4 Transportation and Assignment Problems

4.1 The Transportation Problem

A commodity is produced at m factories or sources $S_1, ..., S_m$ and is sold at n markets or destinations $D_1, ..., D_n$. The annual output or supply available at source S_i is s_i units, the annual demand at destination D_j is d_j units, and the cost of transporting one unit from S_i to D_j is c_{ij}. We wish to determine which sources S_i should supply which destinations D_j so as to minimise transportation costs. We assume that each $s_i > 0$, $d_j > 0$, $c_{ij} \geq 0$.

Let $x_{ij} \geq 0$ be the number of units to be sent from S_i to D_j per year. Then we are to minimise the transportation cost

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to the constraints that the amount taken from source S_i must be at most the supply s_i, that is

$$\sum_{j=1}^{n} x_{ij} \leq s_i \qquad \text{for} \quad i = i, \ldots, m,$$

and the amount taken to market D_j must be at least the demand d_j, that is

$$\sum_{i=1}^{m} x_{ij} \geq d_j \qquad \text{for} \quad j = 1, \ldots, n.$$

If there is any feasible solution x_{ij} then

$$\text{total supply} \ = \sum_i s_i \geq \sum_i \sum_j x_{ij} \geq \sum_j d_j = \ \text{total demand.}$$

If the total supply is at least the total demand, then we can easily adjust the problem to ensure that we have equality here and thus also in each of the inequalities above. We may do this by introducing if necessary an extra market (a *dump*) with demand equal to the excess supply $\sum_i s_i - \sum_j d_j$, and setting all transportation costs to the dump equal to 0. (If the total demand exceeds the total supply, we may introduce a mythical extra source to make up the shortfall, and assign costs which reflect penalties for undersupplying markets.)

We thus define the transportation (or Hitchcock) problem as the following LP, where the $s_i > 0$, $d_j > 0$, $c_{ij} \geq 0$ are given, with $\sum_i s_i = \sum_j d_j$.

$$\min \sum_{i,j} c_{ij} x_{ij}$$

$$(P) \qquad \text{subject to}$$

$$\sum_j x_{ij} = s_i \quad \text{for each } i = 1, \ldots, m$$

$$\sum_i x_{ij} = d_j \quad \text{for each } j = 1, \ldots, n$$

$$x_{ij} \geq 0 \quad \text{for each } i = 1, \ldots, m \ \text{ and } j = 1, \ldots, n.$$

Variable production costs at factories can be built into the transportation costs — see exercise 3.4. Also, seemingly unrelated problems may sometimes be cast into this form, for example the 'caterer's problem' in exercise 3.1 or the production scheduling problem in exercise 3.2.

The above LP may be written as

$$\min \mathbf{c}'\mathbf{x} \quad \text{subject to} \quad A\mathbf{x} = \mathbf{b},\, \mathbf{x} \geq \mathbf{0},$$

where for example in the case $m = 2$, $n = 3$

$$\mathbf{c} = (c_{11}, c_{12}, c_{13}, c_{21}, c_{22}, c_{23})'$$

$$\mathbf{x} = (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23})'$$

$$\mathbf{b} = (-s_1, -s_2, d_1, d_2, d_3)'$$

and

$$A = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

(The minus signs here are just for convenience below, when we consider the dual.)

Dual program

The variable x_{ij} appears with coefficient -1 in the constraint for source S_i, and with coefficient $+1$ in the constraint for market D_j, and appears in no other constraint. Let u_i and v_j be respectively the dual variables corresponding to the S_i and D_j constraints. Then the dual constraint corresponding to the primal variable $x_{ij} \geq 0$ is

$$v_j - u_i \leq c_{ij}.$$

The dual program is thus

$$(D) \qquad \max \sum_j d_j v_j - \sum_i s_i u_i$$

$$\text{subject to}$$

$$\begin{aligned} &v_j - u_i \leq c_{ij} \quad \text{for each } i, j \\ &u_i, v_j \text{ unrestricted in sign.} \end{aligned}$$

An interpretation of the dual program

A haulage contractor wants the transportation contract. He proposes to set local prices u_i, v_j at each point S_i or D_j. Thus he will buy at source S_i at a unit price of u_i and he will sell at market D_j at a unit price of v_j, so that his effective charge for carrying one unit from S_i to D_j is $v_j - u_i$. The dual objective function is his net revenue, which he will try to maximise, and the dual constraints ensure that his effective charges are competitive.

