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On the bandwidth of triangulated triangles☆

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Abstract

We give a technique for obtaining a lower bound on the bandwidth of any planar graph with an embedding in which all bounded faces are triangles. This technique is applied to show that, for each positive integer l, the triangulated triangle T_l with side-length l has bandwidth exactly $l + 1$. This settles a question of Douglas West.

1. Introduction

A *vertex numbering* of a finite graph G is an assignment of distinct integers to the vertices of G, and the *bandwidth of a numbering* is the maximum difference between the numbers assigned to adjacent vertices. The *bandwidth of* G is defined to be the minimum over all numberings of G of the bandwidth of that numbering. Much work has been done towards finding the bandwidth of specific classes of graphs, including grid graphs [2], hypercubes [4] and planar distributive lattices [3]. In this paper we answer a question posed by D. West at the 14th (1993) British Combinatorial Conference (Problem 3 in the original list, Problem 217 on pages 406–407 in this volume): what is the bandwidth of the *triangulated triangle* T_l? Here T_l is the graph whose vertices are the triples of non-negative integers summing to l, with an edge connecting two triples if they agree in one coordinate and differ by 1 in the other two coordinates. The graph T_5 is shown in Fig. 1.

There is an easy upper bound of $l + 1$ on the bandwidth of T_l, obtained from the 'row by row' numbering. Here, we show that this numbering is best possible. This result is obtained as a consequence of a lower bound on the bandwidth of any *plane*

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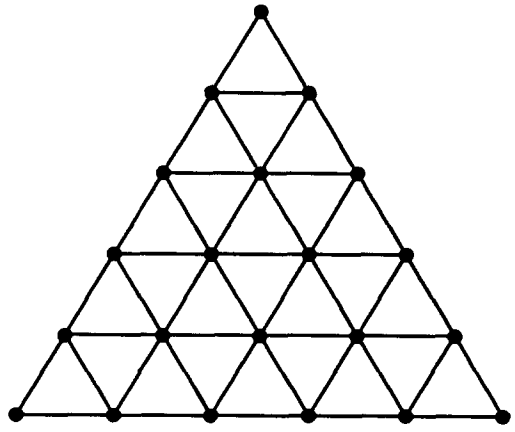

Fig. 1. The graph T_5. The coordinates of the corners are (5, 0, 0), (0, 5, 0), and (0, 0, 5).

near-triangulation, that is, any planar graph embedded in the plane so that all bounded faces are triangles.

2. The main result

Let G be a plane near-triangulation. The following definition generalizes the notion of the 'sides' of the triangle T_l. Let v_1, v_2, v_3 be three distinct vertices that lie on the exterior cycle C. We partition C into 3 paths $P_1 \cup P_2 \cup P_3$ of the form $P_1 = (v_2, \ldots, v_3)$, $P_2 = (v_3, \ldots, v_1)$, $P_3 = (v_1, \ldots, v_2)$ so that v_i is not on P_i. A subset S of $V(G)$ is said to be a *connecting set* (with respect to v_1, v_2, v_3) if the graph induced by G on S has a connected component which contains vertices from each of P_1, P_2 and P_3. Note, for example, that for each i, the set $V(P_i)$ is a connecting set. Our theorem is the following.

Theorem 1. *If G is a plane near-triangulation and v_1, v_2, v_3 are three distinct vertices on the exterior face, then the bandwidth of G is at least as large as the smallest connecting set.*

As an easy consequence of this theorem we have the following corollary.

Corollary 2. *The bandwidth of the triangulated triangle T_l is $l + 1$.*

To prove this, let v_1, v_2 and v_3 be, respectively, the vertices $(l, 0, 0)$, $(0, l, 0)$ and $(0, 0, l)$. Then P_i corresponds to the side of the large triangle whose vertices have 0 in their ith coordinate. We will prove by induction on l that any connecting set H contains at least $l + 1$ vertices. It is clearly true for $l = 1$. Let H' be a connected subgraph of the

graph induced by H which contains vertices from each P_i and is minimal with respect to the number of edges. Note that H' is a tree, and let x be a vertex of degree 1 in H'. By the minimality of H', x must be the unique vertex in $H' \cap P_k$ for some k. $H' - x$ is a connected graph, none of whose vertices has 0 in the kth coordinate, and so the set of vertices of T_{l-1} formed by subtracting 1 from the kth coordinate of each vertex of $H' - x$ is a connecting set of T_{l-1}, which by induction contains at least l vertices.

3. Proof of Theorem 1

We may assume in any given numbering that the vertices get numbered with the integers $1, 2, \ldots, |V|$. Now consider the process of numbering the vertices, starting with 1, then 2, etc., until all vertices are numbered. Let us use 3 colours—red, yellow and blue, to help to see what has been numbered so far. Initially, all vertices are coloured blue. When we number a vertex we colour it yellow, and colour all unnumbered vertices which are adjacent to it red. Thus, at each stage we have a set of yellow numbered vertices, a set of red unnumbered vertices which are the boundary of the yellow set, and perhaps some blue vertices which are unchosen and are not on the boundary. Note that no blue vertex is adjacent to a yellow vertex. Let Y_i, R_i and B_i be the yellow, red and blue sets of vertices, respectively, after i vertices have been numbered.

