On the number of edges in random planar graphs

Stefanie Gerke
TU München
Institut für Informatik
Arcisstraße 21
80290 München

Colin McDiarmid
University of Oxford
Department of Statistics
South Parks Road
Oxford OX1 3TG

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Abstract

We consider random planar graphs on n labelled nodes, and show in particular that if the graph is picked uniformly at random then the expected number of edges is at least $\frac{13}{9}n + o(n)$. To prove this result we give a lower bound on the size of the set of edges that can be added to a planar graph on n nodes and m edges while keeping it planar, and in particular we see that if m is at most $\frac{13}{9}n - c$ (for a suitable constant c) then at least this number of edges can be added.

1 Introduction

Given $0 < p < 1$ and a positive integer n, let $G_{n,p}$ denote the random graph with nodes $v_1, \ldots, v_n$ in which the $\binom{n}{2}$ possible edges appear independently with probability $p$. We denote by $R_{n,p}$ the random graph $G_{n,p}$ conditioned on it being planar. (We may think of repeatedly sampling a graph $G_{n,p}$ until we find one that is planar.) Also, let us denote $R_{n,\frac{1}{2}}$ by $R_n$. Thus $R_n$ is uniformly distributed over all labelled planar graphs on n nodes.

Rather little is known about random planar graphs, even about the number of edges in such graphs, which is our focus here. Let us denote the number
of edges in a (simple) graph $G$ by $m(G)$. Thus we are interested in the random variable $m(R_n)$ and more generally in $m(R_{n,p})$. Of course $m(G) \leq 3n-6$ for any planar graph $G$ on $n$ nodes. The expected value $E[m(R_n)]$ is at least $(3n-6)/2$ – see [1]. It is shown in [4], by using results on counting planar graphs and triangulations, that $m(R_n) \leq 2.56n$ asymptotically almost surely (aas), that is with probability tending to 1 as $n \to \infty$. We will show here in particular that $m(R_n) \geq \frac{13}{7} n + o(n)$ aas, thereby improving on the result from [1] mentioned above.

We now introduce two functions $f(\alpha)$ and $g(p)$ which are needed to state our two main results – see also Figure 1.

Given $1 < \alpha \leq 3$, let $k = k(\alpha) = \lfloor \frac{2\alpha}{2 - \alpha} \rfloor$, and let

$$f(\alpha) = \frac{1}{4} \left( k^2 + k + 6 - (k^2 - 3k + 6)\alpha \right).$$

It is not hard to verify that $f(\alpha)$ is continuous and decreasing on $1 < \alpha \leq 3$, and satisfies $f(\alpha) \to \infty$ as $\alpha \to 1$ and $f(3) = 0$, see also the end of Section 4. (The function $f$ is also piecewise-linear and convex.) For $0 < p < 1$ we may define $g(p)$ to be the unique value $\rho \in (1, 3)$ such that $f(\rho)/\rho = (1 - p)/p$. The function $g$ is continuous and increasing on $0 < p < 1$, and satisfies
\( g(p) \to 1 \) as \( p \to 0 \), \( g(\frac{1}{2}) = \frac{13}{7} \) and \( g(p) \to 3 \) as \( p \to 1 \). We are now able to state our theorem concerning the number of edges of random planar graphs.

**Theorem 1** Let \( 0 < p < 1 \). Then as \( n \to \infty \),

\[
\mathbb{E}[m(R_{n,p})] \geq g(p)n + o(n);
\]

and indeed for any \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that

\[
\Pr(m(R_{n,p}) < (g(p) - \varepsilon)n) = o(e^{-\delta n}).
\]

In particular, since \( g(\frac{1}{2}) = \frac{13}{7} \), this theorem shows that the expected number of edges in a planar graph sampled uniformly at random from all labelled planar graphs on \( n \) nodes is at least about \( \frac{13}{7}n \).

To prove this result we will consider the number of edges that can be added to a planar graph of \( n \) nodes and \( m \) edges while keeping the graph planar. Given a planar graph \( G \), we call a non-edge \( f \) *addable* in \( G \) if the graph \( G + f \) obtained by adding \( f \) as an edge is still planar; and we let \( \text{add}(G) \) denote the set of addable non-edges of \( G \). Let \( \mathcal{P}(n) \) denote the set of all (simple) planar graphs with \( n \) nodes \( v_1, \ldots, v_n \); let \( \mathcal{P}(n,m) \) denote the set of all graphs \( G \in \mathcal{P}(n) \) with \( m \) edges; and let \( \text{add}(n,m) \) denote the minimum value of \( |\text{add}(G)| \) over all graphs \( G \in \mathcal{P}(n,m) \). Observe that by Kuratowski’s theorem, if \( m \leq 7 \) then \( \text{add}(n,m) = \binom{n}{2} - m \), and if \( n \geq 6 \) and \( m \geq 8 \) then \( \text{add}(n,m) < \binom{n}{2} - m \). Also, \( \text{add}(n,m) > 0 \) if \( m < 3n - 6 \) and \( \text{add}(n,3n-6) = 0 \).

**Theorem 2** Let \( 1 < \alpha \leq 3 \), and suppose that \( m = m(n) = \alpha n + O(1) \) as \( n \to \infty \). Then \( \text{add}(n,m) = f(\alpha)n + O(1) \).

In Section 2 we prove Theorem 1 by using Theorem 2. In Section 3 we introduce the concept of \( 2^* \)-connected graphs and show some properties of \( 2^* \)-connected planar graphs. These properties will be used in Section 4 to prove Theorem 2. Analogous results on the number of edges of random outerplanar graphs are given in Section 5.