Note that all the local prices u_i, v_j may be made > 0 if desired by adding a large constant to each. This keeps them feasible, and does not change the objective function value, since the total supply equals the total demand.

Example 1

Suppose that we have two sources S_1 and S_2 with supplies $s_1 = 3$ and $s_2 = 5$, and three markets D_1, D_2 and D_3 with demands $d_1 = 4$, $d_2 = 2$ and $d_3 = 2$. We must use the sources to satisfy the demands as cheaply as possible, where the unit transportation costs are given by the matrix

$$(c_{ij}) = \begin{pmatrix} 3 & 5 & 4 \\ 1 & 2 & 3 \end{pmatrix}.$$

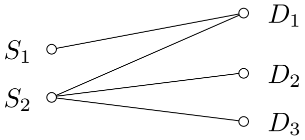
Preliminary comments

(i) A transportation problem always has an optimal solution. For we may easily find a feasible solution for the primal (see below), and setting each $u_i = v_j = 0$ gives a feasible solution for the dual; and so by the duality theorem both programs have optimal solutions.

(ii) The $(m+n) \times mn$ constraint matrix A has rank $m+n-1$. For the sum of all the rows of A is $\mathbf{0}$, and so A has rank $\leq m+n-1$. Also it is not hard to see that after deleting any one row the remaining rows are linearly independent.

We may suppose that we have dropped one equation, say the first. We now have a system of $m+n-1$ independent equations, and any solution to them automatically satisfies the dropped equation. A basic solution to our system will specify $m+n-1$ basic variables. If we do drop the first equation from the primal, then the variable u_1 should not appear in the dual, and this is equivalent to setting $u_1 = 0$.

(iii) Corresponding to a basic feasible solution, we may draw a graph with nodes $S_1, \ldots, S_m$ and $D_1, \ldots, D_n$, and with an edge joining S_i and D_j wherever x_{ij} is basic, so that there are exactly $(m+n-1)$ edges. In example 1, a basic feasible solution is $(x_{ij}) = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & 2 \end{pmatrix}$ with basic variables x_{11}, x_{21}, x_{22}, x_{23}; and the corresponding graph is



It may be shown that a basic feasible solution always corresponds to a ‘spanning tree’ as here, that is to a graph without cycles which connects all the nodes.

(iv) Recall that a linear program is said to be degenerate if, in some basic feasible solution, some basic variable equals 0. It turns out that a transportation problem is degenerate if and only if a partial sum of the supplies s_i equals a partial sum of the demands d_j, so that the routes used could fall into two separate systems. Thus example 1 above is non-degenerate, and example 2 below is degenerate.

Solving the transportation problem

Our method is essentially a slick version of the (revised) simplex method, and is based on the complementary slackness theorem. This says here that feasible solutions x_{ij} for the primal (P) and u_i, v_j for the dual (D) are optimal if and only if

$$v_j - u_i = c_{ij} \quad \text{whenever} \quad x_{ij} > 0.$$

We list four steps.

(1) Find an initial basic feasible solution to the primal (P) (specifying exactly $m+n-1$ basic variables).

(2) Find temporary local prices u_i, v_j by solving the $m+n-1$ equations

$$v_j - u_i = c_{ij} \quad \text{for} \quad x_{ij} \ \text{basic}.$$

These determine the $(m+n)$ prices u_i, v_j uniquely, provided that we set $u_1 = 0$. [The u_i, v_j are the simplex multipliers, and the reduced cost for x_{ij} is $\bar{c}_{ij} = c_{ij} + u_i - v_j$.]

