New Upper Bounds on Harmonious Colourings

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Abstract

We present an improved upper bound on the harmonious chromatic number of an arbitrary graph. We also consider “Fragmentable” classes of graphs (an example is the class of planar graphs) which are, roughly speaking, graphs which can be decomposed into bounded sized components by removing a small proportion of the vertices. We show that for such graphs of bounded degree the harmonious chromatic number is close to the lower bound \((2m^2)^{1/2}\), where \(m\) is the number of edges.

1 Introduction

A harmonious colouring of a simple graph \(G\) is a proper vertex colouring such that each pair of colours appears together on at most one edge. The harmonious chromatic number \(h(G)\) is the least number of colours in such a colouring. A recent survey on harmonious colourings is by B.Wilson [7].
There are two easy but important lower bounds for \( h(G) \). Denote the maximum vertex degree of \( G \) by \( \Delta \) and the number of edges of \( G \) by \( m \). First, any vertex and all of its neighbours must receive distinct colours, and thus

\[
h(G) \geq \Delta + 1.
\]

Second, there must be at least as many pairs of colours as there are edges, hence

\[
m \leq \binom{h(G)}{2}
\]

so that

\[
h(G) > (2m)^{\frac{1}{2}}.
\]

In a recent paper, McDiarmid and Luo [6] improved on a result of Lee and Mitchem [3] and showed that for a graph \( G \) with \( n \) vertices and maximum degree \( \Delta \),

\[
h(G) \leq 2\Delta(n - 1)^{\frac{1}{2}},
\]

and that for trees,

\[
h(G) \leq 2(\Delta n)^{\frac{1}{2}}.
\]

Z.Lu [5] proves a similar result. In the present paper, we improve these results and obtain some asymptotically good results for some classes of graphs with maximum degree bounded by a fixed constant.

## 2 Results for General Graphs

In this section we derive an upper bound for the harmonious chromatic number which is valid for all graphs. This bound shows that for certain classes of graphs \( G \) (including all planar graphs) we have \( h(G) = O(n^{\frac{1}{2}} + \Delta) \).

**Definition** For any graph \( G = (V, E) \), the \( k \)-core of \( G \) is the set \( V_{(k)} \) of vertices which remain after vertices of degree < \( k \) have been repeatedly removed. Note that \( V_{(\Delta + 1)} = \emptyset \).

**Lemma 2.1** [6] For any \( a \) there is an integer \( t > 0 \) such that

\[
\frac{a^2}{t} + t \leq (4a^2 + 1)^{\frac{1}{2}}.
\]
The function \( f(x) = \frac{a^2}{x} + x \) for \( x > 0 \) is convex, with minimum at \( x = a \) where \( f(a) = 2a \). For \( c > 2a \) the line \( y = c \) meets the curve \( y = f(x) \) at the two roots \( x_1 < x_2 \) of \( \frac{a^2}{x} + x = c \). Now \( x_2 - x_1 = (c^2 - 4a^2)^{\frac{1}{2}} \).

Hence if \( c^2 - 4a^2 = 1 \) then some integer \( t \) satisfies \( x_1 < t \leq x_2 \), and \( f(t) \leq c \). So

\[
f(t) \leq c = (4a^2 + 1)^{\frac{1}{2}}.
\]

**Theorem 2.2** Let \( G \) be any graph with \( m \) edges and with maximum degree \( \Delta \). Then for any integer \( k \geq 2 \),

\[
h(G) \leq \max\{|V(k)|, 2(2(k-1)m)^{\frac{1}{2}} + (2k - 3)\Delta\}.
\]

**Proof** A “partial” harmonious colouring of \( G \) is a colouring of a subset of the vertices of \( G \) which is harmonious on the graph induced by the coloured vertices and such that no uncoloured vertex has two or more coloured neighbours with the same colour. List the vertices of \( G \) as \( v_1, v_2, \ldots, v_n \) so that for each \( j > a \), where \( a = |V(k)| \), each \( v_j \) is adjacent to at most \( (k - 1) \) of the vertices \( v_1, \ldots, v_{j-1} \).

