**Random Channel Assignment in the Plane**

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Received 17 August 2001; accepted 13 July 2002  
DOI 10.1002/rsa.10077

**ABSTRACT:** In the model for random radio channel assignment considered here, points corresponding to transmitters are thrown down independently at random in the plane, and we must assign a radio channel to each point but avoid interference. In the most basic version of the model, we assume that there is a threshold $d$ such that, in order to avoid interference, points within distance less than $d$ must be assigned distinct channels. Thus we wish to color the nodes of a corresponding scaled unit disk graph. We consider the first $n$ random points, and we are interested in particular in the behavior of the ratio of the chromatic number to the clique number. We show that, as $n \to \infty$, in probability this ratio tends to 1 in the “sparse” case [when $d = d(n)$ is such that the average degree grows more slowly than $\ln n$] and tends to $2\sqrt{3}/\pi \approx 1.103$ in the “dense” case (when the average degree grows faster than $\ln n$). We also consider related graph invariants, and the more general channel assignment model when assignments must satisfy “frequency-distance” constraints. © 2003 Wiley Periodicals, Inc. Random Struct. Alg., 22: 187–212, 2003

1. **INTRODUCTION**

In the channel assignment problem for cellular radio communication systems (see, for example, [8, 9, 15, 18]), we are to assign radio frequencies or channels to a given collection of sites or transmitters under constraints which ensure that the resulting
interference will be acceptable, and subject to these constraints we wish to minimize the range of channels used. We assume that each site needs just one channel. We are interested in the usual behavior of such problems.

It is not easy to model satisfactorily the distribution of sites in the plane. Sometimes it is assumed that they form part of a regular lattice, but we take quite a different approach here. We assume that the sites are generated by picking \( n \) points independently according to some fixed probability distribution on the plane, and we let \( n \to \infty \).

In the basic model, there is a threshold distance \( d = d(n) \) such that, in order to avoid interference, sites within distance less than \( d \) must be assigned distinct channels. Thus we are interested in coloring the corresponding “proximity” or “scaled unit disk” graph. The chromatic number of the graph is the least number of colors that can be used, and the clique number is a natural lower bound on the chromatic number. (These quantities are defined later.) We examine the asymptotic behavior of the ratio of the chromatic number to the clique number.

We focus on two cases, the “sparse” case when \( d = d(n) \) is such that the average degree grows more slowly than \( \ln n \), and the “dense” case when the average degree grows faster than \( \ln n \). (This is not quite in line with the usual meanings for random graphs, where “dense” often means having average degree growing linearly with the number of nodes, and “sparse” may mean having bounded average degree or at least having average degree growing sublinearly.) The main result on random scaled unit disk graphs is Theorem 2.1 below. This states that, under some mild extra conditions, in the sparse case the ratio of the chromatic number to the clique number tends to 1 in probability as \( n \to \infty \), and in the dense case the ratio tends to \( 2 \sqrt{3/\pi} \approx 1.103 \).

We consider also the more general “frequency-distance constraints” model, when we take into consideration not just “cochannel interference” but also interference arising from nearby channels in the radio spectrum. Now we are interested in the span of channels needed. There are natural lower bounds for the span, corresponding to the clique number lower bound for the chromatic number. The main result is Theorem 3.1 below. This states essentially that, in the sparse case the ratio of the span to the “clique bound” tends to 1 in probability as \( n \to \infty \), and in the dense case the ratio tends to a certain constant at least 1.103.

The above results indicate that usually the chromatic number and span are not much greater than clique-based lower bounds. This may help to explain the success in practice of heuristic coloring and channel assignment methods based on starting from a maximum clique or something similar, and to explain why it is generally easier to make fairly good estimates of the spectral requirements for a radio network than might have been feared, given the link with graph coloring problems.

The plan of the paper is as follows. We describe the results on scaled unit disk graphs in Section 2 and those on frequency distance models in Section 3. The intention is that these first three sections may be easily read by a wide audience. The proofs of the results, together with some refinements, are presented in the next three sections. First in Section 4 we collect some useful preliminary results on distributions in the plane. Then Section 5 contains the proofs for random scaled unit disk graphs, and Section 6 does the same for frequency distance models. Finally, Section 7 contains some brief concluding remarks. The main results given here were first presented at the workshop on Radio Channel Assignment at Brunel University in July 2000.
2. SCALED UNIT DISK GRAPHS

2.1. Unit Disk Graph Models for Channel Assignment

We suppose that the available radio channels are evenly spaced in the radio spectrum, and so they may be taken to be the positive integers. Interference arises when the same channel is reused at sites that are not sufficiently far apart—this is called “cochannel” interference—and similarly for first adjacent channels and second adjacent channels and so on, with diminishing effect.

In the most basic version of the channel assignment problem, we consider only cochannel interference, and assume that we are given a threshold distance \( d \) or \( d_0 \) such that interference will be acceptable as long as no channel is reused at sites less than distance \( d \) apart. Suppose that we are given a family \( \mathcal{V} = (x_i : i \in V) \) of points in the plane and a diameter \( d \geq 0 \). We let \( G(\mathcal{V}, d) \) denote the graph with set \( V \) of nodes in which distinct nodes \( i \) and \( j \) are adjacent whenever the Euclidean distance \( d(x_i, x_j) \) is less than the diameter \( d \). Another way of specifying the same graph is to associate with each node \( i \in V \) an open disk of diameter \( d \) centered on \( x_i \), and to let distinct nodes \( i \) and \( j \) be adjacent when the associated disks intersect. The graph \( G(\mathcal{V}, d) \) is called a scaled unit disk graph, since each disk has the same diameter. (It is often easier to work with the supremum norm; see, for example, [1, 2], but it is the Euclidean norm that is appropriate here.)

Our basic version of the channel assignment problem involves coloring such scaled unit disk graphs. We need to assign colors (radio channels or frequencies) to the nodes (sites or transmitters), using as few colors as possible but avoiding (excessive) interference. Recall that a coloring of the nodes of a graph \( G \) is proper if no two adjacent nodes receive the same color. Thus we are interested in proper colorings of the scaled unit disk graph \( G = G(\mathcal{V}, d) \) using few colors. The least possible number of colors is the chromatic number \( \chi(G) \). This quantity is always at least the clique number \( \omega(G) \), which is the largest number of nodes which are pairwise adjacent.

It is of interest to consider the ratio \( \chi(G)/\omega(G) \), and in particular to investigate how large this ratio is “usually.” We always have \( \chi(G)/\omega(G) \leq 3 \) for a unit disk graph \( G \) (see [20]). To give some meaning here to the word “usually,” we need (a) much empirical data, or (b) a suitable random model together with either (b1) many simulations—see, for example, [34], or (b2) theoretical analysis. We adopt the last approach, and take some first steps in investigating random models for channel assignment problems in the plane.

2.2. Results on Coloring Random Scaled Unit Disk Graphs

Let the random variable \( X \) be distributed in the plane with some distribution \( \nu \). Let \( X_1, X_2, \ldots \), be independent random variables, each distributed like \( X \). Let \( X^{(n)} \) denote the family consisting of the first \( n \) random points \( X_1, \ldots, X_n \). [We could equally consider independent Poisson processes \( \Pi_1, \Pi_2, \ldots \) each with some mean measure \( \mu \) such that the total measure of the whole plane is 1, and let \( X^{(n)} \) be the union of \( \Pi_1, \ldots, \Pi_n \) (which is a Poisson process with total measure \( n \)) (see, e.g., [13]). Let \( d = d(n) \to 0 \) for \( n = 1, 2, \ldots \), and let \( G_n \) denote the random (embedded) scaled unit disk graph \( G(X^{(n)}, d(n)) \). We shall always assume that \( d(n) \to 0 \) as \( n \to \infty \).

For some previous results on random scaled unit disk graphs, see [6, 7, 21–27], and see also the forthcoming book [29]. Note in particular the importance of the ratio \( d^2n/\ln n \). For example, suppose that the underlying distribution \( \nu \) is the uniform distribution on the
unit square, so that for large $n$ the average degree of a node is close to $\pi d^2 n$ [indeed, given that $X_i$ is not within distance $d$ of the boundary of the square, its expected degree equals $\pi d^2(n - 1)$]. Then as $n \to \infty$, the probability that $G_n$ is connected tends to 0 if $d^2 n / \ln n \to 0$ and tends to 1 if $d^2 n / \ln n \to \infty$. For a more precise result, see [23]. These background results indicate that we might expect a change in behavior depending on whether the average degree grows more slowly or faster than $\ln n$. For modeling channel assignment problems, slow growth of some sort in the average degree seems reasonable, but it is not at all clear for example how this should compare with $\ln n$. We focus on two cases, the “sparse” case when $d^2 n / \ln n \to 0$ and so the average degree grows more slowly than $\ln n$; and the “dense” case when $d^2 n / \ln n \to \infty$ and so the average degree grows faster than $\ln n$.