(3) If the local prices u_i, v_j also satisfy

$$v_j - u_i \leq c_{ij} \quad \text{for each non-basic} \quad x_{ij}$$

then they are feasible for the dual problem (D), and so by complementary slackness our solutions are optimal.
(4) If $v_j - u_i > c_{ij}$ for some non-basic x_{ij} (so that the currently unused route from S_i to D_j looks attractive, with reduced cost $\bar{c}_{ij} = c_{ij} + u_i - v_j < 0$), then introduce x_{ij} into the basis, dropping exactly one current basic variable, and return to step (2).

Step (1) is explained below. Steps (2), (3), (4) are best explained by examples.

In order to find an initial basic feasible solution in step (1), we first focus on a particular route (for example the cheapest route available), and allocate as much flow along it as possible. Then reduce the corresponding supply and demand appropriately, and 'kill off' the one reduced to zero. If two are reduced to zero simultaneously we kill off just one, except that we stop after introducing the $(m+n-1)\,st$ basic variable (which must reduce to zero the last active supply and demand). When we always choose a cheapest available route this is called the 'matrix minimum' method.

Example 1 continued (non-degenerate)
We use the matrix minimum method to get started. First we set

$$x_{21} = \min(s_2, d_1) = \min(5, 4) = 4,$$

reduce s_2 to 1 and kill off D_1 leaving a 2 × 2 active tableau.

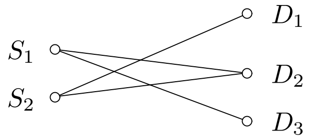
3	5	4	$s_1 = 3$
1 **4**	2	3	$s_2 = \cancel{5}\,1$
$d_1 = \cancel{4}\,X$	$d_2 = 2$	$d_3 = 2$	

Next set $x_{22} = \min(1, 2) = 1$, kill off S_2 and reduce the demand for D_2 to 1. Then set $x_{13} = \min(2, 3) = 2$ reduce the supply for S_1 to 1 and kill off D_3. Our tableau now is

3	5	4 **2**	$\cancel{3}\,1$
1 **4**	2 **1**	3	$\cancel{5}\ \cancel{1}\,X$
X	$\cancel{2}\,1$	$\cancel{2}\,X$	

Finally we are forced to set $x_{12} = 1$ which automatically satisfies both supply and demand simultaneously. We now present our initial basic feasible solution (with cost 19).

3	5 **1**	4 **2**
1 **4**	2 **1**	3



Next we set $u_1 = 0$ and use the tableau to determine the rest of the temporary local prices u_i, v_j. First $v_2 = 5$ (since x_{12} is basic and so $v_2 - u_1 = c_{12}$) and $v_3 = 4$ (since $v_3 - u_1 = c_{13}$). Then $u_2 = 3$ (since $v_2 - u_2 = c_{22}$) and finally $v_1 = 4$ (since $v_1 - u_2 = c_{21}$).

	$v_1 = 4$	$v_2 = 5$	$v_3 = 4$
$u_1 = 0$	3	5 **1**	4 **2**
$u_2 = 3$	1 **4**	2 **1**	3

We know that $v_j - u_i = c_{ij}$ whenever x_{ij} is basic, so that $\bar{c}_{ij} = c_{ij} + u_i - v_j = 0$. For each non-basic x_{ij}, we compute $\bar{c}_{ij}$. We find that $\bar{c}_{11} = -1 < 0$, so that the route S_1 to D_1 is attractive.

We follow step 4, and introduce x_{11} into the basis at level $t \geq 0$. There is always a unique cycle involving the entering variable x_{11} and some of the basic variables (but no other non-basic variable), and we can alternately add and subtract t around the cycle, so that all supplies and demands stay satisfied.

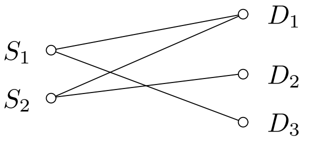
3 +t	5 **1**-t	4 **2**
1 **4**-t	2 **1**+t	3

Note that the change in cost here is

$$\begin{aligned} t(c_{11} - c_{12} + c_{22} - c_{21}) &= t(c_{11} - (v_2 - u_1) + (v_2 - u_2) - (v_1 - u_2)) \\ &= t(c_{11} - (v_1 - u_1)) \;=\; t\bar{c}_{11} \;=\; -t \end{aligned}$$

(as we should expect!). We make t as large as possible, whilst keeping all $x_{ij} \geq 0$. Thus here $t = 1$, and we obtain the new basic feasible solution below (in which x_{12} has left the basis). Also, we calculate the prices u_i, v_j as before, starting with $u_1 = 0$.