Let \( \lambda_k = \max\{|V(k)|, 2(2(k-1)m)^{\frac{1}{2}} + (2k - 3)\Delta\} \). (Note that \( \lambda_k \) need not be an integer.)

First assume that \( V(k-1) \neq \emptyset \). Thus \( k \leq \Delta + 1 \), and \( m \geq \binom{k}{2} \) so \( \frac{2m}{k-1} \geq k > 1 \). Now apply Lemma 2.1 with \( a^2 = \frac{2m}{k-1} - 1 > 0 \). We see that there is a non-negative integer \( t \) such that

\[
(t + 1) + \{2m/(k - 1) - 1\}/(t + 1) \leq \{4(2m/(k - 1) - 1) + 1\}^{\frac{1}{2}}.
\]

We shall now give a harmonious colouring of \( G \) using at most \( \lambda_k \) colours. Colour each of \( v_1, \ldots, v_a \) with a distinct colour. Now we colour \( v_{a+1}, v_{a+2}, \ldots \) in order, ensuring that

\[
\text{for each colour set } S, \sum_{v \in S} d(v) \leq \Delta + t. \quad (*)
\]

Suppose that for some \( a < j \leq n \) we have coloured \( v_1, \ldots, v_j-1 \), giving a partial harmonious colouring of \( G \) satisfying (\( * \)), and we now wish to colour vertex \( v_j \). We shall show that we can extend this colouring to \( v_j \) with (\( * \)) still holding.
We bound the number $U$ of colours unavailable for $v_j$. It will suffice to show that $|U| \leq \lambda_k - 1$. Suppose that $v_j$ is adjacent to a set $A$ of vertices amongst $v_1, \ldots, v_{j-1}$, so that $0 \leq x \leq k - 1$ where $x = |A|$. We have to exclude colours for four reasons:

a) We must exclude the at most $x$ colours on the vertices in $A$.

b) If $w \in A$, and has colour $c$, then we cannot use any colour which is used on a vertex adjacent to one of colour $c$. By assumption, the total degree of the vertices of colour $c$ is at most $\Delta + t$, so we have to exclude at most $x(\Delta + t - 1)$ colours in this way.

c) For any uncoloured neighbour $w$ of $v_j$, we must exclude the colour of any coloured neighbour of $w$. Now $v_j$ has at most $\Delta - x$ uncoloured neighbours and each has at most $k - 1$ neighbours amongst $v_1, \ldots, v_j$, one of which is $v_j$ itself. Hence we have to exclude at most $(\Delta - x)(k - 2)$ colours in this way.

d) We must exclude a colour $c$ if using it would cause the degree sum of vertices coloured $c$ to exceed $\Delta + t$. In this case the degree sum must already be at least $t + 1$, and since the total degree of all the vertices so far coloured is at most $2m - x$, we must exclude at most $(2m - x)/(t + 1)$ in this last way.

Hence we have

$$U \leq x(\Delta + t) + (\Delta - x)(k - 2) + (2m - x)/(t + 1).$$

Now the coefficient of $x$ is $(\Delta + t - k + 2 - 1/(t + 1))$. Since $k \leq \Delta + 1$, this is at least $t + 1 - 1/(t + 1) \geq 0$. Hence substituting $k - 1$ for $x$ we obtain

$$U \leq (k - 1)(\Delta + t) + (\Delta - (k - 1))(k - 2) + (2m - (k - 1))/(t + 1)$$

$$= (k-1)((t+1) + \{2m/(k-1) - 1\}/(t+1)) - (k-1) + (k-1)\Delta + (\Delta-k+1)(k-2)$$

$$\leq (k - 1)(4(2m/(k - 1) - 1) + 1)^{1/2} + (2k - 3)\Delta - (k - 1)^2$$  (by choice of $t$)

