

Valuation with meromorphic Lévy processes

Kees van Schaik¹

University of Manchester

¹Part of this talk is joint work with Alexey Kuznetsov, Andreas Kyprianou and Juan Carlos Pardo

Outline

- 1** Setting & two types of problems
- 2** Short intro into meromorphic Lévy processes (mLP's)
- 3** A new Monte Carlo simulation technique for (a.o.) ruin problems with mLP's
- 4** An approximation technique for American option pricing with mLP's

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What are Lévy processes?

- A (one dimensional, real valued) Lévy process X is a Markov process starting from 0 with:
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- Jump structure described by Lévy measure Π :

$$\Pi(A) = \mathbb{E}[\#\{t \in [0, 1] \mid \Delta X_t \in A\}] \text{ for Borel sets } A \subset \mathbb{R}.$$

Note: $\#\{t \in [0, 1] \mid \Delta X_t > \varepsilon\} < \infty$, yet

$$\sum_{t \in [0, 1], \Delta X_t \leq \varepsilon} |\Delta X_t| \text{ might be infinite!}$$

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- Law of X determined by characteristic exponent Ψ :

$$\begin{aligned} \Psi(\theta) &:= -\frac{1}{t} \log \mathbb{E}[e^{i\theta X_t}] \\ &= a i \theta + \frac{1}{2} \sigma^2 \theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x \mathbf{1}_{\{|x| \leq 1\}}) \Pi(dx) \end{aligned}$$

where $a \in \mathbb{R}$, $\sigma \in \mathbb{R}$, Π Lévy measure

Ruin problem

- Recall classic Cramer-Lundberg model. Imagine an insurance company with:
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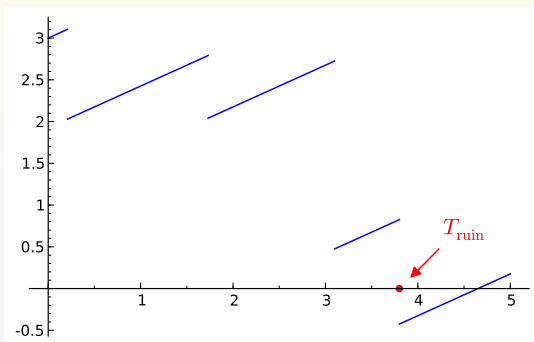
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- Quantities of interest (a.o.): ruin time and overshoot (=deficit at ruin):

$$T_{\text{ruin}} = \inf\{t > 0 \mid X_t < 0\} \quad \text{and} \quad X_{T_{\text{ruin}}}$$

Ruin problem

$$X_t = x + bt - \sum_{i=1}^{N_t} Y_i, \quad T_{\text{ruin}} := \inf\{t > 0 \mid X_t < 0\}$$



A typical path of X with $x = 3$, $b = 1/2$, $Y_i \sim \exp(1)$

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 - much more data fitting possibilities
 - more complicated jump structure (incoming claims)
 - Brownian part: aggregated 'high intensity' small premiums/claims (might be set to 0 as well)

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 - Brownian part: aggregated 'high intensity' small premiums/claims (might be set to 0 as well)
- In this talk: introduce a new simulation method (so-called Wiener-Hopf Monte Carlo simulation method) for a.o. ruin time and overshoot which can be used with meromorphic Lévy processes e.g.

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- Important question: what is the price of an American option in this model? That is, how to find

$$V(T, x) = \sup_{\tau} \mathbb{E}_x [e^{-r(\tau \wedge T)} f(X_{\tau \wedge T})]$$

where τ a stopping time, $T > 0$ expiry date, $r > 0$ interest rate, f the payoff function, \mathbb{P}_x means X starts from x

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- In this talk: introduce an algorithm to approximate $V(T, x)$ which can be used with meromorphic Lévy processes e.g.

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Meromorphic Lévy processes (mLP's)

$$\Psi(\theta) = -\log \mathbb{E}[e^{i\theta X_t}]/t$$

- A mLP X is defined as having:
 - a meromorphic Ψ with poles in $\{-i\rho_n, i\hat{\rho}_n\}_{n \geq 1}$
 - for all $q \geq 0$, $\theta \mapsto q + \Psi(\theta)$ has roots in points $\{-i\zeta_n, i\hat{\zeta}_n\}_{n \geq 1}$
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 - plus some additional technicalities
- Which is essentially equivalent to X having any σ , any a and Lévy measure Π of the form

$$\Pi(dx) = \left(\mathbf{1}_{\{x < 0\}} \sum_{n \geq 1} \hat{c}_n e^{\hat{\rho}_n x} + \mathbf{1}_{\{x > 0\}} \sum_{n \geq 1} c_n e^{-\rho_n x} \right) dx$$