## 2 Proof of Theorem 1

**Proof of Theorem 1.** Fix \( 0 < p < 1 \) throughout. Let \( n \) be a positive integer. Given a graph \( G \in \mathcal{P}(n) \) let

\[
\pi(G) = \Pr(R_{n,p} = G) = \bar{c} \cdot p^{m(G)} (1 - p)^{\binom{n}{2} - m(G)},
\]

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where
\[
\tilde{c} = \left( \sum_{H \in \mathcal{P}(n)} p^m(H)(1 - p)^{\binom{n}{2} - m(H)} \right)^{-1}.
\]

We now define a Markov chain \((X_t)\) with state space \(\mathcal{P}(n)\) which has unique stationary distribution \(\pi\) as above, following the development in [1] for the case \(p = \frac{1}{2}\). The transition probabilities \(p_{G,G'} = \Pr(X_{t+1} = G'|X_t = G)\) are as follows. Let \(G\) be the graph at step \(t\). Choose two distinct nodes \(u\) and \(v\) uniformly at random from \(v_1, \ldots, v_n\). If \(uv\) is not an edge of \(G\) and \(G + uv\) is planar then add the edge \(uv\) to \(G\) with probability \(p\) to obtain \(G'\); if \(uv\) is an edge of \(G\) then remove \(uv\) with probability \(1 - p\); and otherwise do nothing to obtain \(G'\).

This Markov chain is clearly ergodic. We may show that it (is reversible and) has stationary distribution \(\pi\) as required by checking the detailed balance equation
\[
\pi(G)p_{G,G'} = \pi(G')p_{G',G},
\]
whenever \(G'\) is obtained from \(G\) by adding an edge. But if \(m(G) = m\) then both sides equal
\[
\tilde{c} \cdot p^m(1 - p)^{\binom{n}{2} - m} \cdot \frac{1}{\binom{n}{2}} \cdot p = \tilde{c} \cdot p^{m+1}(1 - p)^{\binom{n}{2} - m-1} \cdot \frac{1}{\binom{n}{2}} \cdot (1 - p).
\]

Thus the Markov chain indeed has the stationary distribution \(\pi\) as required.

For brevity we write \(S_m\) for \(\mathcal{P}(n,m)\), and we let \(\sigma_m = \Pr(m(R_{n,p}) = m) = \sum_{G \in S_m} \pi(G)\). Then for each \(0 \leq m < 3n - 6\) and each \(G \in S_m\),
\[
\frac{\text{add}(n,m)p}{\binom{n}{2}} \leq \sum_{G' \in S_{m+1}} p_{G,G'}.
\]

and for each \(G' \in S_{m+1}\),
\[
\sum_{G \in S_m, G \preceq G'} p_{G',G} = \frac{(m + 1)(1 - p)}{\binom{n}{2}}.
\]

Therefore, since the distribution \(\pi\) is stationary,
\[
\frac{\text{add}(n,m)p}{\binom{n}{2}} \sigma_m \leq \sum_{G \in S_m, G' \in S_{m+1}} p_{G,G'} \pi(G) = \sum_{G \in S_m, G' \in S_{m+1}} p_{G',G} \pi(G') = \frac{(m + 1)(1 - p)}{\binom{n}{2}} \sigma_{m+1},
\]

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and hence 
\[
\frac{\sigma_m}{\sigma_{m+1}} \leq \frac{(m + 1)(1 - p)}{\text{add}(n, m)p}.
\]

Let \( m^* = m^*(n) \) be the greatest value of \( m \) such that 
\[
m \leq \frac{\text{add}(n, m)p}{1 - p}.
\]

We shall see shortly that 
\[
m^* = g(p)n + O(1). \tag{1}
\]

For each \( 0 \leq m \leq m^* \), \( \text{add}(n, m) \geq \text{add}(n, m^*) \), so \( \text{add}(n, m)\frac{p}{1 - p} \geq m^* \), and thus 
\[
\frac{\sigma_m}{\sigma_{m+1}} \leq \frac{m + 1}{m^*}.
\]

Hence for each \( 1 \leq j \leq m^* \), 
\[
\frac{\sigma_{m^* - j}^*}{\sigma_{m^*}} = \prod_{m = m^* - j}^{m^* - 1} \frac{\sigma_m}{\sigma_{m+1}} \leq \prod_{m = m^* - j}^{m^* - 1} \frac{m + 1}{m^*} = \prod_{i=0}^{j-1} \left( 1 - \frac{i}{m^*} \right) \leq \exp\left(-\frac{j}{2}m^*\right).
\]

Therefore, 
\[
\Pr(m(R_{n,p}) = m^* - j) \leq \exp\left(-\frac{j}{2}m^*\right).
\]

But now, for some constant \( c > 1 \) 
\[
\Pr(m(R_{n,p}) \leq m^* - c(n \log n)^{1/2}) = o(1),
\]

and it follows that 
\[
\mathbb{E}[m(R_{n,p})] \geq m^* + o(n).
\]

Furthermore, given \( \varepsilon > 0 \), since \( m^* \leq 3n \), 
\[
\Pr(m(R_n) \leq m^* - \varepsilon n) \leq m^* \ e^{-\left(\frac{2\varepsilon}{2} + o(1)\right)n} = o(e^{-\frac{2}{2}n}).
\]

It remains to prove (1). Let \( m = \lceil g(p)n \rceil \). By Theorem 2, 
\[
\text{add}(n, m) = f(g(p))n + O(1) = \frac{1 - p}{p}g(p)n + O(1) = \frac{1 - p}{p}m + O(1).
\]

Thus \( \frac{p}{1 - p} \text{add}(n, m) = m + O(1) \), that is, there is a constant positive integer \( c \) such that 
\[
m - c \leq \frac{p}{1 - p} \text{add}(n, m) \leq m + c.
\]
Then
\[ \frac{p}{1-p} \text{add}(n, m + c + 1) \leq m + c < m + c + 1 \]
and so \( m^* \leq m + c \); and
\[ \frac{p}{1-p} \text{add}(n, m - c) \geq m - c \]
and so \( m^* \geq m - c \). This completes the proof of (1), and thus of the theorem. \( \square \)

### 3 2*-connected graphs

A central role in the proof of Theorem 2 is played by 2*-connected graphs, which are defined as follows.

