Title: Random Independent Sets

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Prerequisite courses: Part B Applied Probability

Theoretical project in probability.

Brief Description:

A graph is a collection of vertices, and edges between certain pairs of vertices. An independent set in the graph is a subset of the vertices which does not contain any pair of neighbours in the graph (two vertices are neighbours if they are joined by an edge).

The concept of an independent set is relevant in many applications. For example in a broadcast network, the vertices may represent transmitters at different locations, with two transmitters joined by an edge in the graph if they interfere with each other. To avoid interference, the set of active transmitters at a given time should be an independent set. In a database system, the vertices might represent different processes, and two vertices are joined by an edge if the corresponding processes need to modify the same file. To avoid conflicts, the set of active processes at a given time should again be an independent set.

Often the underlying graph may have a regular structure; for example a square lattice in which a vertex is joined by an edge to each of its four nearest neighbours (independent sets of lattices are of considerable interest in statistical physics), or a tree (a graph with no cycles).

The project will look at random independent sets of a graph, that is, probability distributions on the set of independent sets. These can arise in various ways, e.g.

(1) uniform distribution over all the independent sets of a graph;

(2) stationary distributions of Markov chains representing the processes in communication theory or operations research mentioned above;

(3) as outputs of randomized algorithms for finding large independent sets in graphs (e.g. a greedy algorithm, which considers all the vertices of the graph in a random order; at each step it "accepts" a vertex if none of its neighbours have yet been accepted).

Things to look at could include:

(i) what does a "typical independent set" from a given distribution look like? For example, on the square lattice, does it tend to form a "chessboard" pattern favouring either "even" or "odd" sites, or does it look more chaotic?

(ii) how to sample from the uniform distribution over independent sets (or from other relevant distributions), either approximately or precisely?

(iii) how to count the total number of independent sets of a graph? (This can sometimes be done approximately using probabilistic methods.)

This is a theoretical project which would suit a student with a strong background in probability, including Part B Applied Probability. It would fit well with Part C Graph Theory and Part C Probabilistic Combinatorics but these are not essential at all. Running simulations could form an important part of the project, so programming ability would be a big plus.