	3	4	4
0	3 **1**	5	4 **2**
2	1 **3**	2 **2**	3



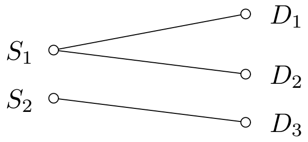
We now find that $v_2 - u_1 \leq c_{12}$ and $v_3 - u_2 \leq c_{23}$, and find that we have reached an optimal solution, with cost 18.

Comments (i) Since here $\bar{c}_{ij} > 0$ for each non-basic x_{ij}, the optimal solution is unique. (ii) We met only a cycle of length four, but in larger problems we could meet longer cycles. (iii) If all supplies s_i and demands d_j are integers, as above, then the solutions x_{ij} we generate will always be integers, which is handy if you are transporting cars.

Example 2

2	3	1	$s_1 = 5$
4	5	2	$s_2 = 2$
$d_1 = 3$	$d_2 = 2$	$d_3 = 2$	

Note that the problem is degenerate, since for example $s_1 = d_1 + d_2$, and so there is a feasible solution such that the graph corresponding to the positive variables x_{ij} is as below.



We use the matrix minimum method to obtain an initial basic feasible solution, specifying exactly $m + n - 1 = 4$ basic variables. Firstly we set $x_{13} = 2$, reduce the first supply to 3 and kill off the last column. Then set $x_{11} = \min\{3, 3\} = 3$. Both the first supply and first demand are now reduced to zero, and we must kill off just one, say the first supply.

2 **3**	3	1 **2**	$\cancel{5}$ $\cancel{3}$ X
4	5	2	2
$\cancel{3}$ 0	2	$\cancel{2}$	

Next set $x_{21} = \min\{2, 0\} = 0$ and kill off the first column, and finally $x_{22} = 2$. We now have an initial basic feasible solution, with cost 18. We obtain the five local prices u_i, v_j as before, by setting $u_1 = 0$ and using the four equations corresponding to the basic variables (including x_{21}).

	2	3	1
0	2 **3**+t	3	1 **2**-t
-2	4 **0**-t	5 **2**	2 +t

Since $v_3 - u_2 = 3 > c_{23}$, we introduce x_{23} into the basis. We must, however, set $t = 0$: we 'stall', and simply swap x_{23} for x_{21} in the basis.

	2	4	1
0	2 **3**	3 +t	1 **2**-t
-1	4	5 **2**-t	2 **0**+t

Now $v_2 - u_1 = 4 > c_{12}$ and so we introduce x_{12} into the basis. This time we set $t = 2$ and find that both x_{13} and x_{22} drop to zero. We must drop exactly one from the basis, say x_{13}.

	2	3	0
0	2 **3**	3 **2**	1
-2	4	5 **0**	2 **2**

We find that we have reached an optimum solution, with cost 16.

Sensitivity analysis

Our optimal solution in example 1 does not use x_{12}. What if the transportation cost c_{12} were reduced? Note that at present $\bar{c}_{12} = c_{12} + u_1 - v_1 = 1$. If we reduce c_{12} to $c_{12} - p$ then clearly the current optimal solution remains optimal if $0 \le p \le 1$, but not if $p > 1$.

What if the cost of a route currently used increases, say c_{21} increases to $c_{21} + p$?

	3	$4-p$	4
0	3 **1**	5	4 **2**
$2-p$	1+p **3**	2 **2**	3

The current solution remains optimal if $4 - p - 0 \le 5$ and $4 - (2 - p) \le 3$, that is if $0 \le p \le 1$, but not if $p > 1$.

We have seen above that it is straightforward to deal with changes in transportation costs c_{ij}. Now consider changes in supplies and demands. The local prices u_i, v_j will guide us about resulting changes in the total transportation costs.

