Expected numbers at hitting times

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Abstract

We determine exactly the expected number of hamilton cycles in the random graph obtained by starting with $n$ isolated vertices and adding edges at random until each vertex degree is at least two. This complements recent work of Cooper and Frieze. There are similar results concerning expected numbers for example of perfect matchings, spanning trees, hamilton paths and directed hamilton cycles.
1 Introduction

Let $K_n$ denote the complete graph on the vertex set $V_n = \{1, \ldots, n\}$ with set $E$ of $N = \binom{n}{2}$ edges. Given a permutation or linear order $\pi$ on $E$ the corresponding graph process $\tilde{G}_n = \tilde{G}_n(\pi)$ is the sequence $G_0, G_1, \ldots, G_N$ of graphs on $V_n$ where $G_i$ contains the first $i$ edges in $E$ under $\pi$. We assume that each of the $N!$ permutations on $E$ is equally likely, so that we obtain a random graph process $\tilde{G}_n$ on $V_n$ (see Bollobás [3], page 39).

Let $\mathcal{P}$ be any increasing property of graphs such that $K_n \in \mathcal{P}$. The hitting time $\tau(\tilde{G}_n, \mathcal{P})$ is the least $m$ such that $G_m \in \mathcal{P}$. For $i = 1, 2$ let $\mathcal{D}^{(i)}$ be the property of having each vertex degree at least $i$. We define $G_n^{(i)}$ to be the graph $G_m$ in the random graph process $\tilde{G}_n$ where $m = \tau(\tilde{G}_n, \mathcal{D}^{(i)})$. Thus $G_n^{(i)}$ is the random graph obtained by adding edges at random until each vertex degree is at least $i$.

Let $\mathcal{H}$ be the property of having a hamilton cycle. Clearly $\tau(\tilde{G}_n, \mathcal{D}^{(2)}) \leq \tau(\tilde{G}_n, \mathcal{H})$ always. One of the most attractive results in the theory of random graphs is that

\begin{equation}
(1.1) \quad P(\tau(\tilde{G}_n, \mathcal{D}^{(2)}) = \tau(\tilde{G}_n, \mathcal{H})) \to 1 \text{ as } n \to \infty.
\end{equation}

This result appeared in Komlós and Szemerédi [6] and was proved by Bollobás [2] (see also Ajtai, Komlós and Szemerédi [1]). Another way of stating (1.1) is

\[ P(G_n^{(2)} \in \mathcal{H}) \to 1 \text{ as } n \to \infty. \]

Cooper and Frieze [4] extended this result as follows. For any graph $G$ let $hc(G)$ be the number of hamilton cycles in $G$. Thus $hc(K_n) = \frac{1}{2} (n - 1)! = \frac{1}{2} (n - 1)!$.
In [4] it is shown that, for any $\epsilon > 0$,

\[(1.2) \quad P(hc(G_n^{(2)}) \geq (\log n)^{(1-\epsilon)n}) \to 1 \text{ as } n \to \infty.\]

Does this result give the right order of magnitude for $hc(G_n^{(2)})$? It is well known (see Bollobás [3] page 61) that if $m_i = m_i(n) = \frac{1}{2}n\{\log n + (i - 1)\log \log n + w(n)\}$ where $w(n) \to \infty$ as $n \to \infty$, then

\[(1.3) \quad P(\tau(\tilde{G}_n, D^{(i)}) \leq m_i) \to 1 \text{ as } n \to \infty.\]

(Here log means natural logarithm.) Also, if a graph $G$ has $n$ vertices and $m$ edges then of course $hc(G) \leq \binom{m}{n} \leq \left(\frac{me}{n}\right)^n$. In fact we shall see (assuming that $m \geq n/2$) that

\[(1.4) \quad hc(G) \leq \left(\frac{2m}{n} - 1\right)^n.\]

It follows immediately from (1.3) and (1.4) that

\[P(hc(G_n^{(2)}) \leq (\log n + 2 \log \log n)^n) \to 1 \text{ as } n \to \infty.\]

Thus indeed the inequality (1.2) of Cooper and Frieze is about best possible.