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- Very broad class! One more explicit example is the β -class with Lévy measure of the form

$$\Pi(dx) = \left(\mathbf{1}_{\{x < 0\}} \gamma_1 \frac{e^{\alpha_1 \beta_1 x}}{(1 - e^{\beta_1 x})^{\lambda_1}} + \mathbf{1}_{\{x > 0\}} \gamma_2 \frac{e^{-\alpha_2 \beta_2 x}}{(1 - e^{-\beta_2 x})^{\lambda_2}} \right) dx$$

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- With $e(q)$ exp. distributed with mean $1/q$, indep. of X and

$$\bar{X}_t = \sup_{s \leq t} X_s \quad \text{and} \quad \underline{X}_t = \inf_{s \leq t} X_s$$

we have

$$\phi_q^+(iz) := \mathbb{E} \left[e^{-z\bar{X}_{e(q)}} \right] = \prod_{n \geq 1} \frac{1 + z/\rho_n}{1 + z/\zeta_n},$$

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- Fourier inversion may be applied to deduce the laws of $\overline{X}_{e(q)}$ and $\underline{X}_{e(q)}$. For example, in the β -class we get

$$\mathbb{P}(\overline{X}_{e(q)} \in dx) = \left(\sum_{n \geq 1} k_n \zeta_n e^{\zeta_n x} \right) dx$$

and similar for $\underline{X}_{e(q)}$.

Main points about mLP's

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- Parameters σ , a , Π are typically used to fit data and therefore need to be known
- Knowing the laws of $\overline{X}_{e(q)}$ and $\underline{X}_{e(q)}$ is used in the methods in this talk. Note: as

$$X_{e(q)} \stackrel{d}{=} \underline{X}_{e(q)} + \overline{X}_{e(q)}$$

the law of $X_{e(q)}$ is a mixture of exponentials on the negative and positive axis

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- Let X be an mLP and let I_q and S_q be independent rv's with

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$$X_{\mathbf{e}(q)} \stackrel{d}{=} S_q + I_q$$

and as a consequence

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- Q2: how can we extend this to obtain (approximate) samples from quantities like

$$T_{\text{ruin}} = \inf\{t > 0 \mid X_t < 0\} \quad \text{and} \quad X_{T_{\text{ruin}}}$$

for our ruin problem?

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Idea²: use a 'stochastic time grid'. That is to say:

- Let

$$\mathbf{g}(n, q) := \sum_{i=1}^n \mathbf{e}^{(i)}(q) \text{ for } n \geq 0,$$

where $(\mathbf{e}^{(i)}(q))$ iid, $\mathbf{e}^{(i)}(q) \sim \exp(q)$

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- Idea:

1. we can approximate any $t > 0$ by setting $q = n/t$ and let $n \rightarrow \infty$. Indeed by the law of large numbers we have $\mathbf{g}(n, n/t) \rightarrow t$ a.s. as $n \rightarrow \infty$
2. exploit the homogeneity of X to use (1) above on all grid intervals $[\mathbf{g}(k-1, q), \mathbf{g}(k, q)]$ to get an expression for $(X_{\mathbf{g}(n, q)}, \overline{X}_{\mathbf{g}(n, q)})$

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$$\mathbf{g}(n, q) = \sum_{i=1}^n \mathbf{e}^{(i)}(q) \text{ for } n \geq 0$$

More precisely, let $(I_q^{(i)})$ (resp. $(S_q^{(i)})$) be iid copies of I_q (resp. S_q).

Then for $n = 2$:

$$\begin{aligned} (X_{\mathbf{g}(2,q)}, \overline{X}_{\mathbf{g}(2,q)}) &= (X_{\mathbf{g}(2,q)}, \max \{ \overline{X}_{0, \mathbf{g}(1,q)}, \overline{X}_{\mathbf{g}(1,q), \mathbf{g}(2,q)} \}) \\ &\stackrel{(d)}{=} (X_{\mathbf{g}(1,q)} + Y_{\mathbf{g}(1,q)}, \max \{ \overline{X}_{0, \mathbf{g}(1,q)}, X_{\mathbf{g}(1,q)} + \overline{Y}_{\mathbf{g}(1,q)} \}) \\ &\quad \uparrow \text{by stat. indep. incr., } Y \text{ indep. copy of } X \\ &\stackrel{(d)}{=} (I_q^{(1)} + S_q^{(1)} + I_q^{(2)} + S_q^{(2)}, \max \{ S_q^{(1)}, I_q^{(1)} + S_q^{(1)} + S_q^{(2)} \}) \\ &\quad \uparrow \text{by eq. (1) above} \end{aligned}$$