An important parameter of the distribution will be the maximum density $\nu_{\max}$. This may be defined in many equivalent ways, for example, as

$$
\nu_{\max} = \sup_{B} \nu(B) / \lambda(B),
$$

where the supremum is over all open balls $B$, $\nu(B) = P[X \in B]$, and $\lambda(B)$ is the area of $B$, that is $\lambda(B) = \pi r^2$ if $B$ has radius $r$. We may think of $\nu_{\max}$ as the maximum value of the density function corresponding to $\nu$ (see Lemma 4.2 below and the discussion around it to make this precise). We shall be interested in the case when $\nu_{\max}$ is finite. Here is our main result on random scaled unit disk graphs.

**Theorem 2.1.** Let the distribution $\nu$ have finite maximum density. Let $d = d(n)$ satisfy $d(n) \to 0$ as $n \to \infty$.

(a) (Sparse case) As $n \to \infty$, if $d^2 n$ is $o(\ln n)$ and is $n^{o(1)}$, then

$$
\chi(G_n) / \omega(G_n) \to 1 \quad \text{in probability.}
$$

(b) (Dense case) As $n \to \infty$, if $d^2 n / \ln n \to \infty$, then

$$
\chi(G_n) / \omega(G_n) \to 2 \sqrt{3}/\pi \sim 1.103 \quad \text{a.s.}
$$

In the sparse case, part (a) above, since $d^2 n$ is $o(\ln n)$ the further condition that $d^2 n$ is $n^{o(1)}$ is equivalent to the condition that $d^2 n \geq n^{-\epsilon(n)}$ for some function $\epsilon(n) = o(1)$, and thus says that $d^2 n$ should not tend to zero too quickly. For applications to channel assignment, it probably would suffice to consider the simpler stronger condition that the average degree is at least 1 say. In part (b) above we use “a.s.” or “almost surely” in the standard sense in probability theory, that is, we are asserting that

$$
P[\chi(G_n) / \omega(G_n) \to 2 \sqrt{3}/\pi \quad \text{as } n \to \infty] = 1.
$$

The constant $2 \sqrt{3}/\pi$ above arises as follows. Pack as many as possible disjoint unit disks in a large square, of side $s$. Then the ratio of the area of the square to the total area of the disks tends to $2 \sqrt{3}/\pi$ as $s \to \infty$. Let us amplify Theorem 2.1, and split the sparse and dense cases into separate results. This will help to explain a little more about where the results in the theorem come from.

Given a graph $G$, let $\Delta(G)$ denote the maximum degree of any node, and let $\delta^*(G)$
denote the degeneracy of $G$, that is, the maximum over all induced subgraphs $H$ of $G$ of the minimum degree in $H$. Then $\chi(G) \leq \delta^*(G) + 1 \leq \Delta(G) + 1$—see, for example, [12, 33].

Given a unit disk graph $G$ with $n$ nodes, it is possible in $O(n^{4.5})$ steps to determine the clique number $\omega(G)$, even without being given an embedding as $G(V, d)$ (see [5, 30]). In practice for problems arising in radio channel assignment (which do not necessarily give rise to a scaled unit disk graph) usually it turns out to be easy to determine or at least approximate it closely, and coloring methods that start from large cliques or near cliques have proved to be very successful [10]. See also [3] where a similar phenomenon is discussed for coloring problems which arise in quite a different context, namely, examination timetabling.

It is of interest also to consider a natural lower bound on $\omega(G)$. The hitting number of $G(V, d)$ is the maximum over all points $x$ in the plane of the number of disks in the representation which contain $x$, or equivalently the maximum over all open disks of diameter $d$ of the number of points of $V$ in the disk. Let us denote this quantity by $\omega^-(G(V, d))$. It is a “scan statistic” measuring the “local density” of points at a certain scale. Of course, we always have

$$\omega(G(V, d)) \geq \omega^-(G(V, d)).$$

It is straightforward to compute the quantity $\omega^-$ in $O(n^3)$ steps. We shall see that with high probability $\omega$ and $\omega^-$ are very close, which may help somewhat to explain why it has been found to be easy to calculate $\omega$. (It may be shown that always $\omega^-/\omega > 1/3$, and the value $1/3$ may not be increased [4].) Recall that always

$$\omega^- \leq \omega \leq \chi \leq \delta^* + 1 \leq \Delta + 1.$$

**Theorem 2.2** (On sparse random scaled unit disk graphs). Let $d = d(n)$ satisfy $d^2n$ is $o(\ln n)$ and is $n^{o(1)}$. Let

$$k = k(n) = \frac{\ln n}{\ln\left(\frac{\ln n}{d^2 n}\right)}.$$

Then $k \to \infty$ as $n \to \infty$, and in probability $\Delta(G_n)/k \to 1$ and $\omega^-(G_n)/k \to 1$, and so $\chi(G_n)/\omega(G_n) \to 1$.

**Theorem 2.3** (On dense random scaled unit disk graphs). Let $d = d(n)$ satisfy $d \to 0$ and $d^2n/\ln n \to \infty$ as $n \to \infty$. Let

$$k = k(n) = \nu_{\max}(\pi/4)d^2n.$$

Then as $n \to \infty$, almost surely $\omega^-(G_n)/k \to 1$, $\omega(G_n)/k \to 1$, $\chi(G_n)/k \to 2\sqrt{3}/\pi$, $\delta^*(G_n)/k \to 2$, and $\Delta(G_n)/k \to 4$.

The lemmas used to prove these two theorems give bounds on the rates of convergence.

There is an unfortunate gap between the sparse and dense cases above. It would be interesting to learn about the behavior of $\chi(G_n)/\omega(G_n)$ when $d^2n/\ln n \to \beta$ where $0 < \beta < 3$. 

**References:**

[12] [Article 12].

[33] [Article 33].

[5] [Article 5].

[30] [Article 30].

[10] [Article 10].

[4] [Article 4].
$\beta < \infty$. See [29] for the behavior of $\omega(G_n)$ in this case, and for further related results. Also, see [28] for very precise results on scan statistics with a fixed scanning set.

3. FREQUENCY-DISTANCE MODELS

In the last section we discussed the case when only cochannel interference is taken into account. However, for channel assignment problems we might wish to consider more than just this source of interference. Suppose still that each site needs one channel, but now we are given a vector $d = (d_0, d_1, \ldots, d_{l-1})$ of $l \geq 1$ distances, where $d_0 \geq d_1 \geq \cdots \geq d_{l-1} > 0$. We call such a vector a distance $l$-vector. Given a family $\mathcal{V} = (x_v : v \in V)$ of points (sites) in the plane, an assignment $\phi : V \to \{1, 2, \ldots, t\}$ is called $d$-feasible if it satisfies the frequency-distance constraints

$$d(x_u, x_v) < d_i \Rightarrow |\phi(u) - \phi(v)| > i$$

for each pair of distinct points $u, v$ in $V$ and for each $i = 0, 1, \ldots, l - 1$. The span of the problem span $(\mathcal{V}, d)$ is the least integer $t$ for which there is such an assignment. (Some authors call $t - 1$ the span.)

This frequency-distance model is a popular standard model for channel assignment (see, e.g., [8, 9, 17]), with $l$ typically equal to 2 or 3 or 4. Note that it is implicitly assumed that all the transmitters have the same power, are omnidirectional, and are located in an otherwise featureless plain. Further, only pairwise effects are considered, leading to binary constraints—there is no attempt to add up contributions to interference. However, even though it is still rather simplified, the frequency-distance model does significantly extend the basic scaled unit disk graph model discussed earlier.

When $l = 1$, so that there is just one distance $d_0$ given, we are back to coloring scaled unit disk graphs as discussed above. For an example with $l = 2$, suppose that $d = (\sqrt{2}, 1)$ and the sites are the integer points $(i, j)$ in the plane. Then we may obtain a $d$-feasible assignment $\phi$ by setting $\phi((i, j)) = 1$ if $i + j$ is odd, and $\phi((i, j)) = 2$ if $i + j$ is even. Clearly, the span is 2 in this case. The values $d_0, d_1, \ldots, d_{l-1}$ are prescribed with the intention that any $d$-feasible assignment will lead to acceptable levels of interference. As discussed above, the $d_0$-constraint limits cochannel interference, while the $d_1$-constraint limits the interference from first adjacent channels, and so on.

We wish to develop an understanding of the quantity span $(\mathcal{V}, d)$, and in particular to see how it compares with certain natural lower bounds. One of these lower bounds on the span comes from considering the “distance-$s$ cliques.” A family of points forms a distance-$s$ clique if each pair of points in the set is at (Euclidean) distance less than $s$. If there is a distance-$d_j$ clique [sometimes called a level-$(j + 1)$ clique] with $k$ elements then, since any two channels assigned to points in this set must be at least $j + 1$ apart, we have

$$\text{span}(\mathcal{V}, d) \geq 1 + (k - 1)(j + 1) = (j + 1)k - j.$$

Let us call the maximum value of these bounds over all $j = 0, \ldots, l - 1$ and all distance-$d_j$ cliques the clique bound for the problem and denote it by cliquebound $(\mathcal{V}, d)$.