But now let us consider expected numbers of hamilton cycles. Suppose that we look at the random graph $G_{n,m_2}$, with vertex set $V_n$ and $m_2$ edges, where all $\binom{N}{m_2}$ such graphs are equally likely; or at the random graph $G_{n,p}$ with vertex set $V_n$ where the $N$ possible edges appear independently with probability $p = m_2/N$. Note that if $C$ is any fixed hamilton cycle in $K_n$ then

\[P(C \text{ in } G_{n,m_2}) = \Pi_{i=0}^{n-1} \left(\frac{m_2 - i}{N - i}\right) \leq \left(\frac{m_2}{N}\right)^n = p^n.\]
Hence
\[
E[hc(G_{n,m})] \leq E[hc(G_{n,p})] = hc(K_n)p^n = (\log n)^n e^{-n+o(n)},
\]
as suggested in [4], following the theorem.

However our interest is in the expected number of hamilton cycles in 
\(G_{n}^{(2)}\), and here the story is quite different.

\(\text{(1.5) Theorem}\)

\[
E[hc(G_{n}^{(2)})] = hc(K_n)n! \left(\frac{2(n-3)!}{(2n-3)!} - \frac{(2n-6)!}{(3n-6)!}\right)
\sim hc(K_n)16(\pi n)^{1/2}4^{-n}.
\]

We prove this result in the next section. It was of course trivial to determine \(E[hc(G_{n,p})]\) : we shall see that it is surprisingly easy to determine \(E[hc(G_{n}^{(2)})]\). The same approach allows us to obtain similar results concerning expected numbers of perfect matchings in \(G_{n}^{(1)}\), of spanning trees and hamilton paths in \(G_{n}^{(1)}\) and in \(G_{n}^{(C)}\), and of directed hamilton cycles in certain random digraphs. Here \(G_{n}^{(C)}\) denotes the random graph obtained by adding edges at random until the graph is connected. These further results are presented in sections 3 to 6 below.

Further possible interesting investigations of a similar nature may occur to the reader, for example concerning directed hamilton paths.
2 Hamilton cycles in $G_{n}^{(2)}$

Bounding $hc(G)$

We must prove the inequality (1.4) concerning the number $hc(G)$ of hamilton cycles in a graph $G$.

(2.1) **Proposition.** Suppose that the graph $G = (V, E)$ has $n$ vertices and $m$ edges, and each vertex degree $d_v > 0$. Then

$$hc(G) \leq \Pi_{v \in V} (d_v - 1) \leq \left(\frac{2m}{n} - 1\right)^n.$$ 

**Proof.** For each vertex $u \in V$ let $hp_u(G)$ be the number of hamilton paths starting at $u$. Let us prove first that

(2.2) $$hp_u(G) \leq d_u \Pi \{ (d_v - 1) : v \in V \setminus \{u\} \}.$$ 

Suppose that $G$ is a counterexample with $|V|$ minimal. Clearly each $d_v \geq 2$. Let $u \in V$ and let $W$ be the set of neighbours of $u$. The inequality (2.2) must hold for the graph $G'$ obtained from $G$ by deleting vertex $u$. Hence

$$hp_u(G) = \sum_{w \in W} hp_w(G')$$

$$\leq \sum_{w \in W} d'_w \Pi \{ (d'_v - 1) : v \in V \setminus \{u, w\} \}$$

$$\leq \sum_{w \in W} (d_w - 1) \Pi \{ (d_v - 1) : v \in V \setminus \{u, w\} \}$$

$$= d_u \Pi \{ (d_v - 1) : v \in V \setminus \{u\} \}.$$ 

We have now proved (2.2). It follows that

(2.3) $$hc(G) \leq \Pi_{v \in V} (d_v - 1)$$

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For to establish (2.3) we may assume that each \( d_v \geq 2 \). Fix some vertex \( u \in V \) and let \( W \) and \( G' \) be as above. Then since \( d_u \geq 2 \),
\[
\text{hc}(G) \leq \frac{1}{2} \sum_{w \in W} hp_w(G') \leq \Pi_{v \in V} (d_v - 1).
\]
Finally to complete the proof of proposition (2.1), we may use the inequality that ‘geometric mean \( \leq \) arithmetic mean’.

\[\square\]

**Two standard results**

We shall need the following two standard results below.

\begin{equation}
(2.4) \quad \text{For any positive integers } \alpha \text{ and } \beta \quad \int_0^1 x^\alpha (1-x)^\beta \, dx = \frac{\alpha! \beta!}{(\alpha + \beta + 1)!}.
\end{equation}

\begin{equation}
(2.5) \quad \text{(Stirlings formula, see for example [3] page 4.) As } n \to \infty, \quad n! \sim (2\pi n)^{1/2} (n/e)^n.
\end{equation}

**Proof of theorem (1.5)**

It is important to generate the random graph process \( \tilde{G}_n \) in a convenient way. We may assume that we start with a family \((X_e : e \in E)\) of independent random variables, each uniformly distributed on \([0, 1]\); and that \( \tilde{G}_n \) corresponds to ordering the edges \( e \) by increasing value of \( X_e \). Note that if \( 0 < p < 1 \) then we may form a random graph \( G_{n,p} \) by including those edges \( e \) with \( X_e \leq p \).