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Generalised for any $n \geq 1$ we get:

$$(X_{\mathbf{g}(n, q)}, \overline{X}_{\mathbf{g}(n, q)}) \stackrel{d}{=} (V(n, q), J(n, q))$$

where $V(0, q) = J(0, q) = 0$,

$V(n, q) := V(n-1, q) + S_q^{(n)} + I_q^{(n)}$ and $J(n, q) := \max\{J(n-1, q), V(n-1, q) + S_q^{(n)}\}$.

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Using that $\mathbf{g}(n, n/t) \rightarrow t$ a.s. our first **main result** follows:

$$(V(n, n/t), J(n, n/t)) \xrightarrow{d} (X_t, \overline{X}_t) \text{ as } n \rightarrow \infty$$

Hence, for X an mLP, we can produce N samples from $(V(n, n/t), J(n, n/t))$ for big n, N and use

$$\mathbb{E}[f(X_t, \overline{X}_t)] \approx \frac{1}{N} \sum_{i=1}^N f(V^{(i)}(n, n/t), J^{(i)}(n, n/t))$$

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- Using the same arguments as before it follows that

$$\begin{aligned} & \left((X_{\mathbf{g}(0, n/t)}, \overline{X}_{\mathbf{g}(0, n/t)}), \dots, (X_{\mathbf{g}(n, n/t)}, \overline{X}_{\mathbf{g}(n, n/t)}) \right) \\ & \stackrel{d}{=} \left((V(0, n/t), J(0, n/t)), \dots, (V(n, n/t), J(n, n/t)) \right) \end{aligned}$$

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$$\begin{aligned} & \left((X_{\mathbf{g}(0,n/t)}, \bar{X}_{\mathbf{g}(0,n/t)}), \dots, (X_{\mathbf{g}(n,n/t)}, \bar{X}_{\mathbf{g}(n,n/t)}) \right) \\ & \stackrel{d}{=} \left((V(0, n/t), J(0, n/t)), \dots, (V(n, n/t), J(n, n/t)) \right) \end{aligned}$$

- Using that we may write $\tau_u = \inf\{t > 0 \mid \bar{X}_t > u\}$, with the definitions

$$\hat{k}^{(n)} := \inf\{k \in \{0, \dots, n\} \mid \bar{X}_{\mathbf{g}(k,n/t)} > u\} \quad \text{and}$$

$$\kappa^{(n)} := \inf\{k \in \{0, \dots, n\} \mid J(k, n/t) > u\}$$

it can be shown that

$$\tau_u \wedge t = \inf\{t > 0 \mid \bar{X}_t > u\} \wedge t \approx \frac{t}{n} (\hat{k}^{(n)} \wedge n) \stackrel{d}{=} \frac{t}{n} (\kappa^{(n)} \wedge n)$$

and similarly

$$X_{\tau_u \wedge t} \approx X_{\mathbf{g}(\hat{k}^{(n)}, n/t)} \stackrel{d}{=} V(\kappa^{(n)} \wedge n, n/t)$$

The Wiener-Hopf Monte Carlo simulation technique (WHMC)

- Hence we arrive at

$$\left(\frac{t}{n} (\kappa^{(n)} \wedge n), V(\kappa^{(n)} \wedge n, n/t) \right) \xrightarrow{d} (\tau_u \wedge t, X_{\tau_u \wedge t} - u)$$

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- If the technicalities are confusing, recall in practice all you need to do is:
 - pick a mLP
 - write down the corresponding I_q and S_q
 - for the path functional that you are interested in, write down the corresponding expression in terms of $V(\cdot, n/t)$ and $J(\cdot, n/t)$
 - write a bit of computer code to run your Monte Carlo simulations

The Wiener-Hopf Monte Carlo simulation technique (WHMC)