Now let $c = (c_0, c_1, \ldots, c_{l-1})$ be a fixed distance $l$-vector and let $d = d(n) \to 0$
as \( n \to \infty \). We shall use \( d \) to scale the vector \( c \) appropriately, and focus on the problem generated by the family \( X^{(n)} \) consisting of the first \( n \) random points \( X_1, \ldots, X_n \) together with the distance vector \( dc = d(n)c \). Denote the corresponding span by \( \text{span}(X^{(n)}, dc) \). Are there results for these random frequency-distance problems corresponding to the earlier results on scaled unit disk graphs?

The quantity \( \text{cliquebound}(\mathcal{V}, d) \) introduced above may be defined by

\[
\text{cliquebound}(\mathcal{V}, d) = \max_{j} \{(j + 1)\omega(G(\mathcal{V}, d_j)) - j\},
\]

where the maximum is over \( j = 0, \ldots, l - 1 \). We consider also

\[
\text{colorbound}(\mathcal{V}, d) = \max_{j} \{(j + 1)\chi(G(\mathcal{V}, d_j)) - j\}.
\]

Clearly \( \text{colorbound}(\mathcal{V}, d) \equiv \text{cliquebound}(\mathcal{V}, d) \) since \( \chi(G) \equiv \omega(G) \) for every graph \( G \), and Lemma 6.1 below shows that \( \text{span}(\mathcal{V}, d) \equiv \text{colorbound}(\mathcal{V}, d) \). Thus

\[
\text{span}(\mathcal{V}, d) \equiv \text{colorbound}(\mathcal{V}, d) \equiv \text{cliquebound}(\mathcal{V}, d).
\]

The next theorem will summarize our results on random frequency-distance models. Before stating it, we recall some definitions from [17], in particular the definition of the inverse channel density \( \chi(c) \).

Let \( A \) be any set of points in the plane. For \( x > 0 \) let \( g(x) \) be the supremum of the ratio \( |A \cap B|/\pi x^2 \) over all open disks \( B \) of radius \( x \). The upper density of \( A \) is \( \sigma^+(A) = \inf_{x>0} g(x) \). For example, the set of integer points in the plane has upper density 1.

Given a distance \( l \)-vector \( d \), for each \( i = 1, 2, \ldots, \) the \( i \)-channel density \( \alpha_i(d) \) is the supremum of the upper density \( \sigma^+(A) \) over all sets \( A \) of points in the plane for which there is a \( d \)-feasible assignment using channels 1, \ldots, \( i \). The 1-channel density \( \alpha_1(1) \) is thus the maximum density of a packing of pairwise disjoint unit-diameter open disks in the plane, and so \( \alpha_1(1) = 2/\sqrt{3} \). The channel density \( \alpha(d) \) is the infimum over all positive integers \( i \) of \( \alpha_i(d)/i \). It is not hard to see that \( \alpha(1) = \alpha_1(1) \) and so \( \alpha(1) = 2/\sqrt{3} \); and that always \( 0 < \alpha(d) < \infty \). Further, define the inverse channel density \( \chi(d) \) to be \( 1/\alpha(d) \).

**Theorem 3.1.** Suppose that the distribution \( \nu \) has finite maximum density. Let \( c = (c_0, c_1, \ldots, c_{l-1}) \) be a fixed distance \( l \)-vector. Let \( d = d(n) \) satisfy \( d(n) \to 0 \) as \( n \to \infty \).

(a) **(Sparse case)** As \( n \to \infty \), if \( d^2 n \) is \( o(\ln n) \) and is \( n^{o(1)} \), then in probability

\[
\text{span}(X^{(n)}, dc)/\text{cliquebound}(X^{(n)}, dc) \to 1.
\]

(b) **(Dense case)** As \( n \to \infty \), if \( d^2 n/\ln n \to \infty \) then a.s.

\[
\text{span}(X^{(n)}, dc)/\text{colorbound}(X^{(n)}, dc) \to (2/\sqrt{3})\chi(c)/\max_{j} \{(j + 1)c_j\},
\]

where the maximum is over \( j = 0, \ldots, l - 1 \), and

\[
\text{colorbound}(X^{(n)}, dc)/\text{cliquebound}(X^{(n)}, dc) \to 2\sqrt{3}/\pi.
\]

By Theorems 2.2 and 2.3, it would make no difference in the above result if we replaced \( \omega \) by \( \omega^{-1} \) in the definition of the clique bound.
Suppose for example that \( l = 2 \) and \( c = (1, 1/\sqrt{2}) \). Then from [17] we have \( \chi(c) = 1 \) and \( \max\{c^2_0, 2c^2_1\} = 1 \), and so the limit in (2) above is \( 2/\sqrt{3} \); and hence in the dense case

\[
\text{span}(X^{(n)}, dc)/\text{cliquebound}(X^{(n)}, dc) \to 4/\pi
\]
a.s. as \( n \to \infty \). If \( c = (1, c_1) \), where \( 0 < c_1 \leq 1/\sqrt{3} \), then we have \( \chi(c) = \sqrt{3}/2 \) and \( \max\{c^2_0, 2c^2_1\} = 1 \), and so the limit in (2) above is 1. See Theorems 6.2 and 6.5 in section 6 below for more detailed results extending Theorem 3.1, much as Theorems 2.2 and 2.3 extend Theorem 2.1.

4. RANDOM PRELIMINARIES

In this section we collect some useful preliminary results on distributions in the plane, probability inequalities and such matters. We start with two basic lemmas concerning “high density” sets, that is, sets \( A \) such that \( \nu(A)/\lambda(A) \) is large. We shall always use “measurable” to mean Lebesgue measurable, and \( \lambda(A) \) to denote the Lebesgue measure (area) of a measurable set \( A \).

For a real number \( t \), a point \( x \) and sets \( A \) and \( B \) of points in the plane, we let \( tA = \{ta : a \in A\} \), \( x + A = \{x + a : a \in A\} \), \( A + B = \{a + b : a \in A, b \in B\} \) and \( A - B = A + (-1)B \).

**Lemma 4.1.** Let \( \nu \) be any probability distribution on the plane. For any measurable sets \( A \) and \( B \) such that \( A - B \) is a measurable set with \( 0 < \lambda(A - B) < \infty \), there is a translate \( B' = y + B \) of \( B \) such that \( A \cap B' \neq \emptyset \) and

\[
\nu(B') \geq \nu(A)\lambda(B)/\lambda(A - B).
\]

**Proof.** Let \( X \) have distribution \( \nu \), and let \( Y \) be uniformly distributed over \( A - B \), independently of \( X \). Note that for each \( x \in A \), we have \( x - B \subseteq A - B \) and so \( P[Y \in x - B] = \lambda(B)/\lambda(A - B) \). Hence

\[
E_Y \nu(Y + B) = E_Y E_X I_{(X \in Y + B)}
= E_X(E_Y I_{(Y \in X - B)})
\geq E_X(\lambda(B)/\lambda(A - B)I_{(X \in A)})
= \nu(A)\lambda(B)/\lambda(A - B),
\]

which yields the desired result. Here \( E_X \) denotes taking expectation with respect to the random variable \( X \), and \( 1_F \) denotes the indicator of the event \( F \).

The next lemma provides what we need about the maximum density, see also the comments following its proof.
Lemma 4.2. Let \( \nu \) be any probability distribution on the plane. Let \( \mathcal{A} \) denote the set of all measurable sets \( A \) with \( 0 < \lambda(A) < \infty \). Then

\[
\nu_{\text{max}} = \sup \{ \nu(A)/\lambda(A) : A \in \mathcal{A} \}.
\]

Further, let \( C \) be any fixed bounded measurable set with \( \lambda(C) > 0 \), and for each \( t \) let \( f_c(t) \) be the supremum of \( \nu(D)/\lambda(D) \) over all translates \( D \) of \( tC \). Then

\[
f_c(t) \to \nu_{\text{max}} \quad \text{as } t \to 0.
\]

Proof. Let \( 0 < \varepsilon < 1 \), and let \( A_1 \in \mathcal{A} \). It suffices for us to show that

\[
f_c(t) \geq (1 - \varepsilon)^3 \nu(A_1)/\lambda(A_1)
\]

for all sufficiently small \( t \). There is an open set \( A_2 \supseteq A_1 \) with \( \lambda(A_2) < (1 - \varepsilon)^{-1} \lambda(A_1) \) (see, e.g., [14] or [32]). There is a finite union \( A_3 \subseteq A_2 \) of open disks with \( \nu(A_3) \geq (1 - \varepsilon) \lambda(A_2) \). Thus

\[
\frac{\nu(A_3)}{\lambda(A_3)} \geq (1 - \varepsilon) \frac{\nu(A_2)}{\lambda(A_2)} \geq (1 - \varepsilon)^2 \frac{\nu(A_1)}{\lambda(A_1)}.
\]

There exists \( \delta > 0 \) such that \( \lambda(A_3 - tC) \leq (1 - \varepsilon)^{-1} \lambda(A_3) \) whenever \( |t| \leq \delta \). By Lemma 4.1, for each such \( t \) there is a translate \( D \) of \( tC \) (meeting \( A_3 \)) with

\[
\frac{\nu(D)}{\lambda(D)} \geq \frac{\nu(A_3)}{\lambda(A_3 - D)} \geq (1 - \varepsilon) \frac{\nu(A_3)}{\lambda(A_3)} \geq (1 - \varepsilon)^3 \frac{\nu(A_1)}{\lambda(A_1)}.
\]

Thus we have shown (4) as required.