**Definition 3** We call a graph \( G \) 2*-connected if the graph \( \hat{G} \) obtained by deleting any isolated nodes of \( G \) satisfies the following conditions.

- \( \hat{G} \) is 2-node-connected,
- for any 2-node cut \( u, v \) in \( \hat{G} \), the nodes \( u \) and \( v \) are not adjacent, and \( \hat{G} - u - v \) has precisely two components.

We will show in this section that there is a 2*-connected planar graph \( G \) on \( n \) nodes with \( m \) edges and with \( |\text{add}(G)| = \text{add}(n, m) \), see Lemma 10. This is the main step in proving Theorem 2. Before we can show this we need some more notation.

Given a planar graph \( G \), a non-edge which is not addable is called excluded. We denote by \( \text{ex}(G) \) the set of excluded edges of \( G \). Clearly, for every graph \( G \in \mathcal{P}(n, m) \)
\[ |\text{add}(G)| + |\text{ex}(G)| + m = \binom{n}{2}. \]

Let us denote by \( \mathcal{P}(n, \leq m) \) the set of graphs \( G \in \mathcal{P}(n) \) with at most \( m \) edges which maximise \( |\text{ex}(G)| \). Observe that these are the graphs \( G \in \mathcal{P}(n) \) with at most \( m \) edges which satisfy
\[ |\text{ex}(G)| = \binom{n}{2} - \text{add}(n, m) - m. \]
Finally, we denote by $P_{\min}(n, \leq m)$ the set of graphs $G \in \hat{\mathcal{P}}(n, \leq m)$ which minimise $m(G)$.

Let us assume that $n \geq 6$ and $8 \leq m \leq 3n - 6$. Then $\text{ex}(G) \neq \emptyset$ for all graphs $G \in \hat{\mathcal{P}}(n, \leq m)$ (and hence for all $G \in P_{\min}(n, \leq m)$).

Let us start by stating a simple observation concerning excluded edges.

**Observation 1** If $G$ has components $H_1, H_2, \ldots, H_k$, then $\text{ex}(G) = \text{ex}(H_1) \cup \text{ex}(H_2) \cup \ldots \cup \text{ex}(H_k)$.

By Observation 1 we can delete all the edges of a component $H$ with $\text{ex}(H) = \emptyset$ to obtain a graph with the same set of excluded edges. This leads to the following observation.

**Observation 2** If $G \in P_{\min}(n, \leq m)$ has components $H_1, H_2, \ldots, H_k$, then for any component $H$ with at least one edge we have $\text{ex}(H) \neq \emptyset$.

The next observation concerns the faces of a $2^*$-connected plane graph, more precisely it states that every face is an induced cycle. This is true because if there were a short-cut $uv \in E(G)$ of a face, then $u, v$ would be a cut which contradicts the fact that no two adjacent nodes in a $2^*$-connected graph form a cut.

**Observation 3** If $G \in \mathcal{P}(n, m)$ is $2^*$-connected, then every face is an induced cycle.

**Lemma 4** Let $G \in \mathcal{P}(n, m)$ be a $2^*$-connected graph such that $\text{ex}(G) \neq \emptyset$. Then for every induced cycle $C$ of $G$ with $|V(C)| \geq 4$, there is a path connecting two non-adjacent nodes of $C$ which is internally disjoint with $C$.

**Proof.** Since $\text{ex}(G) \neq \emptyset$, there is an edge $uv$ in $G$ with $u \in V(C)$ and $v \notin V(C)$. Let $x, y$ be the neighbours of $u$ in $C$, i.e. $ux \in E(C)$ and $uy \in E(C)$. Since $G$ is $2^*$-connected, $u, x$ do not form a 2-cut, i.e. there is a path $p$ internally disjoint to $C$ connecting $v$ to a node $a \in V(C) \setminus \{u, x\}$. If $a \neq y$ then this path together with the edge $uv$ yields a path connecting $u$ with $a$ and $au \notin E(C)$. If $a = y$ then consider the nodes $u$ and $y$. Again, since $G$ is $2^*$-connected $u, y$ do not form a 2-cut, and hence there is a path $p'$ internally disjoint to $C$ connecting $v$ to $b \in V(C) \setminus \{u, y\}$. If $b \neq x$, then $p'$ together with the edge $uv$ forms a path connecting $u$ with $b$ and $ub \notin E(G)$; otherwise the paths $p$ and $p'$ together contain a path internally disjoint to $C$ connecting $x$ and $y$, and since $|V(C)| \geq 4$, we have $xy \notin E(G)$.  

\[\blacksquare\]
Lemma 5 Let $G \in \mathcal{P}(n,m)$ be a $2^*$-connected graph such that $ex(G) \neq \emptyset$. Suppose that $G$ has $i$ isolated nodes and a plane embedding with $n_k$ faces of size $k$. Then we have

$$|add(G)| \geq \frac{1}{4} \sum_{k=3}^{n} (k-3)(k+2)n_k + \frac{i(i-1)}{2} + i(n-i).$$

Proof. Let $\hat{G}$ be the graph obtained from $G$ by deleting all isolated nodes. Fix a plane embedding of $\hat{G}$ with $n_k$ $k$-faces as above. Consider a $k$-face $F$ of $\hat{G}$ bounded by a cycle $C$ with $k \geq 4$ nodes. By Observation 3, the cycle $C$ is induced; and so, by Lemma 4, there is a path $p$ connecting two non-adjacent nodes $u$ and $v$ of $C$ which is internally disjoint from $C$. Fix such a path $p$ for each face $F$. We call a non-edge $f$ with both end nodes in $C$ crossing for $F$ if it connects the two components of $C - u - v$, and otherwise non-crossing.