$$\leq 2(2(k - 1)m)^{1/2} + (2k - 3)\Delta - (k - 1)^2$$

$$\leq \lambda_k - 1,$$

as required.
Finally suppose that $V_{(k-1)} = \emptyset$. Then either $V_{(1)} = \emptyset$, in which case $G$ is null and $h(G) = 0 \leq \lambda_k$, or $V_{(1)} \neq \emptyset$, in which case let $k'$ be the least positive integer such that $V_{(k)} = \emptyset$. Then we have $k \geq 2$ and $V_{(k'-1)} \neq \emptyset$, so $h(G) \leq \lambda_{k'} \leq \lambda_k$.

Putting $k = 2$ we obtain:

**Corollary 2.3** For any tree $T$ with $m$ edges,

$$h(T) \leq 2(2m)^{1/2} + \Delta.$$  

Since for any planar graph the $6-core$ $V_{(6)}$ is empty, we also obtain

**Corollary 2.4** For any planar graph $G$ with $m$ edges,

$$h(G) \leq (40m)^{1/2} + 9\Delta.$$  

For any graph $G$ of genus $\gamma \geq 1$, the $7-core$ $V_{(7)}$ of $G$ has at most $12(\gamma - 1)$ vertices. Hence we obtain

**Corollary 2.5** For any graph $G$ of genus $\gamma \geq 1$ with $m$ edges,

$$h(G) \leq \max\{12(\gamma - 1), (48m)^{1/2} + 11\Delta\}.$$  

### 3 Fragmentable Graphs

We now consider classes of graphs which we call *fragmentable*. Roughly speaking this means that the graphs can be decomposed into components of bounded size by removing a small proportion of the non-isolated vertices. Examples include trees, graphs of bounded genus, and rectangular lattices; we give details at the end of this section. For graphs of bounded degree in these classes we show that harmonious colouring is possible with a near optimal number of colours.
Definition of Fragmentable Graphs

A class $\Gamma$ of graphs is **fragmentable** if: For any $\varepsilon > 0$, there are positive integers $n_0$, $c(\varepsilon)$ such that if $G \in \Gamma$ is a graph with $n \geq n_0$ non-isolated vertices then there is a set $S$ of vertices, with $|S| \leq \varepsilon n$, such that each component of $G \setminus S$ has $\leq c(\varepsilon)$ vertices.

Before proving our result on fragmentable graphs, we first quote a theorem of R.M. Wilson and derive several easy corollaries.

**Theorem** (R.M. Wilson, 1975 [8]) Given a graph $G$, then there exist subgraphs $G_1, \ldots, G_t$ of the complete graph $K_n$, such that each edge of $K_n$ belongs to exactly one of the graphs $G_i$ and such that each $G_i$ is isomorphic to $G$ provided that (i) $n$ is sufficiently large, (ii) the number of edges of $G$ divides $\binom{n}{2}$ and (iii) the greatest common divisor of the degrees of the vertices of $G$ divides $n - 1$.

We will say that a graph $H$ covers a collection $H_1, \ldots, H_t$ of graphs if there are subgraphs $G_1, \ldots, G_t$ of $H$ such that $G_i$ is isomorphic to $H_i$ for $i = 1, \ldots, t$ and each edge of $H$ belongs to at most one $G_i$.

**Corollary 3.1** Let $G$ be a graph, and let $\varepsilon > 0$. Then there is an $n_0$ such that for all $n \geq n_0$, $K_n$ covers $r$ disjoint copies of $G$, where

$$r \geq (1 - \varepsilon) \frac{\binom{r}{2}}{|E(G)|}.$$ 

**Corollary 3.2** Let $G$ be a graph, and let $\varepsilon > 0$. Then there is an $n_0$ such that for all $n \geq n_0$, and for all $r \geq 1$, the number of disjoint copies of $K_n$ needed to cover $r$ disjoint copies of $G$ is at most

$$(1 + \varepsilon) \frac{r |E(G)|}{\binom{n}{2}} + 1.$$ 

**Proof** From Corollary 3.1, all but one of the copies of $K_n$ can be tightly packed with copies of $G$, the extra $K_n$ covers any remaining copies of $G$. 