Aside. There is another equivalent definition of \( \nu_{\text{max}} \) which is perhaps more natural for some. Suppose that \( \nu_{\text{max}} \) is finite. Then \( \nu \) is absolutely continuous with respect to \( \lambda \), and so \( \nu \) has a density function \( f \). Then

\[
\nu_{\text{max}} = \inf \{ t : \lambda([x : f(x) \geq t]) = 0 \}.
\]

To see why this is true, call the right-hand side \( \rho \). Let \( t < \rho \), and let \( A = \{ x : f(x) \geq t \} \). Then \( \lambda(A) > 0 \), and so by the lemma above we have \( \nu_{\text{max}} \geq \nu(A)/\lambda(A) \geq t \). Thus \( \nu_{\text{max}} \geq \rho \). Conversely, for any measurable set \( A \), \( \nu(A) = \int_A f(x)dx \leq \rho \lambda(A) \), and so \( \nu_{\text{max}} \leq \rho \). For related results, see, for example, Chapter 8 of [32].

Next we give a general packing result for distributions with finite maximum density, see also [29] for related results.

Lemma 4.3. Let \( \nu \) be a distribution on the plane with finite maximum density \( \nu_{\text{max}} \), and let \( 0 < \sigma < \nu_{\text{max}} \). Then there exist \( \rho > 0 \) and \( \eta > 0 \) such that for all \( 0 < r \leq \rho \), the following holds: there is a family of at least \( \eta r^{-2} \) pairwise disjoint open disks such that each disk \( D \) in the family has radius \( r \) and satisfies \( \nu(D)/\lambda(D) \geq \sigma \).
Proof. Let \( \delta > 0 \) be such that \((1 - 2\delta)/(1 + \delta)\nu_{\text{max}} \geq \sigma\). Let \( S_1 \) denote the unit square, and let \( B_1 \) denote the unit ball. There exists \( \beta \) with \( 0 < \beta \leq 1/2 \) such that \( \lambda(S_1 + \beta B_1) \leq (1 + \delta) \). By Lemma 4.2, there is a square \( S \), with side \( s \) say, such that \( \nu(S)/s^2 \geq (1 - \delta)\nu_{\text{max}} \).

We claim that for each positive integer \( n \) and each real \( r \) with \( 0 < r \leq \beta s/n \), there is a family of at least \( n^2/10 \) pairwise disjoint open disks \( D \) each of radius \( r \) and such that \( \nu(D)/\lambda(D) \geq \sigma \).

Suppose for the meantime that this claim is true. Let \( \rho = \beta s \), and let \( \eta = \rho^2/40 \). Let \( 0 < r \leq \rho \). There is a unique positive integer \( n \) such that \( p/(n + 1) < r \leq \rho/n \). By the claim, there is a family of at least \( n^2/10 \) pairwise disjoint open disks \( D \) each of radius \( r \) and with \( \nu(D)/\lambda(D) \geq \sigma \). But

\[
r^{-2} < \left( \frac{n + 1}{\rho} \right)^2 \leq \left( \frac{2}{\rho} \right)^2 n^2,
\]

and so \( n^2/10 \geq \eta r^{-2} \), as required.

It remains then to prove the claim. Let \( n \) be a positive integer and let \( 0 < r \leq \beta s/n \). Partition the square \( S \) into \( n^2 \) subsquares \( T_i \) of side \( s/n \). Call a subsquare \( T_i \) good if \( \nu(T_i)/\lambda(T_i) \geq (1 - 2\delta)\nu_{\text{max}} \). Then at least \( n^2/2 \) subsquares are good. For, if not, then

\[
\nu(S) = \sum_i \nu(T_i) \\
\leq (n^2/2)\lambda(T_i)\nu_{\text{max}} + (n^2/2)\lambda(T_i)(1 - 2\delta)\nu_{\text{max}} \\
= n^2\lambda(T_i)\nu_{\text{max}}(1 - \delta) \\
= \lambda(S)\nu_{\text{max}}(1 - \delta),
\]

which contradicts the choice of \( S \).

For each good subsquare \( T_i \), by Lemma 4.1 we may pick an open disk \( D_i \) of radius \( r \) meeting \( T_i \) and with

\[
\frac{\nu(D_i)}{\lambda(T_i)} \geq \frac{\nu(T_i)}{\lambda(T_i) + D_i} \geq (1 + \delta)^{-1} \frac{\nu(T_i)}{\lambda(T_i)} \geq \sigma.
\]

[Note that \( \lambda(T_i + D_i) \leq (1 + \delta)\lambda(T_i) \) by our choice of \( \beta \).] Call these the good disks.

Let \( G \) be the scaled unit disk graph with nodes corresponding to the good disks, where two nodes are adjacent whenever the corresponding disks meet. In any nonempty subset \( W \) of the good disks, the disk corresponding to the leftmost bottom subsquare can meet at most 4 other good disks in \( W \). Thus the graph \( G \) has degeneracy at most 4. Hence it is 5-colorable, and in particular it has a stable set of size at least \((1/5)n^2/2 = n^2/10\). (Recall that a stable set in a graph is a set of nodes no two of which are adjacent, so that all may be given the same color.) This gives us the family of disks as desired.

We shall also need two lemmas concerning the upper tail of the binomial distribution \( B(n, p) \). The first lemma gives convenient nonasymptotic bounds for the upper tail: we include a proof here for completeness, we do not claim that the result is new.
Lemma 4.4. Let $n$ be a positive integer, let $0 \leq p \leq 1$, and let $X \sim B(n, p)$. Then for each positive integer $k$ with $\mu = np \leq k \leq n$,

$$\left(\frac{\mu}{ek}\right)^k \leq P[X \geq k] \leq 2\left(\frac{e\mu}{k}\right)^k.$$

**Proof.** We may assume that $0 < p < 1$. First consider the upper bound. Let $p_k$ denote $P[X = k]$. We use one preliminary result.

**Claim.** Let $\beta > 1$. Then for each integer $k \geq \beta np$,

$$P[X \geq k] \leq p_j/(1 - \beta^{-1}).$$

**Proof of claim.** For $\beta np \leq j \leq n - 1$,

$$\frac{p_{j+1}}{p_j} \leq \frac{n - j}{j} \cdot \frac{p}{1 - p} \leq \frac{n - \beta np}{\beta np} \cdot \frac{p}{1 - p} = \frac{1 - \beta p}{\beta - \beta p} \leq \frac{1}{\beta}.$$

Hence

$$P[X \geq k] = \sum_{j=k}^{n} p_j \leq p_k(1 + \beta^{-1} + \beta^{-2} + \cdots) = p_k/(1 - \beta^{-1}),$$

which completes the proof of the claim.

Now consider an integer $k \geq e\mu$. Then by the claim with $\beta = e$ and the inequality $\binom{n}{k} \leq \left(\frac{enp}{k}\right)^k$, we obtain

$$(1 - e^{-1})P[X \geq k] \leq p_k \leq \left(\frac{enp}{k}\right)^k (1 - p)^{n-k} \leq \left(\frac{e\mu}{k}\right)^k.$$

Since $(1 - e^{-1}) \geq 1/2$, we obtain the upper bound.

Now consider the lower bound. Note that $1 - p \geq e^{-\frac{e}{np}}$. Thus
\[(1 - p)^{n-k} \geq \exp\left(-\frac{(n-k)p}{1-p}\right) \geq e^{-k},\]
since
\[
\frac{(n-k)p}{1-p} = \frac{\mu - kp}{1-p} \leq k.
\]
Hence, since also \((\frac{n}{k}) \geq (\frac{n}{k})^k\), we obtain
\[
P[X \geq k] \geq P[X = k] \geq \left(\frac{np}{k}\right)^k (1 - p)^{n-k} \equiv \mu e^k.
\]

We shall also use the following form of the “Chernoff bound” for large deviations (see, e.g., Theorem 2.1 of [11]).

**Lemma 4.5.** Let the random variable \(Y\) with mean \(\mu\) have either a binomial distribution or a Poisson distribution. Let \(\varepsilon > 0\) and let \(\delta = (1 + \varepsilon)\ln(1 + \varepsilon) - \varepsilon\). Then \(\delta > 0\), and
\[
P[|Y - \mu| \geq \varepsilon \mu] \leq 2e^{-\delta \mu}.
\]

Finally we state a well-known simple lemma that may be used to “introduce independence” into a problem, by replacing a multinomial distribution by independent Poisson random variables. We write \(Y \sim \text{Po}(\mu)\) to indicate that \(Y\) has the Poisson distribution with mean \(\mu\).