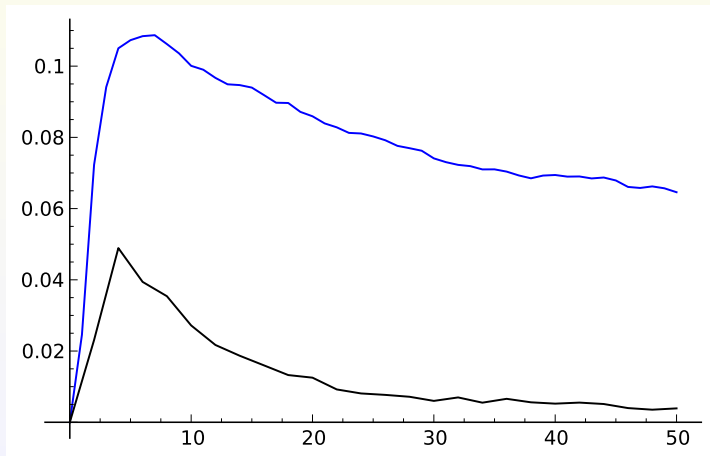
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- Advantages this method over standard Monte Carlo (i.e. a random walk approach)?
 - standard MC requires knowing the law of X_h , in general not available & numerical Fourier inversion of Ψ necessary
 - standard MC is well known to perform poorly for quantities involving running maximum/first hitting times

Example of WHMC simulation technique

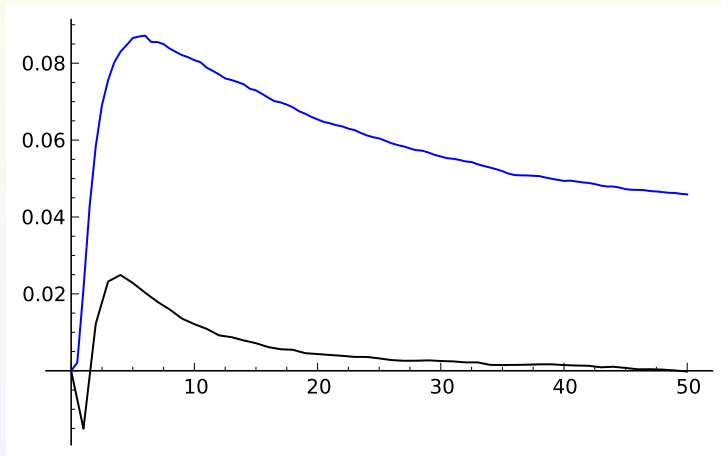
Error (exact value - simulated value) for the simulation of $t \mapsto \mathbb{P}(\tau_1 \leq t)$ where X is a BM, blue for standard Monte Carlo and black for WHMC



Here $n = 25$ and we used 10^4 samples

Example of WHMC simulation technique

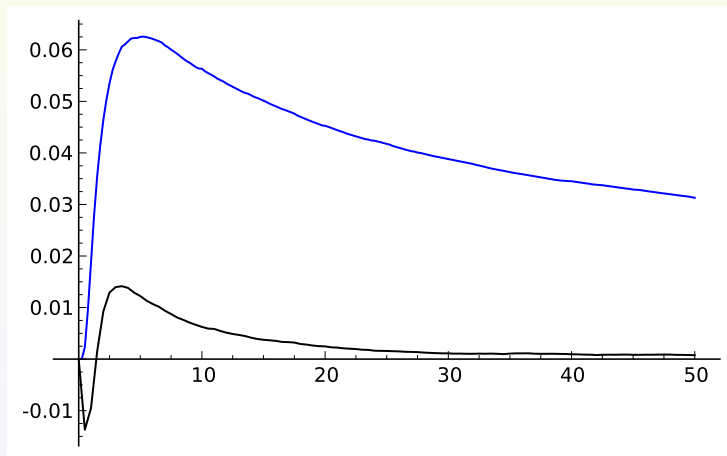
Error (exact value - simulated value) for the simulation of $t \mapsto \mathbb{P}(\tau_1 \leq t)$ where X is a BM, blue for standard Monte Carlo and black for WHMC



Here $n = 50$ and we used 10^5 samples

Example of WHMC simulation technique

Error (exact value - simulated value) for the simulation of $t \mapsto \mathbb{P}(\tau_1 \leq t)$ where X is a BM, blue for standard Monte Carlo and black for WHMC



Here $n = 100$ and we used 10^6 samples

Outline

- 1 Setting & two types of problems
- 2 Short intro into meromorphic Lévy processes (mLP's)
- 3 A new Monte Carlo simulation technique for (a.o.) ruin problems with mLP's
- 4 An approximation technique for American option pricing with mLP's

American option pricing with mLP's

- Recall the problem: f is a cts. and bounded payoff function. Determine for $T > 0$ and $x > 0$ the value function V :

$$V(T, x) = \sup_{\tau} \mathbb{E}_x \left[e^{-r(\tau \wedge T)} f(X_{\tau \wedge T}) \right]$$

where τ is a stopping time. I.e. V is the value function of the American option with payoff function f in a market driven by a mLP X .