Now let \( C \) be any fixed hamilton cycle in \( K_n \) say \( C = (1, 2, \ldots, n) \) and let us focus on the edge \( e = \{1, 2\} \). Write ‘\( e \) last in \( G_n^{(2)} \)’ to denote the event that \( e \) was the last edge added in \( \tilde{G}_n \) to form \( G_n^{(2)} \). For \( i = 1, 2 \) let \( E_i \) be the
set of \( n - 3 \) edges in \( K_n \) given by

\[
E_1 = \{ \{1, j\} : j = 3, \ldots, n - 1\}, \quad E_2 = \{ \{2, j\} : j = 4, \ldots, n\}.
\]

For \( 0 < x < 1 \) and \( i = 1, 2 \) let \( A_i(x) \) be the event that \( X_f \leq x \) for each edge \( f \) in \( C \) other than \( e \), and \( X_f > x \) for each edge \( f \) in \( E_i \). Note that

\[
P(C \text{ in } G_n^{(2)}, \text{ e last in } G_n^{(2)} | X_e = x) = P(A_1(x) \cup A_2(x)).
\]

Also,

\[
P(A_1(x)) = P(A_2(x)) = x^{n-1}(1 - x)^{n-3},
\]

and

\[
P(A_1(x) \cap A_2(x)) = x^{n-1}(1 - x)^{2n-6},
\]

so

\[
P(A_1(x) \cup A_2(x)) = 2x^{n-1}(1 - x)^{n-3} - x^{n-1}(1 - x)^{2n-6}.
\]

Hence

\[
P(C \text{ in } G_n^{(2)}) = n P(C \text{ in } G_n^{(2)}, \text{ e last in } G_n^{(2)})
\]

\[
= n \int_0^1 P(A_1(x) \cup A_2(x)) dx 
\]

\[
= 2n \frac{(n-1)! (n-3)!}{(2n-3)!} - n \frac{(n-1)! (2n-6)!}{(3n-6)!}
\]

by (2.4)

\[
= n! \left\{ \frac{2(n-3)!}{(2n-3)!} - \frac{(2n-6)!}{(3n-6)!} \right\}.
\]

This yields the first part of theorem (1.5), and we may use Stirling’s formula (2.5) to complete the proof. \( \square \)
3 Perfect matchings in $G_n^{(1)}$

Let $pm(G)$ denote the number of perfect matchings in a graph $G$. Thus for $n$ even
\[ pm(K_n) = \frac{n!}{2^{n/2}(n/2)!} = n^{(1/2+o(1))n} \text{ as } n \to \infty. \]

It is known (see Bollobás [3] page 166) that
\[ P \left( pm(G_n^{(1)}) > 0 \right) \to 1 \text{ as } n \to \infty, \text{ } n \text{ even.} \]

(3.1) **Theorem** For $n$ even,
\[ \mathbb{E}[pm(G_n^{(1)})] = pm(K_n)\left(\frac{n}{2}\right)! \left\{ \frac{2(n-2)!}{(n-2)!} - \frac{(2n-4)!}{(2n-4)!} \right\} \]
\[ \sim pm(K_n) \left(\frac{2\pi n}{2}\right)^{1/2} \left(\frac{2}{2\pi n}\right)^{n}. \]

**Proof** Let $M$ be a fixed matching in $K_n$ and focus on a particular edge $e = \{1,2\}$ say in $M$. For $i = 1,2$ let $E_i$ be the set of edges $\{\{i,j\} : j = 3, \ldots, n\}$ in $K_n$.

