Comparison of Parallel Solution Techniques for the Eikonal Equation

T. Prince, P. Crowley, O. Thomas, J. Pitt-Francis, J. W. J. Anderson, and J. Hein‡
Department of Statistics, University of Oxford, 1 South Parks Road, OX1 3TG, United Kingdom
‡ hein@stats.ox.ac.uk

Motivation

The Eikonal equation is well established for modelling wave propagation in the field of cardiac electrophysiology. It offers fast computation speeds in contrast to the bidomain or monodomain models. Consequently, developing effective algorithmic solution techniques to the Eikonal equation is an immediate and valuable task. In this project, we look to apply the "Fast Marching Method" algorithm and more conventional PDE solvers within the Chaste computation environment, with a view to eventually applying this over large clusters.

The world of cardiac modelling

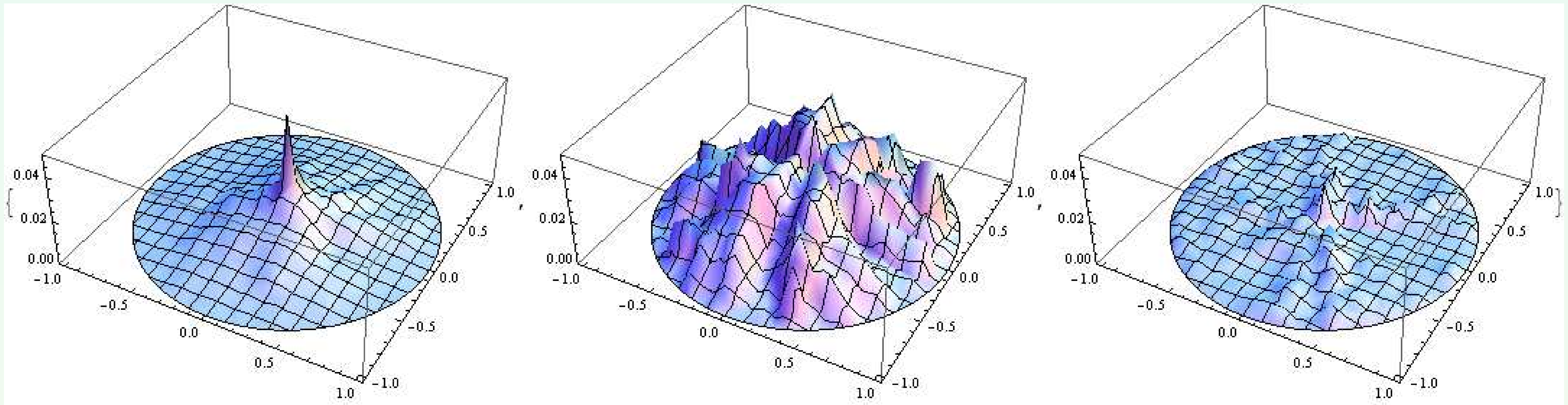
Successful understanding and modelling of cardiac behaviour forms a crucial part of a high-performance computational model of the human body, helping to predict and treat heart disease and inform drug design. A first-principles model proves to be computationally prohibitive but it is possible to take a perturbation from the rest state and take fist order terms to find the Eikonal equation: $|\nabla\phi| = f(\underline{x})$ or $\sqrt{\nabla\phi^T M^T \nabla\phi} = f(\underline{x})$ where M is a tensor related to the conductivity of the medium, ϕ is our scalar which describes the wave at each point and $f(\underline{x})$ is a function which describes the local electrical characteristics of a medium. Solving this equation consists of finding isochronic contours of ϕ and hence finding wavefronts, given initial boundary conditions.

Over the course of the derivatoin of the Eikonal equation we encounter discretisation both in space and time, meaning our solutoin environment will be a discrete mesh of nodes spread over our space, leading us to explore graph-theoretic methods. Similarly, the discretisation in space makes an algorithmic approach very tempting.

A word on Chaste

Chaste (Cancer, Heart and Soft Tissue Environment) is a computation suite written in C++ designed to model cardiac and other biological systems. It follows a "test-driven" approach, and there exists within the extensive codebase methods for approaching solving the Eikonal equation. Of particular relevance to us is a graph-theory-based method using Dijkstra's algorthim and many nonlinear PDE solvers designed for FEM solutions.