³related to Canadisation technique by Peter Carr

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- Idea³:
 - consider again the 'stochastic grid' from before:

$$0 = \mathbf{g}(0, n/T) < \mathbf{g}(1, n/T) < \dots < \mathbf{g}(n, n/T)$$

and recall that by the law of large numbers $\mathbf{g}(n, n/T) \rightarrow T$ as $n \rightarrow \infty$

- perform in a clever way backwards induction over this grid, relying on $\mathbf{g}(k, n/T) - \mathbf{g}(k-1, n/T) \sim \exp(n/T)$ and the fact that we know the law of X at exp. distributed times

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- In order to make this backwards induction as nice as we can we need to
 - allow only for stopping *at* the grid points $\{0 = \mathbf{g}(0, n/T), \mathbf{g}(1, n/T), \dots, \mathbf{g}(n, n/T)\}$ (i.e. not in between)
 - adjust the discounting factor

³related to Canadisation technique by Peter Carr

Setup of the algorithm

$$V(T, x) = \sup_{\tau} \mathbb{E}_x \left[e^{-r(\tau \wedge T)} f(X_{\tau \wedge T}) \right], \quad \mathbf{g}(k, n/T) := \sum_{i=1}^k \mathbf{e}^{(i)}(n/T)$$

In formulae this reads as follows.



$$\sigma_{\text{discr}} \in \{0 = \mathbf{g}(0, n/T), \mathbf{g}(1, n/T), \dots, \mathbf{g}(n, n/T)\}$$

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- We define for $n \geq 1$ and $k = 0, \dots, n$ the functions $v_k^{(n)}$:

$$v_k^{(n)}(x) = \sup_{\sigma_{\text{discr}}} \mathbb{E}_x \left[D(\sigma_{\text{discr}} \wedge \mathbf{g}(k, n/T)) f(X_{\sigma_{\text{discr}} \wedge \mathbf{g}(k, n/T)}) \right],$$

i.e. (compare to V above) the optimal value if you are only allowed to stop at the first k grid points and with adjusted discount factor

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This is a useful setup because:

- We can prove that $v_n^{(n)}(x) \rightarrow V(T, x)$ as $n \rightarrow \infty$. Intuition:
 $\mathbf{g}(n, n/T) \rightarrow T$ and the stochastic grid becomes more and more dense in $[0, T]$ as $n \rightarrow \infty$

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- By an induction argument we have the following straightforward recursive relationship:

$$v_0^{(n)}(x) = f(x), \quad v_k^{(n)}(x) = \max \left\{ f(x), e^{-r/n} \mathbb{E}_x \left[v_{k-1}^{(n)}(X_{\mathbf{e}(n/T)}) \right] \right\}$$

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- which we can use to find pretty explicit formulae for the $v_k^{(n)}$'s! (Thanks to the fact that when X is an mLP the law of $X_{\mathbf{e}(n/T)}$ is a mixture of exponentials on the negative and positive axis).

Practical example

$$V(T, x) = \sup_{\tau} \mathbb{E}_x \left[e^{-r(\tau \wedge T)} f(X_{\tau \wedge T}) \right], \quad \mathbf{g}(k, n/T) := \sum_{i=1}^k \mathbf{e}^{(i)}(n/T)$$

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- Suppose that $f(x) = (K - e^x)^+$ (classic American put) and X an mLP. Then the algorithm (1) above yields the following structure for any $n \geq 1$ and $k = 0, \dots, n$:

$$v_0^{(n)}(x) = (K - e^x)^+, \quad \text{for } k = 1, \dots, n:$$

$$v_k^{(n)}(x) = \begin{cases} (K - e^x)^+ & \text{if } x \leq x_k^{(n)} \\ \text{messy lin. comb. of terms of the form } ax^b e^{cx} & \text{if } x > x_k^{(n)} \end{cases}$$

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- Terrible formulae, but quite explicit (and hence computer friendly, yet computer implementation is still work in progress)

References

- *Randomisation and the American put*. Peter Carr
- *Meromorphic Lévy processes and their fluctuation identities*. Andreas Kyprianou, Alexey Kuznetsov & Juan Carlos Pardo
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- *A note on extending the Wiener-Hopf Monte Carlo simulation technique for Lévy process to hitting times and overshoots*. KvS (In progress)
- *A variation of the Canadisation algorithm for optimal stopping with Lévy processes*. KvS (In progress)