For $0 < x < 1$ and $i = 1,2$ let $A_i(x)$ be the event that $X_f \leq x$ for each edge $f$ in $M$ other than $e$, and $X_f > x$ for each edge $f$ in $E_i$. Note that
\[ P(M \text{ in } G_n^{(1)}, e \text{ last in } G_n^{(1)}|X_e = x) = P(A_1(x) \cup A_2(x)). \]

Also, much as before we find that
\[ P(A_1(x)) = P(A_2(x)) = x^{n-1}(1-x)^{n-2}, \]
and
\[ P(A_1(x) \cap A_2(x)) = x^{n-1}(1-x)^{2n-4}, \]

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so

\[ P(A_1(x) \cup A_2(x)) = 2x^{2-1}(1-x)^{n-2} - x^{2-1}(1-x)^{2n-4}. \]

Hence

\[ P(M \text{ in } G_n^{(2)}) = \frac{n}{2} P(M \text{ in } G_n^{(1)}, e \text{ last in } G_n^{(1)}) \]

\[ = \frac{n}{2} \int_0^1 \{2x^{n/2-1}(1-x)^{n-2} - x^{2-1}(1-x)^{2n-4}\} dx \]

\[ = \left( \frac{n}{2} \right)! \left\{ \frac{2(n-2)!}{(n-2)!} - \frac{(2n-4)!}{(n-4)!} \right\} \]

by (2.4). Now use Stirling’s formula (2.5). \qed
4 Spanning trees

Let $\mathcal{C}$ be the property of being connected, and let $st(G)$ be the number of spanning trees (complexity) of a graph $G$. Thus $st(K_n) = n^{n-2}$. It is well known (see Bollobás [3] page 152) that

$$(4.1) \quad P(st(G_n^{(1)}) > 0) = P(G_n^{(1)} \in \mathcal{C}) \to 1 \text{ as } n \to \infty.$$  

We are interested here in $E[st(G_n^{(1)})]$ and in $E[st(G_n^{(C)})]$ and in the difference between these quantities. Of course $st(G_n^{(C)}) \geq st(G_n^{(1)})$ always, and by (4.1) equality holds with probability $\to 1$ as $n \to \infty$.

**(4.2) Theorem**  For $n \geq 3$,

$$E[st(G_n^{(1)})] = (n - 1)^{n-3}n!(n - 2)!/(2n - 3)! \sim (n^{n-2})(8/e)(\pi n)^{1/2} 4^{-n},$$

and

$$E[st(G_n^{(C)}) - st(G_n^{(1)})]$$

$$= \frac{1}{2} \sum_{k=2}^{n-2} \frac{n-2}{k-1} k^{k-2} (n-k)^{n-k-2} \left\{ \frac{n!(n-k-1)!}{(k(n-k)+n-2)!} \right\}$$

$$\sim (n^{n-2}) \left( \frac{243}{16e^2} \right) (3\pi n)^{1/2} \left( \frac{1}{2\pi} \right)^n.$$  

Now it is not hard to see that for $n \geq 3$, $P(G_n^{(1)}$ is not connected) $> 4/(n + 3)$. (Just consider the process at the stage when the number of components drops to 3.) Thus we have

$$E[st(G_n^{(C)}) - st(G_n^{(1)})]$$

$$= E[st(G_n^{(C)})|G_n^{(1)}$ not connected] P(G_n^{(1)} not connected)$$

$$> \frac{4}{n+3} E[st(G_n^{(C)})|G_n^{(1)}$ not connected].$$
Thus we obtain the surprising (?) result that $E[st(G_n^{(C)})|G_n^{(1)} \text{ not connected}]$ is exponentially smaller than $E[st(G_n^{(1)})]$.

**Proof** Given a spanning tree $T$ of $K_n$ and an edge $e = \{i, j\} \in E$ where $i < j$ define $f(T, e)$ as follows: if $e$ is not in $T$ let $f(T, e) = 0$, and if $e$ is in $T$ let $f(T, e)$ be the number of vertices in the component containing vertex $i$ of the graph obtained from $T$ by deleting the edge $e$. Note that for each $1 \leq k \leq n - 1$ and for each edge $e \in E$, there are

\[ g(n, k) = \binom{n-2}{k-1} k^{k-2} (n - k)^{n-k-2} \]

spanning trees $T$ of $K_n$ such that $f(T, e) = k$.

For each pair $(T, e)$ let $A(T, e)$ be the event that the spanning tree $T$ is in $G_n^{(C)}$ and the edge $e$ is last in $G_n^{(C)}$. If $f(T, e) = k$, $1 \leq k \leq n - 1$, and $0 < x < 1$ then

\[ P(A(T, e)|X_e = x) = x^{n-2}(1 - x)^{k(n-k)-1}; \]

and so

\[ P(A(T, e)) = \int_0^1 x^{n-2}(1 - x)^{k(n-k)-1} \, dx \]

\[ = \frac{(n-2)!((k(n-k)-1)!}{(k(n-k)+n-2)!} \]

by (2.4).

