian random field (GRF). The inverse CDF of a circular distribution is either a closed form expression, or interpolated from the CDF. The set of a CRV transformed from a GRV and the corresponding GRV observation location constitute a simulation of the CRF.

The mathematical properties of the simulated CRF were discussed:

1) The mean cosine at distance zero is defined as one.

2) The cosine at distances where GRV or CRV are uncorrelated is the square of the resultant vector mean length parameter of the CRV ρ as derived in Chapter 3.

3) At all other distances, correlation varies with distance. Spatially correlated observations of a GRF have spatially correlated cumulative probabilities because the CDF is monotonic increasing. Hence, observations which are close together have cumulative probabilities which are close together. Conversely, spatially correlated cumulative probabilities have spatially correlated CRV because the inverse CDF is also monotonic increasing. Depending on which circular distribution is being produced, this process involves 1 to 2 non closed form transformations which reshape the covariance function of the GRF. The resultant cosine curves were characterized as fitted positive define functions adapted from the R package RandomFields (Schlather 2001) function CovarianceFct using the general form of the cosine model (7.2).

The effect of standardizing the observations of the GRF (center by subtracting the mean, and scale by dividing by the standard deviation) prior to evaluating the cumulative probabilities was considered. The effects of standardization include over fitting, bias of the spatial covariance function of the GRF, and inflated type 1 error rates in tests based on over fitted circular distributions. Standardization should not be used for analysis or development of tests. Qualitative evaluations with standardization demonstrated that a CRF was produced with very close and consistent distributional fit, and range consistent with the input specifications. The sill was consistent with both input specifications and expected value derived in Chapter 3.

Chapter 6 provides a comprehensive example combining the results of and connecting Chapters 2 – 5.