Recall that the dual objective function is $\sum_j d_j v_j - \sum_i s_i u_i$. Thus if some s_i and some d_j are increased by 1 then the total cost will increase by $v_j - u_i$ (which could be negative), as long as the current local prices u_i, v_j remain an optimal dual solution, and this will happen as long as we can still use the same basic variables (routes) in the primal problem.

Suppose in example 1 that both s_2 and d_3 are increased by 1. Then the total cost will increase by $v_3 - u_2$ if we can still use the same basic routes. But indeed we can do this, for we may set $x_{23} = 1$ and then 'pivot it out' by alternately adding and subtracting 1 around the cycle formed by x_{23} and the basic variables. We obtain the (degenerate) optimal solution

3 **0**	5	4 **3**
1 **4**	2 **2**	3

which uses the same basic variables. Note that this procedure worked since the old optimal primal solution was non-degenerate, and so each basic variable had value at least 1. If say s_1 and d_3 are both increased by 1, then the cost will increase by $c_{13} = v_3 - u_1 = 4$, since x_{13} is currently basic.

Now suppose that just the capacity limit s_2 for source S_2 is increased by 1, so that we have excess supply. It appears that we should leave the excess unit of supply at the source with the smallest local price u_i (which is source S_1 here), and then the overall cost should decrease by $u_2 - u_1$. We may check this by considering the old optimal tableau with an added dump with demand 1.

	3	4	4	2
0	3 **1**	5	4 **2**	0
2	1 **3**	2 **2**	3	0 **1**

We have set $x_{14} = 0$, $x_{24} = 1$ to obtain a basic feasible solution to the new problem. Note that $v_4 = u_2$. Since $u_1 < u_2$ we have $v_4 - u_1 > 0$. We pivot x_{14} into the basis, and we choose to drop x_{24}.

	3	4	4	0
0	3 **0**	5	4 **2**	0 **1**
2	1 **4**	2 **2**	3	0

The only local price that changes is v_4, which now equals u_1. The current solution is optimal since u_1 is the smallest of the costs u_i.

4.2 The assignment problem

Suppose that we are to assign n jobs to n machines, one per machine. When job i is assigned to machine j this incurs a cost c_{ij}, and we wish to minimise the total cost. We may formulate this problem as the following integer linear program, by using $\{0, 1\}$-valued variables x_{ij}, where $x_{ij} = 1$ corresponds to assigning job i to machine j.

$$\min \textstyle\sum_i \sum_j c_{ij} x_{ij}$$

subject to

$$\begin{aligned} \textstyle\sum_j x_{ij} &= 1 \quad \text{for each } \ i = 1, \ldots, n \\ \textstyle\sum_i x_{ij} &= 1 \quad \text{for each } \ j = 1, \ldots, n \\ x_{ij} &= 0 \text{ or } 1 \quad \text{for each } \ i, j = 1, \ldots, n. \end{aligned}$$

Comments (i) If say there are more machines than jobs, we may add fictitious jobs with zero costs to obtain the above model. (ii) From our discussion of the transportation problem, we see that we may consider the LP 'relaxation' in which we have replaced $x_{ij} = 0$ or 1 above by $x_{ij} \geq 0$. We thus obtain an LP which always has an optimal solution in which each $x_{ij} = 0$ or 1. (iii) The LP relaxation and its dual always have optimal solutions. The dual may be written

$$\max \textstyle\sum_i u_i + \sum_j v_j \quad \text{subject to} \quad u_i + v_j \le c_{ij} \quad \text{for each } i, j.$$

We could solve the problem by using the transportation algorithm described earlier. However, our problem now is highly degenerate: each basic feasible solution has n basic variables equal to 1 and thus has $n-1$ equal to 0. We sketch below a 'primal-dual' approach, which is based on three simple observations.

- Adding a constant γ to any row or column of the cost matrix (c_{ij}) does not change the optimal solutions, since the cost of each feasible solution changes by exactly γ.
- Thus we can ensure that all costs are ≥ 0, and we can introduce many zero costs.
- If we have an assignment using only zero costs it must be optimal.