**Corollary 3.3** Let $G_1, \ldots, G_t$ be a fixed finite collection of graphs, and let $\varepsilon > 0$. Then there exist integers $n_0$, $c$ such that the following holds. Let $r_1, \ldots, r_t$ be non-negative integers and let $H$ be a collection consisting
of \( r_i \) copies of \( G_i \), \( i = 1, \ldots, t \), so that the total number of edges in \( H \) is
\[
\sum r_i |E(G_i)| = m \text{ say.}
\]
If \( n \geq n_0 \) and \( m \geq cn^2 \) then the number of copies of \( K_n \) required to cover \( H \) is at most
\[
(1 + \varepsilon) \frac{m}{\binom{n}{2}}.
\]

**Proof** By the Corollary 3.2, if \( n \) is sufficiently large then for each \( i \), the number of copies of \( K_n \) to cover \( r_i \) copies of \( G_i \) is at most
\[
(1 + \frac{1}{2} \varepsilon) r_i \frac{|E(G_i)|}{\binom{n}{2}} + 1.
\]
Adding these expressions and using the fact that \( t \) is fixed, we choose \( c \) large enough to make
\[
t \leq \frac{1}{2} \frac{m}{\binom{n}{2}}.
\]

**Corollary 3.4** Let \( c \) be a fixed positive integer, and let \( \varepsilon > 0 \). Then there exists \( m_0 \) such that if \( G \) is a graph with \( m \geq m_0 \) edges and with all components having at most \( c \) vertices, then
\[
h(G) \leq (1 + \varepsilon) (2m)^{\frac{1}{2}}.
\]

**Proof** Since all components have at most \( c \) vertices, then each component of \( G \) is isomorphic to one a finite number of graphs \( G_1, \ldots, G_t \). So \( G \) consists of a collection \( H \) of \( r_i \) copies of \( G_i \) for each \( i = 1, \ldots, t \), for some integers \( r_i \). By Corollary 3.3, there are constants \( N, m_0 \) such that if \( m \geq m_0 \), then the number \( Q \) of copies of \( K_N \) required to cover \( H \) is at most
\[
(1 + \frac{1}{2} \varepsilon) \frac{m}{\binom{N}{2}}.
\]
But then by Corollary 3.1, provided \( n \) is sufficiently large, \( K_n \) will cover at least
\[
(1 + \frac{1}{2} \varepsilon)^{-1} \frac{\binom{n}{2}}{\binom{N}{2}}
\]
copies of \( K_N \). But this is at least \( Q \) provided that
\[
(1 + \frac{1}{2} \varepsilon)^{-1} \frac{\binom{n}{2}}{\binom{N}{2}} \geq (1 + \frac{1}{2} \varepsilon) \frac{m}{\binom{N}{2}}
\]
7
and this happens if
\[ n \geq (1 + \frac{1}{2} \varepsilon)(2m)^{\frac{1}{2}} + 1. \]

Clearly provided \( m \) is large enough, \( n \) can be chosen to be at most \((1 + \varepsilon)(2m)^{\frac{1}{2}}\). By identifying the vertices of \( K_n \) with \( n \) colours, we obtain a harmonious colouring of \( G \) with \( n \) colours. The result follows.

Recall that \( h(G) > (2m)^{\frac{1}{2}} \) for any graph \( G \), where \( G \) has \( m \) edges. Our main theorem shows that this bound is asymptotically tight for fragmentable graphs of bounded degree.

