6 Zero-Sum Games

6.1 Introduction

A *two-person zero-sum game* is specified by an $m \times n$ matrix A. Player 1 (Robin) may choose amongst m *actions* or *strategies* and player 2 (Chris) amongst n. If Robin chooses action i and Chris action j then Robin receives payoff a_{ij} and Chris receives payoff $-a_{ij}$.

Example 1 A zero-sum 2×3 game:

$$A = \begin{bmatrix} 3 & 1 & 2 \\ -1 & -1 & 2 \end{bmatrix}$$

A zero-sum game, then, is one in which one player can only gain at the expense of the other. Such games are also called 'strictly competitive'. We can also consider 'constant-sum' games where payoffs add up to a constant, not necessarily zero, but these are equivalent to zero-sum games. Games such as poker or chess might be modelled as zero-sum games. They are less obviously appropriate in an economic context, where there may be mutual gains from cooperation (for example, collusion in an oligopoly).

6.2 Saddlepoints

Example 1 continued

Since whatever Robin wins, Chris loses, Chris would like to make Robin's payoff as small as possible. A very risk-averse approach for Robin would then be to choose the strategy which guarantees him the highest gain. If he chooses strategy 1, then the least that he will obtain is 1 (when Chris chooses 2), whereas if he chooses 2, he may do as badly as -1. Strategy 1 would then be a conservative choice for Robin, called the *maximin* strategy (it maximises the minimum gain).

Similarly, a risk-averse approach for Chris would to choose the strategy which guarantees the smallest gain for Robin (and so the highest gain for himself). If he chooses 1, the best Robin can obtain is 3, if 2 then the best is 1 and if 3 then the best is 2. Strategy 2 would then be a conservative choice for Chris, called the *minimax* strategy (it minimises the maximum gain).

Now if Robin and Chris do play actions 1 and 2 respectively, note that each would have no incentive to revise their choice if they knew the choice of the other: given that Chris is playing action 2, action 1 is the best choice for Robin and vice-versa. Such a pair of strategies is called a *saddlepoint* or *Nash equilibrium.*

In general the greatest payoff the row player can guarantee himself is *maximin* $= \max_i \min_j a_{ij}$ and the lowest payoff the column player can ensure the row player receives is *minimax* $= \min_j \max_i a_{ij}$. A pair of strategies (k, l) is said to a *saddlepoint* or *Nash equilibrium* if $a_{il} \leq a_{kl} \leq a_{kj}$ for all i and j. It is not difficult to show that:

Theorem 1 *In any zero-sum game, maximin $\leq$ minimax. If it has a saddlepoint, (k, l) then maximin $=$ minimax $= a_{kl}$.*

In a game with a saddlepoint, then, the conservative notions of maximin and maximin coincide with that of Nash equilibrium. Unfortunately, not all games have saddlepoints:

Example 2

$$A = \begin{bmatrix} -3 & 4 \\ 2 & -3 \end{bmatrix}$$

It is easy to check that this game has no saddlepoint (or Nash equilibrium). Note that $maximin = -3 < 2 = minimax$.

We can, however, restore the ideal world of Theorem 1 by introducing the idea of *mixed* strategies. In games such as poker it is an advantage to be unpredictable, so players may wish to make random choices. Thus in the case of Example 2, Robin may choose row 1 with probability p and row 2 with probability $1 - p$.

6.3 The Minimax Theorem

In general, Robin chooses a row-vector $x = (x_1, \ldots, x_m)$ of probabilities (where $\sum_i x_i = 1$ and each $x_i \geq 0$), and picks a row i with probability x_i. Similarly, Chris chooses a column-vector $y = (y_1, \ldots, y_n)'$ of probabilities for the columns. The expected return to Robin is then

$$\sum_i \sum_j x_i y_j a_{ij} = xAy.$$

Robin wishes to choose x in order to maximise

$$\min_y xAy = \min_y \sum_j (xA)_j y_j.$$

A little algebra will show that, for any x, this equals $\min_j (xA)_j$.

So Robin wishes to choose $x = (x_1, \ldots, x_m)$ to maximise $\min_j \left(\sum_i x_i a_{ij}\right)$. We obtain the LP

$$\begin{aligned} \max \quad & z \\ \text{subject to} \quad & \\ & z - \sum_{i=1}^{m} a_{ij} x_i \leq 0 \qquad \text{for } j = 1, \ldots, n \\ & \sum_{i=1}^{m} x_i = 1 \\ & x_i \geq 0 \qquad \text{for } i = 1, \ldots, m \end{aligned}$$

Similarly, Chris wants to choose y to minimise $\max_x xAy = \max_i (Ay)_i$ and we obtain the LP

$$
\begin{aligned}
\min \quad & w \\
\text{subject to} \quad & \\
& w - \sum_{j=1}^{n} a_{ij} y_j \geq 0 \qquad \text{for } j = 1, \ldots, n \\
& \sum_{j=1}^{n} y_j = 1 \\
& y_j \geq 0 \qquad \text{for } j = 1, \ldots, n
\end{aligned}
$$

Example 2 (continued)

$$
A = \begin{bmatrix} -3 & 4 \\ 2 & -3 \end{bmatrix}
$$

So we obtain

LP (Robin)

$$
\begin{array}{llll}
\max & z & & \\
\text{subject to} & z - (-3)x_1 - 2x_2 & \leq 0 & y_1 \\
& z - 4x_1 - (-3)x_2 & \leq 0 & y_2 \\
& x_1 + x_2 & = 1 & w \\
& x_1, x_2 & \geq 0 &
\end{array}
$$

and LP(Chris)

$$
\begin{array}{ll}
\min & w \\
\text{subject to} & w - (-3)y_1 - 4y_2 \geq 0 \\
& w - 2y_1 - (-3)y_2 \geq 0 \\
& y_1 + y_2 = 1 \\
& y_1, y_2 \geq 0
\end{array}
$$

These are dual LPs! This is true in general, and using the Duality Theorem for LPs we obtain

Theorem 2 *(Von Neumann Minimax Theorem (1928)) Consider the matrix game with matrix A. There exist (optimal) mixed strategies x^* for the row player and y^* for the column player, such that*

$$
\min_y x^* A y = x^* A y^* = \max_x x A y^*
$$

The common value $x^* A y^*$ is the *value* of the game.