**Lemma 4.6.** Let \(p_1, \ldots, p_k \geq 0\) with \(\sum_i p_i = 1\). Let \(n\) be a positive integer, let \(Y_1, \ldots, Y_k\) be independent random variables with \(Y_i \sim \text{Po}(p_i n)\), and let \(N = \sum_i Y_i\). Then \(N \sim \text{Po}(n)\), \(P[N = n] \geq 1/e \sqrt{n}\), and conditional on \(N = n\), the random variables \(Y_1, \ldots, Y_k\) have the multinomial distribution with parameters \(n\) and \(p_1, \ldots, p_k\).

5. PROOFS FOR RANDOM SCALED UNIT DISK GRAPHS

5.1. Deterministic Preliminaries

In this subsection we give two deterministic lemmas which we shall use to prove part (b) (the dense case) of Theorem 2.1. These lemmas are based on work in [16]. Let \(T\) denote the triangular lattice, that is the set of all integer linear combinations of the points \((0, 0)\) and \((\frac{1}{2}, \frac{\sqrt{3}}{2})\). The fundamental cells of \(T\) are hexagons with area \(\sqrt{3}/2\). When we use the lemma below to consider the random case, we will choose the parameter \(\tau\) to be slightly greater than the maximum density.
Lemma 5.1. Let $\mathcal{V} = (x_v : v \in V)$ be a family of points in the plane, let $d > 0$, and consider the corresponding scaled unit disk graph $G = G(\mathcal{V}, d)$. Let $\delta > 0$, assume that the maximum number of points of $\mathcal{V}$ in a cell of the scaled triangular lattice $(\delta d)T$ is finite, and denote it by $y$. Then

$$\chi(G) < ((1/\delta) + 3)^2 y, \quad (6)$$

and

$$\omega(G) < (\pi/(2\sqrt{3}))((1/\delta) + 3)^2 y. \quad (7)$$

In particular, if $y \leq (\sqrt{3}/2)\pi(\delta d)^2 n$, then

$$\chi(G) < (1 + 3\delta)^2((\sqrt{3}/2)\pi d^2 n,$$

and

$$\omega(G) < (1 + 3\delta)^2\pi/4 \tau d^2 n.$$ 

Proof. We have $\chi(G(T, d)) < (d + 1)^2$ (see Theorem 3 in [16]), where a precise result is given. Let $s = \delta d$. Since the distance from any point in a cell in $sT$ to the center is at most $s/\sqrt{3}$,

$$\chi(G) \leq y\chi(G(sT, d + (2/\sqrt{3})s))$$
$$= y\chi(G(T, (1/\delta) + (2/\sqrt{3})))$$
$$< y((1/\delta) + (2/\sqrt{3}) + 1)^2,$$

and (6) follows.

By Theorem 2 in [16], $\omega(G(T, d)) < (2/\sqrt{3})(\pi/4)(d + (2/\sqrt{3}))^2$, since the triangular lattice $T$ has a cell structure with density $2/\sqrt{3}$ and radius $1/\sqrt{3}$. Hence, arguing as above,

$$\omega(G) \leq y\omega(G(sT, d + (2/\sqrt{3})s))$$
$$= y\omega(G(T, (1/\delta) + (2/\sqrt{3})))$$
$$< y(2/\sqrt{3})(\pi/4)((1/\delta) + (4/\sqrt{3}))^2,$$

and (7) follows. 

When we use the next lemma to consider the random case, we will choose the parameter $\tau$ to be slightly less than the maximum density. Recall that $\lambda(B)$ denotes the area of $B$.

Lemma 5.2. For any $\varepsilon > 0$ there is a $K$ such that for each family $\mathcal{V} = (x_v : v \in V)$ of points in the plane and each $d > 0$, the corresponding scaled unit disk graph
$G = G(\mathcal{V}, d)$ has the following property. If some disk $B$ of diameter at least $Kd$ contains at least $\lambda(B)\pi n$ points of $\mathcal{V}$, then

$$\chi(G) \geq (1 - e)(\sqrt{3/2})\pi d^2 n.$$  

Proof. Let $0 < e < 1$. By Thue’s theorem (see, e.g., [19, 31]), there exists $K$ such that the following holds. For any set $S$ of pairwise disjoint open disks of diameter $d$ each meeting a disk $B$ of diameter at least $Kd$,

$$|S| \leq (1 - e)^{-1}\lambda(B)/((\sqrt{3/2})d^2).$$

Thus, if $H$ denotes the subgraph of $G$ induced by the nodes $v$ with $x_v \in B$, the stability number $\alpha(H)$ of $H$ is at most this bound. Hence

$$\chi(G) \geq \chi(H) \geq |V(H)|/\alpha(H) \geq (1 - e)(\sqrt{3/2})\pi d^2 n,$$

as required. $\blacksquare$

### 5.2. Random Scaled Unit Disk Graphs

Assume throughout this subsection that the maximum density $\nu_{\max}$ is finite. Let us first consider Theorem 2.2, the sparse case. The following lemma will essentially answer all our questions concerning this case.

**Lemma 5.3.** Let $d = d(n)$ satisfy $d^2 n$ is $o(\ln n)$ and is $n^{o(1)}$. Let $M_n$ be the maximum over all open disks $B$ of diameter $d(n)$ of the number of points $X_1, \ldots, X_n$ in $B$. [Thus $M_n$ is $\omega^-(G_n).$] Finally let

$$k = k(n) = \frac{\ln n}{\ln \left(\frac{\ln n}{d^2 n}\right)}.$$

Then $k \to \infty$ as $n \to \infty$, and $M_n/k \to 1$ in probability.

Proof. Note first that $\ln(\ln n/d^2 n) = o(\ln n)$ and so indeed $k \to \infty$ as $n \to \infty$. Also $k/d^2 n \to \infty$ as $n \to \infty$, and so $\ln(k/d^2 n) \sim \ln(\ln n/d^2 n)$; and hence

$$k \ln(k/d^2 n) \sim \ln n.$$  

(8)

Let $e > 0$. Let $k_1 = k_1(n) = (1 + e)k(n)$. Let $p_1 = p_1(n) = \nu_{\max}\pi d^2$, and let $Z_1 \sim B(n, p_1)$. Note that $\mu_1 = np_1$ satisfies $\mu_1 = \Theta(d^2 n)$, and so $k_1 \geq \mu_1$ for $n$ sufficiently large. Then by Lemma 4.4 and (8)

$$P[Z_1 \geq k_1] \geq 2 \left(\frac{e\mu_1}{k_1}\right)^{k_1} = n^{-(1+e+o(1))}.$$
Hence

\[ P[M_n \geq k] \leq nP[Z \geq k] = O(n^{-\varepsilon+o(1)}) = o(1). \]

Now let \( k_2 = k_2(n) = (1 - \varepsilon)k(n) \). Let \( p_2 = p_2(n) = (1 - \varepsilon)\nu_2 \pi d^2/4 \). Let \( Z_2 \sim B(n, p_2) \) and let \( \mu_2 = np_2 \). Then by Lemma 4.4 and (8) as before, for \( n \) sufficiently large

\[ P[Z_2 \geq k_2] \geq \left( \frac{\mu_2}{ek_2} \right)^{k_2} \geq n^{-(1-\varepsilon+o(1))}. \]

But, by Lemma 4.3, there exists \( \eta > 0 \) such that for \( n \) sufficiently large there is a family of at least \( \eta d^{-2} \) pairwise disjoint disks \( B \) of diameter \( d \) with \( \nu(B) \geq p_2 \). Hence

\[
P[M_n < k_2] \leq (1 - P[Z_2 \geq k_2])^\eta d^{-2} \\
\leq \exp(-n^{-(1-\varepsilon+o(1))}\eta d^{-2}) \\
= \exp(-n^{\varepsilon+o(1)})/d^2n \\
= \exp(-n^{\varepsilon+o(1)}) \\
= o(1).
\]

Lemma 5.3 easily yields Theorem 2.2 on sparse random scaled unit disk graphs. It immediately gives \( o^-(G_n)/k \rightarrow 1 \) in probability as \( n \rightarrow \infty \). Further, suppose that we replace \( d(n) \) by \( 2d(n) \) in the lemma. Then \( k(n) \) is replaced by \( (1 + o(1))k(n) \), and now \( \Delta(G_n) \leq M_n \). Hence we see that, for any \( \varepsilon > 0 \),

\[ P[\Delta_n < (1 + \varepsilon)k] \rightarrow 1 \quad \text{as} \quad n \rightarrow \infty. \]

We require several lemmas to prove Theorem 2.3, on dense random scaled unit disk graphs. We give first upper bounds and then lower bounds on \( \chi(G_n) \) and \( \omega(G_n) \). After that we deal similarly with \( \Delta(G_n) \) and \( \delta^*(G_n) \). In each case, for the upper bounds we assume that \( d^2n/\ln n \rightarrow \infty \) as \( n \rightarrow \infty \), and obtain upper bounds that hold with probability \( 1 - e^{-\Omega(d^2n)} \); and for the lower bounds we assume that \( d(n) = o(1) \) and sometimes also that \( d^2n \rightarrow \infty \) as \( n \rightarrow \infty \), and obtain lower bounds that hold with probability \( 1 - e^{-\Omega(n)} \).