**Theorem 3.5**  Let \( \Gamma \) be any fragmentable class of graphs. Then for any \( d \) and any \( \varepsilon > 0 \) there is an \( m_0 \) such that: For any graph \( G \in \Gamma \) with maximum degree \( \Delta \leq d \) and with \( m \geq m_0 \) edges we have
\[ h(G) < (2m)^{\frac{1}{2}} + \varepsilon m^{\frac{1}{2}}. \]

In other words for graphs \( G \) in \( \Gamma \) with \( \Delta \leq d \) we have
\[ h(G) = (\sqrt{2} + o(1))m^{\frac{1}{2}}. \]

**Proof**  Let \( \varepsilon > 0 \), and suppose that \( G \in \Gamma \) with \( |V(G)| = n \), \( |E(G)| = m \), and \( \Delta \leq d \). Note that removing isolated vertices from \( G \) has no effect on the hypothesis or the conclusion of the theorem, so we assume that \( G \) has no isolated vertices. We also assume that \( \varepsilon < 1/(4d) \). Then provided that \( n \geq n_0 \), we find \( S \subseteq V(G) \) such that \( |S| \leq \varepsilon n \), and \( G \setminus S \) has all components of size at most \( c(\varepsilon) \). Note also that since \( G \) has no isolated vertices, and \( \varepsilon < 1/(4d) \), then \( G \setminus S \) has at least \( \frac{n}{2} - \frac{n}{4} = \frac{n}{4} \) edges. By Corollary 3.4, we can find a harmonious colouring of \( G \setminus S \) with at most \((\sqrt{2} + \varepsilon)m^{\frac{1}{2}}\) colours, provided that \( n \geq n_1 \), say. Now set
\[ N(S) = \{ v \mid (v, x) \in E \text{ for some } x \in S \} \setminus S \]
\[ N^2(S) = \{ v \mid (v, x) \in E \text{ for some } x \in N(S) \} \setminus (S \cup N(S)). \]

Note that \( |N(S)| \leq d|S| \leq d\varepsilon n \), and \( |N^2(S)| \leq (d - 1)|N(S)| \leq d^2\varepsilon n \).

We now find a harmonious colouring for \( G \) in three stages.

**Stage 1**  We ensure that no colour occurs more than \( \lceil \varepsilon^\frac{1}{2} n^{\frac{1}{2}} \rceil \) times on \( N^2(S) \), as follows: If colour \( c \) occurs more than \( \lceil \varepsilon^\frac{1}{2} n^{\frac{1}{2}} \rceil \) times then pick \( \lceil \varepsilon^\frac{1}{2} n^{\frac{1}{2}} \rceil \) of the vertices with colour \( c \) and recolour them with a new colour.
not previously used for any vertex. Repeat this until no colour occurs more
than \( \lceil \frac{1}{2} \cdot \frac{n}{d} \rceil \) times. Now since each new colour occurs exactly \( \lceil \frac{1}{2} \cdot \frac{n}{d} \rceil \) times, and \(|N^2(S)| \leq d^2 \cdot \frac{n}{d}\), then we have used at most \( d^2 \cdot \frac{1}{2} \cdot \frac{n}{d} \) new colours.

**Stage 2** Now recolour \( N(S) \) with new colours not previously used. We will ensure that after this has been done, we still have a partial harmonious colouring of \( G \) (as defined in the proof of theorem 2.2, so that in particular, for each \( v \in S \), no two neighbours of \( v \) have the same colour).

We colour \( N(S) \) sequentially. Order the vertices arbitrarily. We ensure that no colour set on \( N(S) \) ever has size \( \lceil \frac{1}{2} \cdot \frac{n}{d} \rceil \). Thus when we colour an element \( v \in N(S) \) some colours are unavailable.

a) The colours on the coloured neighbours of \( v \) are unavailable. This excludes at most \( d \) colours.

b) If \( w \in N^2(S) \cup N(S) \) is adjacent to \( v \) and is already coloured, and \( w' \in N^2(S) \cup N(S) \) is the same colour as \( w \), then we cannot use the colour of any neighbour of \( w' \). There are at most \( d \) choices for \( w \), then at most \( \lceil \frac{1}{2} \cdot \frac{n}{d} \rceil \) for \( w' \), and then at most \( d \) for a neighbour of \( w' \).