Observe that the only way to add a crossing non-edge for $F$ in the given embedding is to draw it within the face $F$. A non-crossing non-edge $xy$ can be drawn in at most two faces in this embedding, since otherwise $\hat{G} - x - y$ would have at least three components (which would contradict the $2^*$-connectedness of $G$). Hence if we assign to a crossing non-edge $f$ of $F$ the weight $w_F(f) = 1$ and to a non-crossing non-edge $g$ the weight $w_F(g) = 1/2$, then the sum of all the non-edge weights over all the faces is a lower bound for $|add(\hat{G})|$.

Let $F$ be a $k$-face, with bounding cycle $C$. We claim that the sum of the weights $w_F(f)$ over all non-edges $f$ of $F$ is at least $\frac{1}{4}(k-3)(k+2)$. Once we have established this claim the lemma will follow easily.

To prove the claim, suppose that the chosen path $p$ for $F$ connects the non-adjacent nodes $u$ and $v$ in $C$. Let $l$ and $k - l - 2$ be the numbers of nodes in the two components of $C - u - v$. By Observation 3, the cycle $C$ is induced. Hence there are $l(k - l - 2)$ crossing non-edges and

$$\binom{k}{2} - k - l(k - l - 2)$$

non-crossing non-edges. Therefore the weight $\sum_f w_F(f)$ of the face $F$ equals

$$\frac{1}{2} \binom{k}{2} - \frac{1}{2} k + \frac{1}{2} l(k - l - 2).$$

As a function of $l$ this is minimised over $1 \leq l \leq k - 3$ at $l = 1$ or $k - 3$, and the value in each case is then $\frac{1}{4}(k - 3)(k + 2)$. □
The remainder of this Section is devoted to showing that there is a 2*-connected graph \( G \in \mathcal{P}(n, m) \) with \(|\text{add}(G)| = \text{add}(n, m)\). To show this we need the following lemmas.

**Lemma 6** There is a graph \( G \in \mathcal{P}_{\min}(n, \leq m) \) with no cut-edges.

**Proof.** Let \( G \) be any graph in \( \mathcal{P}_{\min}(n, \leq m) \) with as few cut-edges as possible. Let \( \hat{G} \) be the graph obtained from \( G \) by deleting all isolated nodes. If each component of \( \hat{G} \) is 2-edge-connected, then there remains nothing to show, so assume that \( \hat{G} \) has a cut-edge. By Observation 2 there is no component \( H \) of \( \hat{G} \) with \( \text{ex}(H) = \emptyset \). In particular none of the components of \( \hat{G} \) is a tree, and hence there is a cut-edge \( e = uv \) such that \( u \) belongs to a cycle. We construct a new graph \( G' \) by choosing an edge \( e' = ux \) which belongs to a cycle, deleting \( e' \), and adding the edge \( vx \), see Figure 2.

It is easy to verify that \( G' \) is planar, has the same number of edges as \( G \), and has fewer cut-edges. We shall show that \( \text{ex}(\hat{G}) \subseteq \text{ex}(G') \), and hence \( G' \in \mathcal{P}_{\min}(n, \leq m) \), which will contradict the choice of \( G \) and thus complete the proof.

Let \( H_1 \) and \( H_2 \) be the components of \( \hat{G} - uv \) with \( u \in H_1 \) and \( v \in H_2 \). Suppose that \( ab \in \text{ex}(\hat{G}) \). If both \( a \) and \( b \) are in some component \( H_i \) of \( \hat{G} - uv \) then \( ab \in \text{ex}(H_i) \), and so \( ab \in \text{ex}(G') \). Now suppose that \( a \in H_1 \) and \( b \in H_2 \), and that \( ab \in \text{add}(G') \). If we start with \( G + ab \) and contract \( H_2 \) onto \( u \) we see that \( au \notin \text{ex}(H_1) \). Similarly, if we start with \( G' + ab \) and contract \( H_1 \) onto \( v \) we see that \( bv \notin \text{ex}(H_2) \). Hence there is a plane embedding of \( H_1 \) with \( a \) and \( u \) on the outer face, and a plane embedding of \( H_2 \) with \( b \) and \( v \) on the outer face. Therefore there is a plane embedding of \( \hat{G} + ab \) which contradicts the assumption that \( ab \in \text{ex}(\hat{G}) \). \[ \square \]

**Lemma 7** Let the planar graph \( G \) be 2-connected and have a 2-node cut \( u, v \). Let \( G - u - v \) have components \( H_1, H_2, \ldots \), and let \( \overline{H_i} \) be the subgraph of \( G \) induced by \( V(H_i) \cup \{u, v\} \). Denote by \( H_i^+ \) the (planar) graph formed from \( \overline{H_i} \)
by adding a new node $z$ and edges $uz$ and $vz$. Consider a non-edge $ab$ of $G$. If both $a$ and $b$ are in some $\overline{H}_i$ then $ab \in \text{ex}(G)$ if and only if $ab \in \text{ex}(H_i^+)$. If $a$ is in $H_i$ and $b$ is in $H_j$ where $i \neq j$ then $ab \in \text{ex}(G)$ if and only if $az \in \text{ex}(H_i^+)$ or $bz \in \text{ex}(H_j^+)$. 

**Proof.** Suppose first that both $a$ and $b$ are in the same $\overline{H}_i$, say $\overline{H}_1$. If $ab \in \text{add}(G)$ then $ab \in \text{add}(H_i^+)$. Conversely if $ab \in \text{add}(H_i^+)$ then there is an embedding in the plane of $\overline{H}_1 + ab$ with $u$ and $v$ on the outer face, and it follows easily that $G + ab$ is planar.