**Lemma 5.4.** Let \( \sigma > \nu_\text{max} \). Let \( d = d(n) \) satisfy \( d^2n/\ln n \rightarrow \infty \) as \( n \rightarrow \infty \). Then

\[ \chi(G_n) \leq \sigma(\sqrt{3}/2)d^2n \]

and

\[ \omega(G_n) \leq \sigma(\pi/4)d^2n \]

with probability \( 1 - e^{-\Omega(d^2n)} \).
Proof. Let \( \delta > 0 \). Let \( s = \delta d \), and consider the scaled triangular lattice \( sT \). Let \( Y_n \) denote the maximum number of points \( X_1, \ldots, X_n \) in a cell. Let \( \sigma > \tau > \nu_{\text{max}} \). By Lemma 5.1 it suffices to show that

\[
P[Y_n > (\sqrt{3}/2)\tau s^2 n] \leq e^{-\Omega(d^2 n)}.
\]

Let \( Z_n \sim B(n, p) \), where \( p = p(n) = (\sqrt{3}/2)\nu_{\text{max}}s^2 \). By the Chernoff bound, Lemma 4.5, since \( \pi/\nu_{\text{max}} > 1 \),

\[
P[Z_n > (\tau/\nu_{\text{max}})np] = e^{-\Omega(d^2 n)}.
\]

By combining sets \( S \) with \( \nu(S) \leq p/2 \), we end up with at most \( 2p + 1 \) sets formed from unions of cells, such that these sets partition the plane and each set \( S \) has \( \nu(S) \leq p \). Hence

\[
P[Y_n > (\sqrt{3}/2)\tau s^2 n] \leq (2/p + 1)P[Z_n > (\sqrt{3}/2)\tau s^2 n]
\]

\[
= (2/p + 1)P[Z_n > (\tau/\nu_{\text{max}})np]
\]

\[
\leq \exp(2 \ln(1/d) + O(1) - \Omega(d^2 n))
\]

\[
= \exp(-\Omega(d^2 n)).
\]

Lemma 5.5. Let \( 0 < \sigma < \nu_{\text{max}} \). Let \( d = d(n) \to 0 \) as \( n \to \infty \). Then

\[
\chi(G_n) \geq \sigma(\sqrt{3}/2)d^2 n
\]

and

\[
\omega^-(G_n) \geq \sigma(\pi/4)d^2 n
\]

with probability \( 1 - e^{-\Omega(n)} \) as \( n \to \infty \).

Proof. We first prove the lower bound on \( \chi(G_n) \). Let \( \sigma < \tau < \tau' < \nu_{\text{max}} \). Let \( K \) be a positive constant, and let \( a = a(n) = \pi K^2 d^2/4 \). Let \( A_n \) be the event that each disk of diameter \( Kd \) contains less than \( a\tau n \) of the points \( X_1, \ldots, X_n \). By Lemma 5.2 it suffices for us to show that \( P(A_n) = e^{-\Omega(n)} \).

By Lemma 4.3 there is a constant \( \eta > 0 \) such that for all \( n \) sufficiently large there is a family of at least \( \eta d^{-2} \) pairwise disjoint disks \( B \) of diameter \( Kd \) each with \( \nu(B) \geq a\tau' \). Let \( Z \sim B(n, a\tau') \). By the Chernoff bound, Lemma 4.5, since \( \tau < \tau' \),

\[
P[Z < a\tau n] \leq e^{-\Omega(m)} = e^{-\Omega(d^2 n)}.
\]

Hence

\[
P[A_n] \leq (P[Z < a\tau n])^{\eta d^{-2}} = e^{-\Omega(n)}.
\]

To prove the lower bound on \( \omega^-(G_n) \) just take \( K = 1 \) above, to see that with probability \( 1 - e^{-\Omega(n)} \) there is a disk of diameter \( d \) containing at least \( \sigma(\pi/4)d^2 n \) of the
points $X_1, \ldots, X_n$.

Now we deal with $\Delta(G_n)$ and $\delta^*(G_n)$: as above, we first give upper bounds then lower bounds. For other results on $\Delta(G_n)$, see [29]. It is straightforward to deal with the upper bounds for these quantities.

**Lemma 5.6.** Let $\sigma > \nu_{\text{max}}$. Let $d = d(n)$ satisfy $d^2 \ln n \to \infty$ as $n \to \infty$. Then

$$\Delta(G_n) \leq \sigma \pi d^2 n$$

and

$$\delta^*(G_n) \leq \sigma (\pi/2) d^2 n$$

with probability $1 - e^{-\Omega(d^2 n)}$.

**Proof.** Let $X \sim B(n, \nu_{\text{max}} \pi d^2)$. Then the probability that the $i$th node has degree at least $x$ is at most $P(X \geq x)$. Hence

$$P[\Delta(G_n) \geq \sigma \pi d^2 n] \leq n P[X \geq \sigma \pi d^2 n] = e^{-\Omega(d^2 n)},$$

by the Chernoff bound, Lemma 4.5. We may handle $\delta^*(G_n)$ similarly. Now let $Y \sim B(n, \nu_{\text{max}} (\pi/2) d^2)$. Note that in any subgraph $H$ of $G_n$ there must be a node with each of its neighbors in the half-plane to the right of it. Hence

$$P[\delta^*(G_n) \geq \sigma (\pi/2) d^2 n] \leq n P[Y \geq \sigma (\pi/2) d^2 n] = e^{-\Omega(d^2 n)},$$

again by Lemma 4.5.

Finally we must deal with the lower bounds on $\Delta(G_n)$ and $\delta^*(G_n)$. First we consider the lower bound on $\Delta(G_n)$. We use two lemmas, with the first being a preliminary “local” result for a single disk.

**Lemma 5.7.** Suppose that $d = d(n) \to 0$ and $d^2 n \to \infty$ as $n \to \infty$. Let $0 < \varepsilon \leq 1$. Then there exist $c > 0$ and $n_0$ such that the following holds.

Let $D$ be an open disk with radius $(1 - \varepsilon)d$ and with $\nu(D) \geq (1 - \varepsilon^2/2) \lambda(D) \nu_{\text{max}}$. Let $N \sim \text{Po}(\nu(D)n)$. Suppose that we throw $N$ points independently into $D$, each with distribution that of $X$ conditional on $X \in D$. Let $\Delta(D)$ denote the maximum degree of the corresponding scaled unit disk graph with diameter $d = d(n)$ [where $\Delta(D) = 0$ if $N = 0$]. Then

$$P[\Delta(D) < (1 - 3\varepsilon) \pi d^2 n \nu_{\text{max}}] \leq e^{-c d^2 n}$$

for each $n \geq n_0$.

**Proof.** Let $D'$ be the open disk of radius $\varepsilon d$ with the same center as $D$. Then
\[ \nu(D') = \nu(D) - \nu(D \setminus D') \]
\[ \geq (1 - \varepsilon^2/2)(1 - \varepsilon)^2 \pi d^2 \nu_{\max} - ((1 - \varepsilon)^2 \pi d^2 - \varepsilon^2 \pi d^2) \nu_{\max} \]
\[ = (\varepsilon^2 - (\varepsilon^2/2)(1 - \varepsilon)^2) \pi d^2 \nu_{\max} \]
\[ \geq (\varepsilon^2/2) \pi d^2 \nu_{\max}. \]

Let \( N' \) be the number of points that land in \( D' \). Then \( N' \sim \text{Po}(\nu(D') n) \), and hence
\[ P[N' = 0] = \exp(-\nu(D') n) \leq \exp\left(-\left(\varepsilon^2/2\right) \pi \nu_{\max} d^2 n \right) = e^{-\Omega(d^n)}. \]

Also,
\[ \nu(D) \geq (1 - \varepsilon^2/2)(1 - \varepsilon)^2 \pi d^2 \nu_{\max} \geq (1 - 2\varepsilon) \pi d^2 \nu_{\max}, \]
and hence by Lemma 4.5,
\[ P[N < 1 + (1 - 3\varepsilon) \pi d^2 n \nu_{\max}] = e^{-\Omega(d^n)}. \]

But since the distance between any point in \( D' \) and any point in \( D \) is less than \( d \), it follows that
\[ P[\Delta(D) < (1 - 3\varepsilon) \pi d^2 n \nu_{\max}] \]
\[ \leq P[N' = 0] + P[N < 1 + (1 - 3\varepsilon) \pi d^2 n \nu_{\max}] \]
\[ = e^{-\Omega(d^n)}. \]

**Lemma 5.8.** Suppose that \( d = d(n) \to 0 \) and \( d^2 n \to \infty \) as \( n \to \infty \). Let \( 0 < \varepsilon \leq 1 \). Then there exist \( c > 0 \) and \( n_0 \) such that
\[ P[\Delta(G_n) < (1 - \varepsilon) \pi d^2 n \nu_{\max}] \leq e^{-cn} \]
for each \( n \geq n_0 \).