Hence this excludes at most \( d^2 \lceil \frac{1}{2} \cdot \frac{n}{d} \rceil \) colours.

c) We must ensure that no uncoloured vertex in \( N(S) \) and no vertex in \( S \) gets two neighbours in \( N(S) \) of the same colour. This excludes at most \( d(d-1) \) colours.

d) We must exclude any colour which has already occurred \( \lceil \frac{1}{2} \cdot \frac{n}{d} \rceil \) times on \( N(S) \); there are at most \( |N(S)| / \lceil \frac{1}{2} \cdot \frac{n}{d} \rceil \) such colours. So we exclude at most \( d \cdot \frac{1}{2} \cdot \frac{n}{d} \).

Hence in total at most \( d^2 \lceil \frac{1}{2} \cdot \frac{n}{d} \rceil + d \cdot \frac{1}{2} \cdot \frac{n}{d} + d(d-1) + d \) colours are excluded, so we can colour \( N(S) \) with at most \( d^2 \lceil \frac{1}{2} \cdot \frac{n}{d} \rceil + d \cdot \frac{1}{2} \cdot \frac{n}{d} + d^2 + 1 \) colours. **Stage 3** We colour the elements of \( S \) sequentially, with brand new colours, never allowing any colour set to become larger than \( \lceil \frac{1}{2} \cdot \frac{n}{d} \rceil \). For reasons as in a) - d) above, we have to exclude at most \( d^2 \lceil \frac{1}{2} \cdot \frac{n}{d} \rceil + \frac{1}{2} \cdot \frac{n}{d} + d^2 \) colours when we colour each vertex, so we can complete the colouring with at most \( d^2 \lceil \frac{1}{2} \cdot \frac{n}{d} \rceil + \frac{1}{2} \cdot \frac{n}{d} + d^2 + 1 \) new colours.
Hence provided \( n \geq \max(n_0, n_1) \), then
\[
h(G) \leq (\sqrt{2} + \varepsilon)m^\frac{1}{2} + 3d^2 \left[ \varepsilon^2 n^\frac{1}{2} \right] + (d + 1)\varepsilon n^\frac{1}{2} + 2d^2 + 2.
\]
Since \( m \geq n/2 \), the result follows.

Classes of Fragmentable Graphs

We use the following lemma to show that some classes of graphs are fragmentable.

**Lemma 3.6** Let \( \Gamma \) be a class of graphs. Suppose that there exist real numbers \( A > 0 \), \( 0 \leq \lambda < 1 \), and \( 0 < \alpha < 1 \), such that for any graph \( G \in \Gamma \), where \( G \) has \( n \) vertices, there is a set of at most \( An^\lambda \) vertices whose removal from \( G \) leaves every component with at most \( \alpha n \) vertices, and each component a member of \( \Gamma \). Then \( \Gamma \) is fragmentable.

**Proof** We show how to split up the graph into small components by a sequence of stages. Before each stage \( i \), no component has more than \( \alpha^{i-1}n \) vertices. (This is obviously true before stage 1.) The number of components with more than \( \alpha^i n \) vertices is at most \( \alpha^{-i} \). By the hypotheses of the theorem, we can remove at most \( A(\alpha^{i-1}n)^\lambda \) vertices from each such component to ensure that after stage \( i \), no component has more than \( \alpha^i n \) vertices. The number of stages is the least \( k \) such that \( \alpha^k n \leq C \), where \( C \) is a constant to be chosen later. Hence \( \alpha^k - 1 > C \), so that \( \alpha^k - 1 < n/C \).