Now suppose that $a \in H_1$ and $b \in H_2$. Assume first that $ab \notin \text{ex}(G)$, that is $ab \in \text{add}(G)$. In $\overline{H}_2$ there are paths from $b$ to $u$ and from $b$ to $v$ with only $b$ in common. Thus $G + ab$ has a minor $H_1^+ + az$, and so $az \notin \text{ex}(H_i^+)$. Similarly $bz \notin \text{ex}(H_2^+)$. Now assume that $az \notin \text{ex}(H_i^+)$ and $bz \notin \text{ex}(H_2^+)$. Then there is an embedding of $\overline{H}_1$ in the plane with each of $u$, $v$, and $a$ on the outer face, and similarly an embedding of $\overline{H}_2$ in the plane with each of $u$, $v$, and $b$ on the outer face. If we put two such embeddings side-by-side and identify the two copies of $u$ and the two copies of $v$ then we may add the edge $ab$, and see that $ab \notin \text{ex}(G)$.

**Lemma 8** Let $G \in \mathcal{P}_{\text{min}}(n, \leq m)$ consist of 2-edge connected components and possibly some isolated nodes. Then $G$ is 2*-connected.

**Proof.** Let $\hat{G}$ be the graph obtained from $G$ by deleting all the isolated nodes.

**Claim 8.1** The graph $\hat{G}$ is connected.

Assume for a contradiction that $\hat{G}$ has (at least) two components $H_1$ and $H_2$. Let $v, w \in V(H_1)$ such that $vw \in \text{ex}(\hat{G})$. Such a non-edge exists by Observation 2. We construct a new graph $G'$ by choosing an edge $uv$ of $E(H_1)$ and an edge $xy$ of $E(H_2)$, removing these edges, and adding $ux$ and $vy$, see Figure 3. The new graph is clearly planar since we can choose a planar embedding of $H_1$ with $uv$ on the outer face and a planar embedding of $H_2$ with $xy$ on the outer face. We shall show that $|\text{ex}(\hat{G})| < |\text{ex}(G')|$ which together with the fact that $|E(\hat{G})| = |E(G')|$ yields $G \notin \mathcal{P}_{\text{min}}(n, \leq m)$, a contradiction.

Let $ab \in \text{ex}(G)$. Then both $a$ and $b$ are in some component $H_i$, and so $ab \in \text{ex}(H_i)$. Hence $ab \in \text{ex}(G')$, since we can contract $G'$ down to $H_i$. Thus $\text{ex}(G') \subseteq \text{ex}(G')$. Further, $wy \in \text{ex}(G')$, for if we start with $G' + wy$ and contract $H_2$ onto $v$ we obtain $H_1 + wv$. 

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Figure 3: Operation to decrease the number of components, the dashed lines are excluded non-edges.

Figure 4: Operation to remove cut nodes.

Claim 8.2 The graph $\hat{G}$ is 2-node connected.

Assume for a contradiction that $v$ is a cut node of $\hat{G}$. Let $H_1, H_2, \ldots, H_k$ be the components of $\hat{G} - v$, where $k \geq 2$. Also for $i = 1, \ldots, k$, let $\hat{H}_i$ be the subgraph of $\hat{G}$ induced by $V(H_i) \cup \{v\}$. Let $u$ be a node in $V(H_i)$ and $w$ be a node in $V(H_j)$ such that $uw, vw \in E(\hat{G})$. Form a new graph $G'$ by removing the edges $uw$ and $vw$, and adding the edge $uv$, see Figure 4. The graph $G'$ is planar, since there exist planar embeddings of $\hat{H}_1$ and $\hat{H}_2$ with $uv$ respectively $vw$ on the outer face. In addition, $G'$ has strictly fewer edges than $\hat{G}$. We now show that $\text{ex}(\hat{G}) \subseteq \text{ex}(G')$ which together with the fact that $|E(G')| < |E(\hat{G})|$ implies that $\hat{G} \notin \mathcal{P}_{\text{min}}(n, \leq m)$.

Let $ab \in \text{ex}(\hat{G})$. If both $a$ and $b$ are in some $\hat{H}_i$, then $ab \in \text{ex}(\hat{H}_i)$. But then $ab \in \text{ex}(G')$, since $\hat{H}_i$ is a minor of $G'$. To see this for $\hat{H}_1$ for example, note that in $\hat{H}_2$ there is a path between $v$ and $w$. Suppose now that $a$ is in $\hat{H}_i$ and $b$ is in $\hat{H}_j$, $i \neq j$. We have already seen that $uv \in \text{add}(\hat{G})$, and $ab \neq uv$ and thus $ab \notin E(G')$. Assume that $ab \in \text{add}(G')$. If we contract all the nodes of $\hat{H}_j$ onto $v$ then we see that $av \notin \text{ex}(\hat{H}_i)$. Similarly $bv \notin \text{ex}(\hat{H}_j)$. Hence there exists a plane embedding of $\hat{H}_i$ with $av$ on the outer face, and a plane embedding of $\hat{H}_j$ with $bv$ on the outer face. Hence $\hat{G} + ab$ is planar, which contradicts the assumption that $ab \in \text{ex}(\hat{G})$. Finally, since $ab$ is not
in $E(G')$ or $\text{add}(G')$, we must have $ab \in \text{ex}(G')$, as required.

**Claim 8.3** There is no 2-node cut $u,v$ in $\hat{G}$ with $uv \in E(\hat{G})$.

Suppose that $\hat{G}$ has a 2-node cut $u,v$ with $u$ and $v$ adjacent, and let $G' = \hat{G} - uv$. For a non-edge $ab$ of $\hat{G}$, the condition given in Lemma 7 for $ab \in \text{ex}(\hat{G})$ is exactly the same as for $ab \in \text{ex}(G')$, and so $\text{ex}(\hat{G}) = \text{ex}(G')$. But $G'$ has fewer edges than $\hat{G}$, which contradicts our choice of $G$.