Now let us use $A_{T,e}$ as the indicator for the event $A(T, e)$. Then

\[ st(G_n^{(C)}) = \sum_e \sum_T A_{T,e} \]

\[ = \sum_e \sum_{k=1}^{n-1} \sum\{A_{T,e} : T \text{ satisfies } f(T, e) = k\}. \]

Hence

\[ E[st(G_n^{(C)})] = \sum_{k=1}^{n-1} a_k, \]
where by the above
\[ a_k = \binom{n}{2} g(n, k) \frac{(n-2)!(k(n-k)-1)!}{(k(n-k)+n-2)!} \]
\[ = \frac{1}{2} (n-2)! k^{n-k-2} (n-k)^{n-k-2} \frac{n!(k(n-k)-1)!}{[k(n-k)+n-2]!}. \]

Now note that
\[ st(G_n^{(1)}) = \sum_e \sum \{ A_{T,e} : T \text{ satisfies } f(T,e) = 1 \text{ or } n-1 \}, \]
and so
\[ E[st(G_n^{(1)})] = a_1 + a_{n-1} = 2a_1. \]

It is easy to check using Stirling’s formula (2.5) that
\[ 2a_1 = (n - 1)^{n-3} n!(n - 2)!/(2n - 3)! \]
\[ \sim n^{n-2} (8/e)(\pi n)^{1/2} 4^{-n}, \]
and
\[ 2a_2 = (n - 2)^{n-3} n!(2n - 5)!/(3n - 6)! \]
\[ \sim n^{n-2} \left( \frac{243}{16e^2} \right)(3\pi n)^{1/2} (\frac{4}{27})^n. \]

It remains only to check that the sum \( \sum_{k=3}^{n-3} a_k \) is \( o(a_2) \) as \( n \to \infty \).

Let \( n \geq 6 \). The total number of trees containing a given edge \( e \) is \( 2n^{n-3} \). Thus certainly \( g(n, k) \leq 2n^{n-3} \) for each \( k \). Also note that
\[ g(n, 2) = (n - 2)^{n-3} \sim e^{-2} n^{n-3}. \]

For \( 3 \leq k \leq n - 3 \),
\[ \frac{g(n, 2)}{2n^{n-3}} \frac{a_k}{a_2} = \frac{g(n, k)}{2n^{n-3}} \frac{(3n-6)_{(n-1)}}{(k(n-k)+n-2)_{(n-1)}} \]
\[ \leq \frac{(3n-6)_{(n-1)}}{(4n-11)_{(n-1)}} \leq (\frac{3n-6}{4n-11})^{n-1}. \]
Hence

\[
\sum_{k=3}^{n-3} \frac{a_k}{a_2} \leq \left( \frac{2n^{n-3}}{(n-2)^{n-3}} \right) (n-5) \left( \frac{3n-6}{4n-11} \right)^{n-1} = o(1).
\]

\[\Box\]
5 Hamilton paths

Let $hp(G)$ denote the number of hamilton paths in a graph $G$. Thus $hp(K_n) = \frac{1}{2} n!$. We might ask about the expected numbers of hamilton paths in $G^{(1)}_n$ and $G^{(C)}_n$. If we argue as in the last section we obtain the following result, which resembles theorem 4.2.

(5.1) Theorem For $n \geq 2$,

$$E[hp(G^{(1)}_n)] = n!((n-2)!)^2/(2n-3)!$$

$$\sim hp(K_n)16(\pi/n)^{\frac{1}{2}}4^{-n},$$

and

$$E[hp(G^{(C)}_n) - hp(G^{(1)}_n)]$$

$$= \sum_{k=2}^{n-2} \frac{1}{2} n!((n-2)!(k(n-k) - 1)!/(k(n-k) + n - 2)!$$

$$\sim hp(K_n)\frac{243}{8}\left(\frac{3n}{n}\right)^{\frac{1}{2}}\left(\frac{4}{27}\right)^n.$$
6 Directed hamilton cycles

Just as we defined a random graph process $\tilde{G}_n$ on $V_n$ we may define a random digraph process $\tilde{D}_n$ on $V_n$. Let $DK_n$ denote the complete digraph on the vertex set $V_n$ with set $A$ of $n(n-1)$ arcs. Given a permutation $\pi$ of $A$ the corresponding digraph process in the sequence $D_0, D_1, \ldots, D_{n(n-1)}$ of digraphs on $V_n$ where $D_i$ contains the first $i$ arcs in $A$ under $\pi$. We assume that each of the $(n(n-1))!$ permutations on $A$ is equally likely, so that we obtain a random digraph process $\tilde{D}_n$ on $V_n$.