**Proof.** By Lemma 4.3, there exists \( \eta > 0 \) such that for \( n \) sufficiently large there is a family \( \mathcal{F}_n \) of at least \( \eta d^{-2} \) pairwise disjoint open disks \( D \) each of radius \( (1 - \varepsilon)d \) and each satisfying
\[ \nu(D) \geq (1 - \varepsilon^2/2) \lambda(D) \nu_{\max}. \]

By Lemma 4.6,
\[ P[\Delta(G_n) < (1 - 3\varepsilon) \pi d^2 n \nu_{\max}] \leq e^{-\sqrt{n} P[\Delta(D) < (1 - 3\varepsilon) \pi d^2 n \nu_{\max} \quad \forall D \in \mathcal{F}_n]}, \]
where the random variables \( \Delta(D) \) for \( D \in \mathcal{F}_n \) are as in Lemma 5.7 and are independent. Hence, with \( c \) as in that lemma, if \( c' < c\eta \), then
\[ P[\Delta(G_n) < (1 - 3\varepsilon)\pi d^2 n \nu_{\text{max}}] \leq e^{\sqrt{n} (e^{-cd^2 n})^n d^2} \leq e^{-c n} \]

for \( n \) sufficiently large.

It remains to handle the lower bound on \( \delta^*(G_n) \). Again we use two lemmas, with the first being a preliminary “local” result.

**Lemma 5.9.** Suppose that \( d = d(n) \to 0 \) and \( d^2 n \to \infty \) as \( n \to \infty \). Let \( 0 < \varepsilon \leq 1 \). Let \( K \) be sufficiently large that for each open disk \( B \) of radius \( K \) and each open disk \( A \) of radius \( 1 \) with center in \( B \), we have

\[ \lambda(A \cap B) \geq (1 - \varepsilon/2)\left(\frac{\pi}{2}\right). \]

Then there exist \( c > 0 \) and \( n_0 \) such that the following holds.

Let \( B \) be an open disk of radius \( Kd \) with

\[ \nu(B) \geq (1 - \varepsilon/(2K^2))\lambda(B) \nu_{\text{max}}, \]

[There is such a disk for all \( n \) sufficiently large.] Let \( N \sim \text{Po}(\nu(B)n) \). Suppose that we throw \( N \) points independently into \( B \), each with distribution that of \( X \) conditional on \( X \in B \). Let \( \delta(B) \) denote the minimum degree of the corresponding scaled unit disk graph with diameter \( d = d(n) \) [where \( \delta(B) = 0 \) if \( N = 0 \)]. Then

\[ P[\delta(B) < (1 - 4\varepsilon)(\pi/2)d^2 n \nu_{\text{max}}] \leq e^{-cd^2 n} \]

for each \( n \geq n_0 \).

**Proof.** Let \( A \) be a disk of radius \( d \) with center in \( B \). Since

\[ \nu(B \setminus A) \leq (\lambda(B) - \lambda(A \cap B)) \nu_{\text{max}}, \]

we have

\[ \nu(A \cap B) = \nu(B) - \nu(B \setminus A) \]

\[ \geq (1 - \varepsilon/2K^2)\lambda(B) \nu_{\text{max}} - (\lambda(B) - \lambda(A \cap B)) \nu_{\text{max}} \]

\[ = (\lambda(A \cap B) - (\varepsilon/2K^2)\lambda(B)) \nu_{\text{max}} \]

\[ \geq ((1 - \varepsilon/2)(\pi/2)d^2 - (\varepsilon/4)\pi d^2) \nu_{\text{max}} \]

\[ = (1 - \varepsilon)(\pi/2)d^2 \nu_{\text{max}} \]

\[ \geq (1 - \varepsilon)\nu(B)/(2K^2). \]

Thus

\[ \nu(A \cap B)/\nu(B) \geq (1 - \varepsilon)/(2K^2). \] (9)
Let $a = a(n) = \pi d^2 n \nu_{\text{max}} = \Theta(d^2 n)$. Let $F$ denote the event that

$$(1 - 2\varepsilon)K^2 a \leq N \leq (1 + \varepsilon)K^2 a.$$ 

Note that for the complement $\bar{F}$ of $F$, by Lemma 4.5 we have $P[\bar{F}] = e^{-\Omega(d^n)}$. Now

$$P[\delta(B) < (1 - 4\varepsilon)a/2] \leq P[\delta(B) < (1 - 4\varepsilon)a/2|F] + P[\bar{F}].$$

It remains to bound the first term on the right-hand side appropriately. Let $Z \sim B\left((1 - 2\varepsilon)K^2 a\right)$, $(1 - \varepsilon)/(2K^2))$. Then by (9)

$$P[\delta(B) < (1 - 4\varepsilon)a/2|F] \leq (1 + \varepsilon)K^2 a P[Z < 1 + (1 - 4\varepsilon)a/2],$$

and the right-hand side is $e^{-\Omega(d^n)}$ by Lemma 4.5, since

$$E(Z) \geq (1 - 2\varepsilon)(1 - \varepsilon)a/2 \geq (1 - 3\varepsilon)a/2.$$ 

**Lemma 5.10.** Suppose that $d = d(n) \to 0$ and $d^n \to \infty$ as $n \to \infty$. Let $0 < \varepsilon \leq 1$. Then there exist $c > 0$ and $n_0$ such that

$$P[\delta^*(G_n) < (1 - 4\varepsilon)(\pi/2)d^2 n \nu_{\text{max}}] \leq e^{-cn}$$

for each $n \geq n_0$.

**Proof.** We proceed much as in the proof of Lemma 5.8. Let $K$, $c$, and $n_0$ be as in Lemma 5.9. By Lemma 4.3, there exists $\eta > 0$ such that for $n$ sufficiently large there is a family $\mathcal{F}_n$ of at least $\eta d^{-2}$ pairwise disjoint open disks $D$ each of radius $Kd$ and each satisfying

$$\nu(D) \geq (1 - \varepsilon/(2K^2))\lambda(D)\nu_{\text{max}}.$$ 

Let

$$b = b(n) = (1 - 4\varepsilon)(\pi/2)d^2 n \nu_{\text{max}}.$$ 

Then, by Lemma 4.6 as before,

$$P[\delta^*(G_n) < b] \leq e \sqrt{n} P[\delta^*(D) < b \quad \forall \ D \in \mathcal{F}_n],$$

where the random variables $\delta^*(D)$ for $D \in \mathcal{F}_n$ are as in Lemma 5.9 and are independent. Hence, by that lemma, if $c' < c\eta$,

$$P[\delta(G_n) < b] \leq e \sqrt{n} (e^{-c'd\varepsilon n})^{\nu_{\text{max}}/2} \leq e^{-c'd\varepsilon n}$$

for $n$ sufficiently large.

The last lemma above yields the required lower bound on $\delta^*(G_n)$, and thus completes the proof of Theorem 2.3, on dense random scaled unit disk graphs.
6. PROOFS FOR THE FREQUENCY-DISTANCE MODEL

The three deterministic lemmas below, Lemmas 6.1, 6.3, and 6.4, are proved using results and ideas from [17]. We use them to prove extended versions of the two parts of Theorem 3.1.

**Lemma 6.1.** Let \( V = (x_v : v \in V) \) be a family of points in the plane, and let \( d = (d_0, \ldots, d_{l-1}) \) be a distance \( l \)-vector. Then

\[
\text{span}(V, d) \leq l\Delta(G(V, d_0)) + 1,
\]

and for each \( j \in \{0, 1, \ldots, l - 1\} \)

\[
\text{span}(V, d) \geq (j + 1)\chi(G(V, d_j)) - j.
\]

**Proof.** Let \( \mathbf{1}_j \) denote the \( j \)-vector of 1’s. We shall use the result that

\[
\text{span}(V, d\mathbf{1}_j) = j\chi(G(V, d)) - j + 1. \tag{10}
\]

This is for example Theorem 9 of [17], with different notation. For the upper bound

\[
\text{span}(V, d) \leq \text{span}(V, d_0\mathbf{1}_j)
\]

\[
= l\chi(G(V, d_0)) - l + 1
\]

\[
\leq l\Delta(G(V, d_0)) + 1.
\]

For the lower bound,

\[
\text{span}(V, d) \geq \text{span}(V, d_j\mathbf{1}_{j+1}) = (j + 1)\chi(G(V, d_0)) - j.
\]

We may use Lemma 6.1 together with Lemma 5.3 to prove part (a) of Theorem 3.1. We state an expanded version of this result as a theorem.