**Claim 8.4** For any two nodes $u$ and $v$, the graph $\hat{G} - u - v$ has at most two components.

Assume that there is a cut $u$, $v$ such that $\hat{G} - u - v$ has components $H_1, H_2, \ldots, H_k$, where $k \geq 3$. By Lemma 7, at least one component $H_i$ satisfies $\text{ex}(H_i^+) \neq \emptyset$, since otherwise $\text{ex}(G) = \emptyset$: we may assume that $i = 1$. Fix a plane embedding of $\hat{G}$ such that $u$ and $v$ are on the outer face. Starting at $u$ follow the leftmost path in $H_1^+$ to $v$ and the rightmost path in $H_1^+$ to $v$. These two paths cannot be identical since $\text{ex}(H_1^+) \neq \emptyset$. Suppose that the two paths first part at $c$ (where perhaps $c = u$) and next join at $d$ (where perhaps $d = v$). Since by Claim 8.3 there are no 2-node cuts consisting of adjacent nodes, there must be a node $a$ on the leftmost path and a node $b$ on the rightmost path strictly between $c$ and $d$. Choose any node $w$ in $H_3$ such that $wz \in \text{add}(H_3^+)$. 

Now, form a new graph $G'$ from $\hat{G}$ by ‘rotating’ $H_2$ so that instead of being attached at $u$ and $v$ it is attached at $b$ and $w$; that is, by replacing each edge $yu$ with $y$ in $H_2$ by the edge $yb$, and each edge $yw$ with $y$ in $H_2$ by the edge $yw$, see Figure 5. Observe that $G'$ is planar, since $bw \in \text{add}(G)$ by Lemma 7, and so $bw \in \text{add}(G - H_2)$. Also $|E(G')| = |E(\hat{G})|$. We shall show that $|\text{ex}(G')| > |\text{ex}(\hat{G})|$, which will contradict our choice of $G$. We do this in three steps.

(a) We first consider the set $A$ of possible edges $xy$ where $x \in \{u,v,b,w\}$ and $y \in H_2$. For each $y \in H_2$ we have

$$yu \in \text{ex}(G) \Leftrightarrow yb \in \text{ex}(G') \quad \text{and} \quad yu \in \text{ex}(G') \Leftrightarrow yb \in \text{ex}(G),$$

and similarly

$$yw \in \text{ex}(G) \Leftrightarrow yw \in \text{ex}(G') \quad \text{and} \quad yv \in \text{ex}(G') \Leftrightarrow yw \in \text{ex}(G).$$

Thus $|A \cap \text{ex}(G)| = |A \cap \text{ex}(G')|$. 

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(b) Now consider a non-edge $f$ other than those in $A$. Suppose first that $f$ has neither endpoint in $H_2$. Then by Lemma 7, if $f \in \text{ex}(\hat{G})$ then $f \in \text{ex}(G - H_2)$, and so $f \in \text{ex}(G')$. Suppose now that $f = pq$ where $p$ in $H_2$, and that $f \in \text{add}(G')$. If $q$ is in $H_2$, then $\overrightarrow{T}_2$ has a plane embedding with $p$ and $q$ on one face and with $u$ and $v$ on the outer face, and so $G + pq$ is planar, that is $pq \in \text{add}(G)$. Suppose finally that $q$ is in $H_i$ for some $i \neq 2$. Then there is a path from $q$ to $b$ in $G' - w - H_2$ and a path from $q$ to $w$ in $G' - b - H_2$. Thus there is a node $q'$ in $G - \overrightarrow{T}_2$ (possibly $q' = q$) and three paths between $q'$ and $b$, $q'$ and $w$, and $q'$ and $q$ in $G - H_2$ such that these paths pairwise have only $q'$ in common. Since $pq \in \text{add}(G')$ it follows that in the notation of Lemma 7, we have $pq \in \text{add}(H_i^+)$. Similarly $qz \in \text{add}(H_i^+)$, and hence $pq \in \text{add}(G)$. So we have $|\text{ex}(G')| \geq |\text{ex}(\hat{G})|$.

(c) To show that the last inequality is strict, consider a node $x$ in $H_2$ such that $ax \in \text{add}(G)$. But $ax$ is not addable in $G'$ since $G' + ax$ contains a homeomorphic of $K_{3,3}$, with the nodes of degree greater than 2 being $a, b, w$ and $c, d, x$.

Combining Lemma 6 and Lemma 8 we obtain the following result.

**Lemma 9** There is a 2*-connected graph $G \in P_{\text{min}}(n, \leq m)$.

**Lemma 10** There is a 2*-connected graph $G \in P(n, m)$ with $|\text{add}(G)| = \text{add}(n, m)$.

**Proof.** By Lemma 9 there is a 2*-connected graph $G^* \in P_{\text{min}}(n, \leq m)$. If $|E(G^*)| = m$, then there remains nothing to prove, because in this case $G^* \in P(n, m)$ and $\text{add}(G^*) = \text{add}(n, m)$. If $|E(G^*)| < m$, then we can repeatedly add edges to obtain 2*-connected planar graphs in the following...
way. Let \( \hat{G} \) be the 2-node connected component of \( G^* \). Suppose first that \( \hat{G} \) is triangulated, and so \( G^* \) has an isolated node. Form \( G' \) by deleting an isolated node and subdividing an arbitrary edge \( uv \), by inserting a new node \( z \). Clearly the number of edges increases and \( G' \) satisfies \(\text{ex}(G^*) \subseteq \text{ex}(G')\).

Further, it is easy to check that \( G' \) is 2*-connected. For clearly \( G' \) is 2-connected. Let \( x, y \) be a 2-node cut in \( G' \). If this is not a cut in \( G^* \), then either (a) it is \( u, v \) and we have \( uv \notin E(G') \) and \( G' - u - v \) consists of the single node \( z \) and one other component; or (b) \( x \) say is \( z \) and \( y \) is not \( u \) or \( v \) (since \( G' - z - u \) and \( G' - z - v \) are the same as \( G^* - u \)), and thus \( x \) and \( y \) are not adjacent in \( G' \), and \( G' - x - y \) is the same as \( G^* - y - uv \) and so has at most two components.