Errors

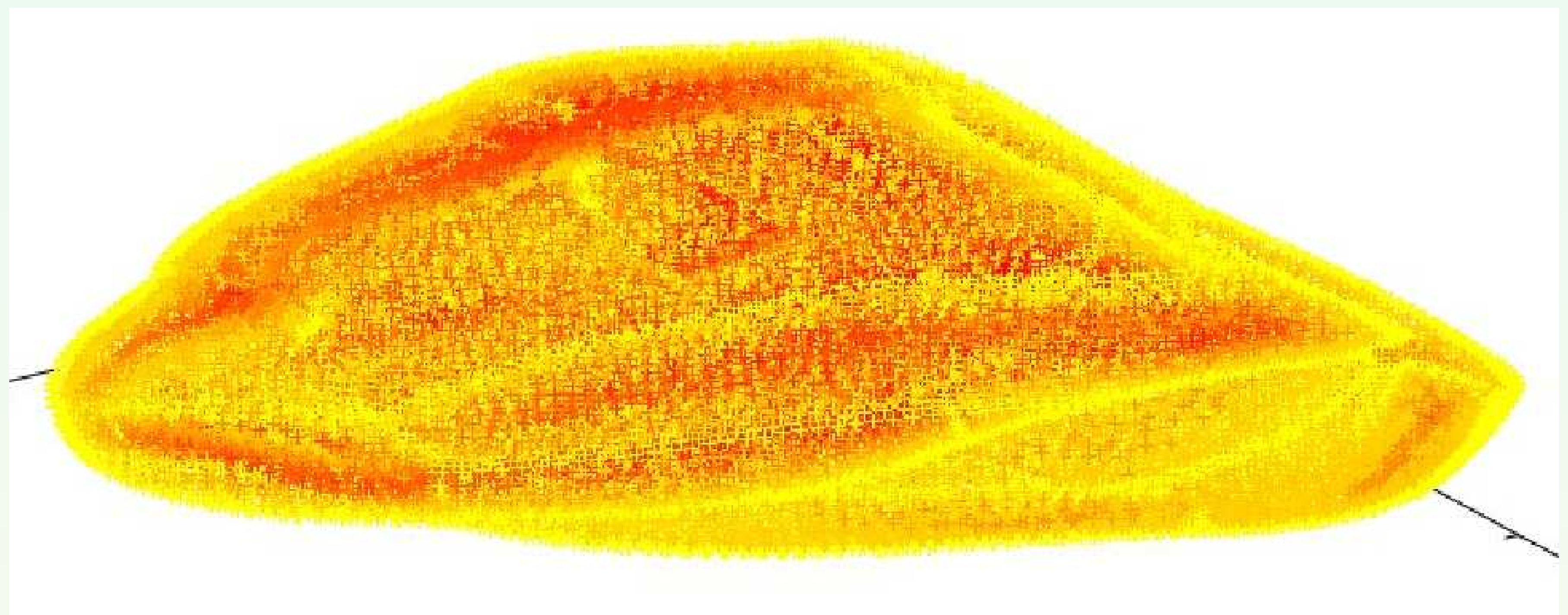


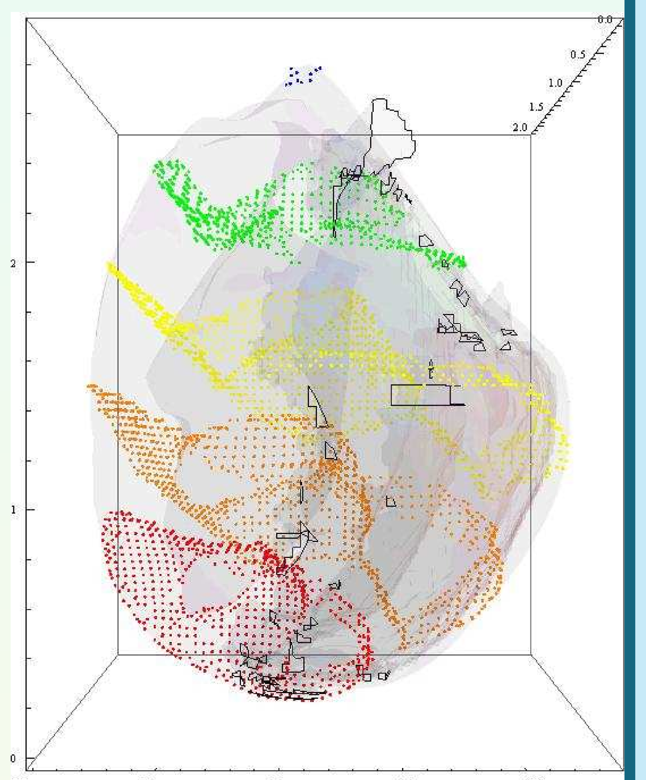
The above figures demonstrate the deviation from the analytical solution when solving the isotropic Eikonal equation on a circular mesh using the Fast-Marching Method, Dijkstra's algorithm and FEM approaches. As we can see, Dijkstra is very noisy, FMM exhibits controlled error growth and FEM has a uniform (adjustable) low tolerance.

In the above solutions the FMM appears to exhibit large error growth towards the centre of the circle; more generally, the solutions grow less accurate in regions of high wavefront curvature. This follows from the approximation that the solution and wavefront is a piecewise flat surface in $(\mathbf{x}, t)$. The only real way to avoid this with the FMM is intelligent mesh refinement in regions of high wavefront curvature.

The FEM solutions have the attractive property of having an adjustable tolerance threshold; our solutions will eventually approach the "best" solution that the mesh discretisation will allow. It is, however, prohibitively slow compared to the graph-theoretic methods of order 10^3 making it only useful when very accurate local results are required. We think that an overall most effective approach would be based on the FMM, hybridised intelligently with the FEM solvers in regions of high wavefront curvature, and possibly Dijkstra in regions of *very* low structural complexity.

Results





These figures have been generated from the output produced by our code. In these figures, we see the Eikonal equation solved over a cuboid and two hearts; the first two show our solution in terms of a spectrum demonstrating when each node is activated, the second highlights some isochrones over the heart.

Acknowledgements

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Finite Element Methods