Let $P$ be an increasing property of digraphs such that $DK_n \in \mathcal{P}$. The hitting time $\tau(\tilde{D}_n, \mathcal{P})$ is the least $m$ such that $D_m \in \mathcal{P}$. For $i, j = 0, 1$, let $\mathcal{D}^{i,j}$ be the property that each vertex has indegree at least $i$ and outdegree at least $j$. We define $D_n^{i,j}$ to be the digraph $D_m$ in the random digraph process $\tilde{D}_n$ where $m = \tau(\tilde{D}_n, \mathcal{D}^{i,j}) = \tau^{i,j}$ say. Thus $D_n^{i,j}$ is obtained by adding arcs at random until each vertex has indegree at least $i$ and outdegree at least $j$.

As with random graph processes, we may assume that we start with a family $(X_a : a \in A)$ of independent random variables each uniformly distributed on $[0, 1]$; and that the random digraph process $\tilde{D}_n$ corresponds to ordering the arcs $a$ by increasing value of $X_a$.

For any digraph $D$ let $hc(D)$ denote the number of (directed) hamilton cycles in $D$. Thus $hc(DK_n) = (n-1)!$ Frieze [5] has shown the beautiful result that

$$P(hc(D_n^{1,1}) > 0) \to 1 \text{ as } n \to \infty.$$  

We are interested in $E[hc(D_n^{0,1})]$ (which equals $E[hc(D_n^{1,0})]$ by symmetry) and $E[hc(D_n^{1,1})]$. 

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Theorem. We have
\[ E[hc(D_{0,1}^n)] = hc(DK_n)/(2n-2) \]
\[ \sim hc(DK_n)4(\pi n)^{1/2}4^{-n}, \]
and
\[ E[hc(D_{1,1}^n)] = hc(DK_n)n! \left\{ \frac{2(n-2)!}{(2n-2)!} - \frac{(2n-4)!}{(3n-4)!} \right\} \]
\[ \sim 2E[hc(D_{0,1}^n)]. \]

Observe that from theorems (1.5) and (6.1) we obtain

\[ E[hc(D_{1,1}^n)] \sim E[hc(G_n^{(2)})] \text{ as } n \to \infty. \]

Observe also from theorem (6.1) that on average as we add arcs in the random digraph process \( \bar{D}_n \), about the same number of hamilton cycles appear in the stage when we form \( D_{0,1}^n \) as in the succeeding stage (if there is one) when we continue to form \( D_{1,1}^n \). This seems intuitively reasonable, since if the random indicator variable \( I = 1 \) when \( \tau_{0,1} = \tau_{1,0} \) and \( I = 0 \) otherwise, then

\[ P(I = 1) = 0(n^{-2}) \]

as is easily checked, and always

\[ hc(D_{0,1}^n) + hc(D_{1,0}^n) = hc(D_{1,1}^n)(1 + I). \]

Proof. Let \( C \) be a fixed hamilton cycle in \( DK_n \) and let \( a \) be an arc in \( C \). Arguing as in the proof of theorem (1.5) we find

\[ P(C \text{ in } D_{n}^{0,1}) \sim n \int_0^1 x^{n-1} (1-x)^{n-2} dx \]

and so

\[ P(C \text{ in } D_{n}^{0,1}) = n \int_0^1 x^{n-1} (1-x)^{n-2} dx \]
\[ = \frac{n(n-2)!}{(2n-2)!}. \]
Hence
\[ E[hc(D_n^{0,1})] = hc(DK_n)/\binom{2n-2}{n} \]
\[ \sim hc(DK_n)4(\pi n)^{1/2}4^{-n}. \]

In a similar way we find
\[ P(C \text{ in } D_n^{1,1}, a \text{ last in } D_n^{1,1}|X_a = x) = 2x^{n-1}(1-x)^{n-2} - x^{n-1}(1-x)^{2n-4}, \]
and so
\[ P(C \text{ in } D_n^{1,1}) = n \int_0^1 (2x^{n-1}(1-x)^{n-2} - x^{n-1}(1-x)^{2n-4})dx \]
\[ = n! \left\{ \frac{2(n-2)!}{(2n-2)!} - \frac{(2n-4)!}{(3n-4)!} \right\}. \]

Hence
\[ E[hc(D_n^{1,1})] = hc(DK_n)P(C \text{ in } D_n^{1,1}) \]
\[ \sim 2E[hc(D_n^{0,1})]. \]

\[ \square \]
References