Now suppose that \( \hat{G} \) is not triangulated, and choose a face bounded by a cycle \( C \) with at least 4 nodes. By Observation \( 3 \) \( C \) is an induced cycle, and by Lemma \( 4 \) there are two non-adjacent nodes \( u \) and \( v \) on \( C \) which are connected by a path internally disjoint from \( C \). Form a new graph \( G' \) by adding an edge \( xy \) such that \( x \) and \( y \) are in different components of \( C - u - v \), see Figure 6. We need to check that \( G' \) is 2*-connected, and to do this it clearly suffices to check that \( x, y \) do not form a 2-cut in \( G' \). But our choice of \( u \) and \( v \) ensures that they do not form a 2-cut in \( \hat{G} \) and hence not in \( G' \) either. \( \blacksquare \)
4 Proof of Theorem 2

Proof of Theorem 2. We show first that \( \text{add}(n, m) \geq f(\alpha)n + c \) for a constant \( c \). Given non-negative integers \( i \) and \( n_3, n_4, \ldots, n_n \), let

\[
h(i, n_3, \ldots, n_n) = \frac{1}{4} \sum_{k=3}^{n} (k - 3)(k + 2)n_k + i(n - \frac{i + 1}{2}).
\]

Let \( G \in \mathcal{P}(n, m) \) be a 2\( \alpha \)-connected graph with \( \text{add}(G) = \text{add}(n, m) \). Such a graph exists by Lemma 10. Let \( i \) be the number of isolated nodes in \( G \). It is perhaps intuitively clear that \( i \) should be 0, but we must allow \( i \) to be positive here. Fix a plane embedding of \( G \), and for \( k = 3, 4, \ldots \) let \( n_k \) be the number of \( k \)-faces (that is, faces bounded by \( k \) edges). Since \( \alpha > 1 \), it now follows from Lemma 5 that

\[
\text{add}(n, m) = |\text{add}(G)| \geq h(i, n_3, \ldots, n_n).
\]

Observe that

\[
\sum_{k=3}^{n} kn_k = 2m,
\]

and by Euler’s formula

\[
\sum_{k=3}^{n} n_k = m - (n - i) + 2.
\]

We shall obtain our lower bound on \( \text{add}(n, m) \) by minimising \( h(i, n_3, \ldots, n_n) \) over all choices of non-negative integers \( i, n_3, \ldots, n_n \), subject to the linear constraints (3) and (4). Denote this minimum value by \( h^*(n, m) \). Observe that the formula \((k - 3)(k + 2)\) yields a convex function of \( k \). Thus the minimum value \( h^*(n, m) \) is attained with either a single non-zero value \( n_k \) or two ‘adjacent’ non-zero values \( n_k \) and \( n_{k+1} \).

From now on, let \( \hat{k}, i^* \) and \( n_k^*, n_{k+1}^*, \ldots, n_n^* \) be integers such that \( 3 \leq \hat{k} < n, i^* \geq 0, n_{\hat{k}}^* > 0, n_{\hat{k}+1}^* \geq 0 \) and \( n_j^* = 0 \) for each \( j \neq \hat{k}, \hat{k} + 1 \); the equations (3) and (4) hold; and

\[
h^*(n, m) = h(i^*, n_{\hat{k}}^*, \ldots, n_n^*)
\]

\[
= \frac{1}{4}(\hat{k} - 3)(\hat{k} + 2)n_{\hat{k}}^* + \frac{1}{4}(\hat{k} - 2)(\hat{k} + 3)n_{\hat{k}+1}^* + i^*(n - \frac{i^* + 1}{2}).
\]

Let \( \bar{n} = n - i^* \). It follows from (3) and (4) that

\[
n_k^* = \hat{k}(m - \bar{n} + 2) - (m + \bar{n} - 2),
\]

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\[ n_{k+1}^* = 2m - \hat{k}(m - \tilde{n} + 2), \]

and, since \( m \leq 3(n - i^*) - 6, \)

\[ i^* \leq n - \frac{1}{3}m - 2. \tag{5} \]

Also

\[ 2m = \hat{k}n_k^* + (\hat{k} + 1)n_{k+1}^* \geq \hat{k}n_k^* + \hat{k}n_{k+1}^* = \hat{k}(m - \tilde{n} + 2) \]

and

\[ 2m = \hat{k}n_k^* + (\hat{k} + 1)n_{k+1}^* < (\hat{k} + 1)n_k^* + (\hat{k} + 1)n_{k+1}^* = (\hat{k} + 1)(m - \tilde{n} + 2). \]

Therefore

\[ \frac{2m}{m - \tilde{n} + 2} - 1 < \hat{k} \leq \frac{2m}{m - \tilde{n} + 2} < \frac{2m}{m - n}. \tag{6} \]

Hence \( \hat{k} = \lfloor \frac{2m}{m - \tilde{n} + 2} \rfloor. \) Observe that \( 2m/(m - n) \) tends to \( 2\alpha/(\alpha - 1) \) as \( n \) tends to infinity, and hence \( \hat{k} \) is bounded.