It is naturally of interest to consider the usual numerical approaches to solving nonlinear PDEs and seeing how these respond to anisotropic meshes and reduced discrete boundary conditions. We used a Navier Stokes form $|\nabla\phi|^2 = 1 - \Gamma\nabla^2\phi$ or $|\sigma\nabla\phi|^2 = 1 - \Gamma\nabla^2\phi$ where $\sigma \in V^* \otimes V^*, V \in \mathbb{R}^3$ and in this investigation the Γ term is constant and iteratively reduced in order to solve the unsmoothed equation. The favoured approach, due to the complex nature of the mesh and boundary conditions, and the need for time efficency is the *Galerkin Finite Element Method* (FEM).

The idea behind this approach is to convert, for the linear case, an infinite dimensional inversion problem into a finite dimensional one by laying a mesh over the domain and taking the vector space formed by the linear span of some basis functions, taken prototypically as piecewise linear over the elements, with no basis function being non zero over a small (mesh dimension dependant) number of elements. This means that the matrix inversion problem one reduces to is a diagonal block matrix that may be easily inverted.

Fast Marching Method

Graph theoretic methods already exist in Chaste, fundamentally based on Dijkstra's algorithm. However, this approach confines our signal to travelling along the edges between nodes, neglecting physical information, and hence gives noisy results. The "Fast Marching Method" is a well established algorithmic approach to find approximate solutions to the Eikonal equation. It resembles Dijkstra's method, but differs in how it uses local information to estimate the travel times to nodes on the mesh. The algorithm begins by asserting a subset of 'source' nodes to be zero distance, then uses this information to estimate the time at which the wavefront will reach neighbouring nodes; each iteration of the algorithm then consists of the finalisation of one of these nodes, and use of its wavefront-time to estimate that of neighbouring nodes. This uses the maximal amount local information to estimate nodes' wavefront-time. The physical interpretation of this implementation is that Fast Marching allows geodesics to pass through the volume of elements, taking us closer to an analytical solution.