The following lemma, which can be found implicitly in [4], is a standard tool for obtaining lower bounds on bandwidth.

Lemma 3. *For each i, $|R_i|$ is a lower bound for the bandwidth of the numbering.*

Proof. All vertices in Y_i are assigned numbers less than or equal to i. When we are done numbering the vertices, some vertex in R_i will be assigned a number at least $i + |R_i|$, and will be adjacent to a vertex in Y_i. The difference across the corresponding edge will be at least $|R_i|$. □

Theorem 1 will now follow from the following lemmas.

Lemma 4. *Let i be the first index for which Y_{i+1} is a connecting set. Then R_i is a connecting set.*

In order to prove this lemma, we first recall a classical result from combinatorial topology, namely Sperner's lemma, and use it to prove a lemma on 2-colourings of G. We then use this lemma to prove Lemma 4, and thus complete the proof of Theorem 1.

Lemma 5 (Sperner's Lemma). *Let G and P_1, P_2, P_3 be as above. If the vertices of G are assigned labels from the set $\{1, 2, 3\}$ so that P_i has no vertices labelled i (and thus v_i is*

labelled i), then there must be a triangular face of G whose vertices have three different labels.

For a proof see, for example, [1].

Lemma 6. *Given any 2-colouring of the vertices of G by black and white, exactly one colour will be a connecting set.*

Proof. We first show that the black and white sets cannot both be connecting sets. For suppose that there is both a black connecting component B and a white connecting component W. On the path P_1 there is both a vertex $b \in B$ and a vertex $w \in W$ and, without loss of generality, we may assume that b comes before w on P_1. But now there cannot be both a black path from b to P_2 and a white path from w to P_3, so we have the contradiction.

The proof that either the black set or the white set is a connecting set follows [5] (see the claim following Lemma 3.3), and uses Sperner's lemma.

Suppose for a contradiction that neither black nor white is connecting set. Then for each vertex v there is at least one path P_1, P_2 or P_3 that cannot be reached from v by vertices the same colour as v. Label each vertex $v \in G$ by the *least* i so that P_i cannot be reached from v along vertices the same colour as v. This yields a labelling of $V(G)$ by $1, 2, 3$ which clearly satisfies the conditions of Sperner's lemma, and hence we can find a triangle with vertices coloured 1, 2 and 3. By the pigeonhole principle, two of these were originally the same colour, black or white. Suppose, without loss of generality, that the vertices now labelled i and j were both black, where $i < j$. Then the black vertex labelled j could reach path P_i via black vertices, and the black vertex labelled i could not reach path P_i via black vertices. But they are adjacent, which give a contradiction, finishing the proof of Lemma 6. □

Proof of Lemma 4. Finally, we prove Lemma 4. Let v be the vertex that is numbered $i + 1$. Note that Y_i becomes a connecting set when we add v to it, so v must be adjacent to some vertex of Y_i, i.e., $v \in R_i$. Now $Y_{i+1} = Y_i \cup \{v\}$ is a connecting set, so by Lemma 6, the set B_i, which is disjoint from Y_{i+1}, is not a connecting set. But Y_i is also not a connecting set by the definition of i, and Y_i has no vertex adjacent to any vertex of B_i; hence, $Y_i \cup B_i$ is not a connecting set. So by Lemma 6, R_i, being the complement of $Y_i \cup B_i$, *is* a connecting set. □

As already remarked, this completes the proof of Theorem 1. Finally, let us note (in answer to a question from a referee) that in this theorem the maximum of the lower bounds corresponding to different choices of the vertices v_i may be strictly less than the bandwidth. For example, if G is the plane near-triangulation obtained from the 6-cycle C_6 by adding three edges incident to a given vertex, then G has bandwidth 3 but we always obtain the lower bound 2.

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory With Applications (North-Holland, New York, 1976).
[2] J. Chvátalová, Optimal labelling of a product of two paths, Discrete Math. 11 (1975) 249–253.
[3] U. Faigle and G. Gierz, The bandwidth of planar distributive lattices, in: Graph Theoretic Concepts in Computer Science: Proc. 12th Internat. Workshop, Berlin, Germany, June 20 22 (Springer, Berlin, 1990).
[4] L.H. Harper, Optimal numberings and isoperimetric problems on graphs, J. Combin. Theory (1966) 385–393.
[5] N. Linial and M. Saks, Low diameter graph decompositions, Combinatorica, to appear. (Preliminary version: Decomposing graphs into regions of small diameter, Proc. 2nd ACM-SIAM Symp. on Disc. Algorithms (1991) 320–330.