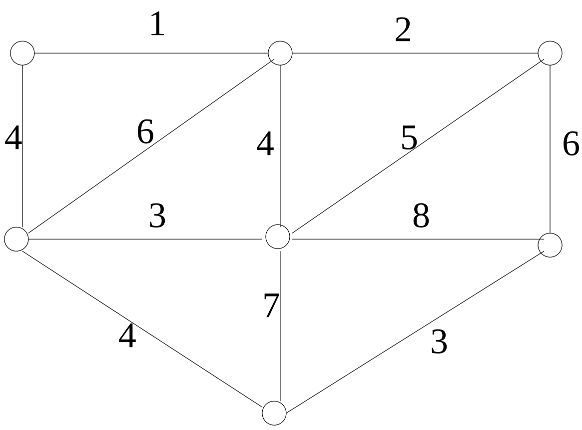

Figure 1: Minimum spanning trees (question 1)

Combinatorial Optimisation HT 2010
Problem set 1, MSc in Applied Statistics, for week 5

1 Consider the graph shown in the figure, with nodes numbered 1 to 7 left to right along the top row, then along the next row and last the bottom node.

(a) Use Kruskal's algorithm to find a minimum spanning tree in the network illustrated in the figure; and then use Prim's algorithm to do the same, starting at node 1. In each case give the sequence of edges added to the tree.

(b) Now ignore the above minimum spanning trees. Use the green rule (once) and the red rule (twice) to show that there is a minimum spanning tree containing the edge 23 (cost 2) and not containing the edges 35 (cost 5) or 36 (cost 6).

2 The matrix A below gives the arc lengths in a network on the six nodes $1, \ldots, 6$ (where a dot indicates the absence of an arc); and similarly for the matrix B.

(a) For the matrix A, find the shortest path lengths from node 1 to each other node in the network, by Dijkstra's method.

(b) For the matrix B, find all shortest path lengths by using Floyd's method. [Why does the second row of the matrix never change? Why can there be no change in the iteration with $k = 2$?]

(c) Now consider the matrix A again, and suppose that the length of arc 53 is changed from 2 to -6. Use Floyd's method to detect a negative cycle. [With $k = 1$ there can be no changes, so (for hand calculation) we can jump to $k = 2$. You should detect a negative cycle with $k = 4$.]

$$A = \begin{pmatrix} . & 3 & . & 2 & . & . \\ . & . & 7 & 2 & . & . \\ . & 1 & . & 1 & . & 3 \\ . & 2 & . & . & 4 & . \\ . & . & 2 & 3 & . & 6 \\ . & . & 1 & . & . & . \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 3 & -2 & 2 \\ . & 0 & . & . \\ . & 5 & 0 & 3 \\ 1 & 1 & . & 0 \end{pmatrix}$$

3 If an organisation allocates i of its sales representatives to region j the profit from that region is given by $a_{ij} \geq 0$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$. Describe *very briefly* how dynamic programming gives an algorithm for finding the optimal allocation of sales representatives, which uses no more than $f(m,n)$ integer comparisons and additions, where f is a polynomial which should be found.

Illustrate the algorithm by using it to find an optimum allocation when the a_{ij} are presented as the following array:

$i \vert j$	1	2	3	4	5
0	0	0	0	0	0
1	21	39	50	37	42
2	39	60	110	53	63
3	73	79	139	72	89
4	90	105	180	120	130

4 Describe an algorithm for the problem P^λ, to maximise

$$\sum_{i=1}^{n} r_i(x_i, y_i) - \lambda \sum_{i=1}^{n} y_i$$

subject to $\sum_{i=1}^{n} x_i = \alpha$, $x_i \geq 0$, $0 \leq y_i \leq \beta$, and x_i and y_i integral for each i. How long will your method take? (Assume that you have a table for the values $r_i(x, y)$.)

5 Consider a flowshop with machines $M_1, \ldots, M_m$.

(i) Show that for a problem $n/m/F/F_{max}$ there is an optimal schedule with the same job sequence on machines M_{m-1} and M_m.

(ii) Show that for a problem $n/3/F/F_{max}$ there is an optimal schedule with the same job sequence on all three machines.

6 Consider a problem $n/1//\bar{T}$. Given a set Q of jobs let $H(Q)$ be the minimum value of the total tardiness for scheduling the jobs in Q. Give a dynamic programming recurrence for $H(Q)$. Use this approach to solve the following $4/1//\bar{T}$ problem:

i	1	2	3	4
p_i	9	12	7	4
d_i	15	19	23	21