Now the total number \( r_i \) of vertices removed at stage \( i \) is at most
\[
a^{-i}A(\alpha^{i-1}n)^\lambda
= An^\lambda \alpha^{\lambda-1-\lambda}
= An^\lambda \alpha^{(1-i)(1-\lambda)}
\]
and since \( \alpha^k - 1 < n/C \), we have \( \alpha^k - 1 < (n/C)\alpha^{k-i} \). Hence
\[
r_i < \frac{An^\lambda}{\alpha} (n/C)^{(1-\lambda)} \alpha^{(k-i)(1-\lambda)}
= \frac{An}{\alpha C^{(1-\lambda)} \beta^{(k-i)}},
\]

where \( \beta = \alpha^{(1-\lambda)} \) (so \( 0 < \beta < 1 \)). Then the total number \( R \) of vertices removed is
\[
\sum_{i=1}^{k} r_i < \frac{An}{\alpha C^{(1-\lambda)}} \sum_{i=1}^{k} \beta^{(k-i)}
\]
\begin{equation*}
< \frac{An}{\alpha C(1-\lambda)} \sum_{i \geq 0} \beta^i
\end{equation*}
\begin{equation*}
= \frac{An}{\alpha C(1-\lambda)} \frac{1}{1 - \beta}.
\end{equation*}

Since $1 - \lambda > 0$, it follows that for any $\varepsilon > 0$ we can choose $C = C(\varepsilon)$ such that $R \leq \varepsilon n$. The result follows.

**Corollary 3.7** The following classes of graphs are fragmentable: (i) trees, (ii) graphs of genus at most $\gamma$, for any fixed $\gamma \geq 0$, (iii) rectangular lattices of dimension at most $d$, for a fixed integer $d$.

**Proof**
(i) Here we can take $\lambda = 0$, $A = 1$ and $\alpha = \frac{1}{2}$.
(ii) This follows from the separator theorems of Lipton and Tarjan [4], and of Gilbert, Hutchinson and Tarjan [1]. The latter gives the required result immediately with $A = 2^{\frac{1}{3}}(\gamma + 2)$, $\lambda = \frac{1}{2}$ and $\alpha = 2/3$.
(iii) If $G$ is a $d$-dimensional rectangular lattice of size $m_1 \times \ldots \times m_d$, where $m_1, \ldots, m_d$ are all integers at least 2, then we can assume without loss of generality that the largest of $m_1, \ldots, m_d$ is $m_1$. Then by removing the vertices on a hyperplane orthogonal to this longest dimension, we can ensure that the components which remain are at most half the size of the original, and the number of vertices removed is $m_2 \times \ldots \times m_d$, which is at most $n^{((d-1)/d)}$. Hence we can take $A = 1$, $\lambda = 1 - (1/d)$, and $\alpha = \frac{1}{2}$.

4 Concluding Remarks

There is still a considerable gap between the upper and lower bounds for $h(G)$ in general. It seems likely that the lower bounds are nearer to the true values. For example, for the simplest case of forests, we have that $h(G) \leq 2(2n)^{\frac{1}{2}} + \Delta$. However for some special cases, such as a) forests whose components are all stars, b) regular $k$-ary trees with all the leaves on the same level, the extra structure allows us to show that $h(G) \leq (2 |E|)^{\frac{1}{2}} + A\Delta$ for some constant $A$. This leads us to conjecture:

**Conjecture** There is a constant $A$ such that for any forest $F$ with $m$ edges and with maximum degree at most $\Delta$,

\begin{equation*}
h(F) \leq (2m)^{\frac{1}{2}} + A\Delta.
\end{equation*}
For general graphs, this bound must be increased slightly, but it seems reasonable to conjecture the following:

**Conjecture**  For any positive integer $\Delta$, there is a constant $C_{\Delta}$ such that if $G$ is any graph with $m$ edges and with maximum degree at most $\Delta$, then

$$h(G) \leq (2m)^{\frac{3}{2}} + C_\Delta.$$

Some further inequalities for $h(G)$ have been found independently by Krasikov and Roditty [2].

**References**