**Theorem 6.2** (On sparse frequency distance models). Suppose that the distribution \( v \) has finite maximum density. Let \( c = (c_0, c_1, \ldots, c_{l-1}) \) be a fixed distance \( l \)-vector. Let \( d = d(n) \) satisfy \( d^2n \) is \( o(\ln n) \) and is \( n^{o(1)} \). Let

\[
k = k(n) = \frac{\ln n}{\ln\left(\frac{\ln n}{d^2n}\right)}.
\]

Then \( k \to \infty \) as \( n \to \infty \), and in probability

\[
\text{span}(X^{(n)}, d\mathbf{c})/k \to 1,
\]

and
cliquebound(\(X^{(n)}\), \(dc)\)/\(k\) \(\rightarrow\) 1;

and hence

\[\text{span}(X^{(n)}, dc)/\text{cliquebound}(X^{(n)}, dc) \rightarrow 1.\]

**Proof.** By Lemma 5.3, \(k \rightarrow \infty\) as \(n \rightarrow \infty\), and in probability

\[\Delta(G(X^{(n)}, dc_0))/k \rightarrow 1,\]

and

\[\omega(G(X^{(n)}, dc_{l-1}))/k \rightarrow 1.\]

Now the theorem follows from Lemma 6.1.

We shall use the next two deterministic lemmas to prove part (b) of Theorem 3.1.

**Lemma 6.3.** Let \(e\) be a distance \(l\)-vector. Let \(\sigma > 0\). Let \(d = d(n)\) satisfy \(d^2n \rightarrow \infty\) as \(n \rightarrow \infty\). Let \(e > 0\). Then there exist \(K\) and \(n_0\) such that for each \(n \geq n_0\) the following holds. Suppose that \(\mathcal{V} = (x_v : v \in V)\) is a family of points in the plane such that for some \(s \geq Kd\) and some \((s \times s)\) square \(S\)

\[|\{v \in V : x_v \in S\}| \geq \sigma s^2 n.\]

Then

\[\text{span}(\mathcal{V}, dc) \geq (1 - \varepsilon)\chi(e)\sigma d^2 n.\]

**Proof.** Let \(\delta > 0\) satisfy \((1 + \delta)^{-3} \geq 1 - \varepsilon\). Since from [19], \(\alpha_i(e)/i \rightarrow \alpha(e)\) as \(i \rightarrow \infty\), we may choose a positive integer \(i_0\) such that \(\alpha_i(e)/i \leq (1 + \delta)\alpha(e)\) for each integer \(i > i_0\). Let \(K = c_0/\delta\). Since \(d^2n \rightarrow \infty\) as \(n \rightarrow \infty\), there exists \(n_0\) such that \(\sigma d^2 n > (1 + \delta)^2 \alpha_{i_0}(e)\) for each \(n \geq n_0\). We must check that these choices of \(K\) and \(n_0\) are as required.

Let \(s \geq Kd\) and \(n \geq n_0\). Let \(\mathcal{V} = (x_v : v \in V)\) be a family of points in the plane which satisfy the condition stated in the lemma. Let \(i = \text{span}(\mathcal{V}, dc)\). Let \(\hat{S}\) denote the square of side \((s + dc_0)\) obtained by adding a border of width \(dc_0/2\) around \(S\). By tiling the plane with copies of \(\hat{S}\), we may obtain a family of points \(\mathcal{V}'\) with \(\text{span}(\mathcal{V}', dc) = i\) and density at least \(\sigma s^2 n / (s + dc_0)^2\). Hence

\[\sigma s^2 n \leq \alpha_i(de)(s + dc_0)^2.\]

But \(\alpha_i(de) = d^{-2} \alpha_i(e)\), and \((s + dc_0)^2 \leq (1 + c_0/K)^2 s^2 = (1 + \delta)^2 s^2\). Hence

\[\sigma d^2 n \leq (1 + \delta)^2 \alpha_i(e).\]

It follows since \(n \geq n_0\) that \(i > i_0\). Now by the above
\[ \sigma d^2 n \leq (1 + \delta)^2 \alpha(t) \leq (1 + \delta)^3 i \alpha(t), \]

and so

\[ i \geq (1 + \delta)^{-3} \sigma d^2 n / \alpha(t) \geq (1 - \varepsilon) \chi(t) \sigma d^2 n, \]

as required. \hfill \blacksquare

**Lemma 6.4.** Let \( c \) be a distance \( l \)-vector, let \( \sigma > 0 \) and let \( \varepsilon > 0 \). Then there exists \( \delta > 0 \) such that the following holds. Let \( d > 0 \) and let \( s = \delta d \). Let \( V = (x_v : v \in V) \) be a set of points in the plane. Consider the scaled triangular lattice \( sT \), and assume that the maximum number \( y \) of points of \( V \) in a cell satisfies

\[ y \leq (\sqrt{3}/2) \sigma s^2 n. \]  

(11)

Then

\[ \text{span}(V, dc) \leq (1 + \varepsilon) \chi(t) \sigma d^2 n. \]

**Proof.** Let \( \beta > 0 \) be sufficiently small that

\[(1 + \beta) (1 + 2\beta/c_{l-1}^2) + 2\beta^2/l \chi(t) \leq 1 + \varepsilon.\]

By Theorem 1 of [17], there is a constant \( t_0 > 0 \) such that

\[ \text{span}(T, tc) \leq (1 + \beta)(2/\sqrt{3}) \chi(t)t^2 \]

for all \( t \geq t_0 \). Let \( \delta = \min(\beta, t_0^{-1}) \). Then, as in the proof of Lemma 5.1, we may see that

\[ \text{span}(V, dc) \leq y \text{span}(sT, dc + (2/\sqrt{3})s1_l) + (y - 1)(l - 1) \]

\[ \leq y \text{span}(T, (1/\delta)c + 21_l) + yl \]

\[ \leq y \text{span}(T, (1 + 2\delta/c_{l-1})(1/\delta)c) + yl \]

\[ \leq y(1 + \beta)(2/\sqrt{3}) \chi(t)(1 + 2\delta/c_{l-1})^2(1/\delta)^2 + yl \]

\[ \leq \sigma d^2 n (1 + \beta) \chi(t)(1 + 2\delta/c_{l-1})^2 + 2\delta^2/l \sigma d^2 n \]

\[ = ((1 + \beta)(1 + 2\delta/c_{l-1})^2 + 2\delta^2/l \chi(t)) \chi(t) \sigma d^2 n \]

\[ \leq (1 + \varepsilon) \chi(t) \sigma d^2 n, \]

as required. \hfill \blacksquare

We may now prove part (b) of Theorem 3.1. In fact, we prove an extended version of that result.
Theorem 6.5 (On dense frequency distance models). Suppose that the distribution \( \nu \) has finite maximum density \( \nu_{\text{max}} \). Let \( \mathbf{e} = (c_0, c_1, \ldots, c_{l-1}) \) be a fixed distance \( l \)-vector. Let \( d = d(n) \) satisfy \( d(n) \to 0 \) as \( n \to \infty \) but \( d^2 n / \ln n \to \infty \). As \( n \to \infty \), a.s.

\[
\text{span}(X^{(n)}, \mathbf{de}) / d^2 n \to \nu_{\text{max}} \chi(\mathbf{e}),
\]

\[
\text{colorbound}(X^{(n)}, \mathbf{de}) / d^2 n \to \nu_{\text{max}}(\sqrt{3}/2) \max_j \{ (j + 1) c_j^2 \},
\]

and

\[
\text{cliquebound}(X^{(n)}, \mathbf{de}) / d^2 n \to \nu_{\text{max}}(\pi/4) \max_j \{ (j + 1) c_j^2 \},
\]

where the maxima are over \( j = 0, \ldots, l - 1 \).

**Proof.** We may prove just as in the proof of Lemma 5.5 that if \( \sigma < \nu_{\text{max}} \), then for any \( K > 0 \) there is a \( (Kd \times Kd) \) square \( S \) containing at least \( \sigma(Kd)^2 n \) points of \( X^{(n)} \) with probability \( 1 - e^{-\Omega(n)} \). Similarly we may prove just as in the proof of Lemma 5.4 that the condition (11) in Lemma 6.4 holds with probability \( 1 - e^{-\Omega(d^2 n)} \). These two results yield the first part of the theorem. The last two parts follow from Theorem 2.3.

7. CONCLUDING REMARKS

We have taken a natural first step in investigating random models for radio channel assignment problems in the plane, by considering the “independent” model, where transmitter sites appear independently according to some general probability distribution. We have been able to analyze the asymptotic behavior in the “sparse” and the “dense” cases. This has cast some light on the important question of how close the chromatic number or more generally the span is typically to lower bounds based on cliques, and has helped to explain why finding approximate solutions to channel assignment problems is less difficult than might have been expected.

It would be interesting to investigate the intermediate case between sparse and dense (see the comments at the end of Section 2) and to explore, perhaps by simulations, how soon the asymptotic behavior sets in. It would be interesting also to consider less simplified probabilistic models, perhaps allowing some form of “repulsion” between possible sites. We have confined our attention here to models in the plane with Euclidean distances, since that is the natural setting for the motivating application to radio channel assignment. Extensions of some of the present results to higher dimensions and different norms will appear in [29].

ACKNOWLEDGMENTS

I would like to acknowledge helpful comments from Mathew Penrose and from the referees.
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