Example 1

$$(c_{ij}) = \begin{pmatrix} 5 & 7 & 9 \\ 14 & 10 & 12 \\ 15 & 13 & 16 \end{pmatrix}$$

We subtract the row minimum u_i from each row i, and then the net column minimum v_j from each column j.

$$\begin{array}{r|ccc} & v_1 = 0 & v_2 = 0 & v_3 = 2 \\ \hline u_1 = 5 & 5 & 7 & 9 \\ u_2 = 10 & 14 & 10 & 12 \\ u_3 = 13 & 15 & 13 & 16 \end{array}$$

The resulting reduced costs $\bar{c}_{ij} = c_{ij} - u_i - v_j$ are given by

$$(\bar{c}_{ij}) = \begin{pmatrix} 0 & 2 & 2 \\ 4 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

The (unique) optimal solution for these costs is

$$(x_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and this is then the (unique) optimal solution for the original costs, with total cost 30.

The u_i, v_j always form a feasible solution to the dual problem (D), since all reduced costs $\bar{c}_{ij} = c_{ij} - u_i - v_j$ are ≥ 0. Indeed, they form an optimal dual solution here, by complementary slackness. (Check also that $\sum_i u_i + \sum_j v_j = 30$.)

By a complementary slackness theorem, there must always be some dual-feasible u_i, v_j such that there is an assignment just using zero net costs $\bar{c}_{ij}$. We were lucky above but ...?

Example 2

$$(c_{ij}) = \begin{pmatrix} 5 & 2 & 3 & 4 \\ 7 & 8 & 4 & 5 \\ 6 & 3 & 5 & 6 \\ 2 & 2 & 3 & 5 \end{pmatrix}$$

We proceed as before.

$$\begin{array}{r|cccc} & v_1 = 0 & v_2 = 0 & v_3 = 0 & v_4 = 1 \\ \hline u_1 = 2 & 5 & 2 & 3 & 4 \\ u_2 = 4 & 7 & 8 & 4 & 5 \\ u_3 = 3 & 6 & 3 & 5 & 6 \\ u_4 = 2 & 2 & 2 & 3 & 5 \end{array}$$

Now

$$(\bar{c}_{ij}) = \begin{pmatrix} 3 & \mathbf{0} & 1 & 1 \\ 3 & 4 & 0 & \mathbf{0} \\ 3 & 0 & 2 & 2 \\ \mathbf{0} & 0 & 1 & 2 \end{pmatrix}$$

We focus on the positions where the $\bar{c}_{ij}$ are zero. We may spot a partial assignment using 3 zeros, shown in bold above. We may also spot 3 lines, for example row 2 and columns 1 and 2, which cover all the zeros. This shows that we can do no better, since we can pick at most one zero from any line.

König's theorem states that, if we can assign only k zeros, then there are k lines which cover all the zeros. Indeed, there is a good method for finding a largest partial assignment, using say k zeros, and a corresponding cover with k lines. For small problems this step can be done by inspection.

Our next step is to improve the u_i, v_j by using the cover by lines that we have found. Let δ be the minimum uncovered element in the $\bar{c}_{ij}$ matrix, so that $\delta > 0$. Here $\delta = 1$. Subtract δ from each uncovered row (to introduce zeros into the previously barren region) and add δ to each covered column (to ensure that the new $\bar{c}_{ij}$ are ≥ 0). Note that this is the same as subtracting δ from each uncovered element and adding δ to each doubly covered element. We obtain

$$(\bar{c}_{ij}) = \begin{pmatrix} 3 & 0 & 0 & \mathbf{0} \\ 4 & 5 & \mathbf{0} & 0 \\ 3 & \mathbf{0} & 1 & 1 \\ \mathbf{0} & 0 & 0 & 1 \end{pmatrix}$$

The bold entries yield an optimal solution, with cost 13. (As a check, note that the final u_i, v_j values are 3, 4, 4, 3, -1, -1, 0, 1 which sum to 13.)

This whole procedure can be organised to run in $O(n^3)$ steps. Note that, in terms of the linear programs, we maintain dual feasibility and complementary slackness throughout, and seek primal feasibility.