We first show that \( i^* = 0. \) To this end we assume for a contradiction that \( i^* \geq 1. \) Now

\[ n_k^* + n_{k+1}^* = m - n + i^* + 2 \geq (\hat{k} + 1) + (\hat{k} + 2) \]

for \( n \) sufficiently large, since \( \hat{k} \) is bounded, and \( m/n \to \alpha > 1 \) as \( n \to \infty. \)

Then for some \( 3 \leq j < n, j = \hat{k} \) or \( j = \hat{k} + 1, \) we have \( n_j^* \geq j + 1. \) Choose such a \( j. \) Define new values \( i', n_3', n_4', \ldots, n_n', \) which equal the old values except that \( i' = i^* - 1, n_j' = n_j^* - (j + 1) \) and \( n_{j+1}' = n_{j+1}^* + j. \) Then (3) and (4) still hold for the new values, and the old value of \( h \) minus the new value equals

\[ \frac{1}{4}(j - 3)(j + 2)(j + 1) - \frac{1}{4}(j - 2)(j + 3)j + n - i^* > 0 \]

for \( n \) sufficiently large. This contradicts the fact that we started from the minimum value of \( h, \) and it follows that \( i^* = 0, \) for \( n \) sufficiently large. We can now replace \( \tilde{n} \) by \( n \) in the formulae above for \( \hat{k}, n_k^* \) and \( n_{k+1}^*. \)

Set \( \alpha_n = m/n. \) Then

\[ h^*(n, m) \]

\[ = \frac{1}{4}(\hat{k} - 3)(\hat{k} + 2)n_k^* + \frac{1}{4}(\hat{k} - 2)(\hat{k} + 3)n_{k+1}^* \tag{7} \]

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\[
\frac{1}{4} (\hat{k}^2 - \hat{k} - 6)n_k^* + \frac{1}{4} (\hat{k}^2 + \hat{k} - 6)(-n_k^* + m - n + 2)
\]

\[
= \frac{-2\hat{k}}{4}n_k^* + \frac{1}{4} (\hat{k}^2 + \hat{k} - 6)(m - n + 2)
\]

\[
= \frac{-2\hat{k}}{4}((\hat{k} - 1)\alpha_n - (\hat{k} + 1))n + 2(\hat{k} + 1) + \frac{1}{4}(\hat{k}^2 + \hat{k} - 6)(m - n + 2)
\]

\[
= \frac{1}{4}n(\hat{k}^2 + \hat{k} + 6 - (\hat{k}^2 - 3\hat{k} + 6)\alpha_n) - \frac{1}{2}(\hat{k}^2 + \hat{k} + 6).
\]

Recall that \( \hat{k} = \left\lfloor \frac{2m}{m - n + 2} \right\rfloor \). Let \( \tilde{k} = \left\lfloor \frac{2\alpha}{\alpha - 1} \right\rfloor \). By the above, if \( \hat{k} = \tilde{k} \), then \( \text{add}(n, m) \geq f(\alpha)n + c \) for a constant \( c \), since \( \alpha_n = \alpha + O(\frac{1}{n}) \). If \( 2\alpha/(\alpha - 1) \) is not integral, then \( \hat{k} = \tilde{k} \) for all sufficiently large \( n \). If \( 2\alpha/(\alpha - 1) \) is integral, and so equals \( \tilde{k} \), then \( \hat{k} = \tilde{k} \) or \( \hat{k} = \tilde{k} - 1 \) for sufficiently large \( n \). If \( \hat{k} = \tilde{k} - 1 \) then with \( c_k = -\frac{1}{2}(\hat{k}^2 + \hat{k} + 6) \)

\[
h^*(n, m)
\]

\[
= \frac{1}{4}n((\tilde{k} - 1)^2 + (\tilde{k} - 1) + 6 - ((\tilde{k} - 1)^2 - 3(\tilde{k} - 1) + 6)\alpha_n) + c_k
\]

\[
= \frac{1}{4}n(\tilde{k}^2 + \tilde{k} + 6 - (\tilde{k}^2 - 3\tilde{k} + 6)\alpha_n) + \frac{1}{4}n(-2\tilde{k} + 2\tilde{k}\alpha_n - 4\alpha_n) + c_k.
\]

But

\[
-2\tilde{k} + 2\tilde{k}\alpha_n - 4\alpha_n = -\frac{4\alpha}{\alpha - 1} + \frac{4\alpha\alpha_n}{\alpha - 1} - \frac{4\alpha\alpha_n - 4\alpha_n}{\alpha - 1} = \frac{4(\alpha_n - \alpha)}{\alpha - 1}.
\]

It now follows that \( \text{add}(n, m) \geq f(\alpha)n + c' \) for an appropriate constant \( c' \), since \( \alpha_n = \alpha + O(1/n) \).

We will now show that \( \text{add}(n, m) \leq f(\alpha)n + c \) for an appropriate constant \( c \). Since \( \text{add}(n, m) \leq \text{add}(n, m - 1) \), it suffices to show that \( \text{add}(n, m) \leq f(\alpha)n + c \) if \( m - n \) is even (and still \( m = \alpha n + O(1) \)). Let

\[
k = \left\lfloor \frac{2m}{m - n + 2} \right\rfloor,
\]

\[
n_k = k(m - n + 2) - (n + m - 2),
\]

and

\[
n_{k+1} = 2m - k(m - n + 2).
\]
Figure 7: The graph $G^*$

Observe that $k \geq 3$ since $m \leq 3n - 6$, and either $k = \lceil 2\alpha/(\alpha - 1) \rceil$ or $k = \lfloor 2\alpha/(\alpha - 1) \rfloor - 1$ for $n$ sufficiently large. Also

$$n_k > \left( \frac{2m}{m - n + 2} - 1 \right) (m - n + 2) - (n + m - 2) = 0$$

and

$$n_{k+1} \geq 2m - \frac{2m}{m - n + 2} (m - n + 2) = 0.$$

Further, both $n_k$ and $n_{k+1}$ are even, since $m - n$ is even.

We will construct a planar graph $G$ which has $n$ nodes, $m$ edges, $n_k$ faces of size $k$, and $n_{k+1}$ faces of size $k+1$. This construction follows an idea from [2] and simplifies an earlier construction of ours.

Let $t$ be the integer $(n_k + n_{k+1} - 2)/2$, which is positive for all $n$ sufficiently large. Let the graph $G^*$ have nodes $v_0, \ldots, v_t, u$ and $w$; and edges $v_i v_{i+1}$ for $i = 0, \ldots, t - 1$, $uw_j$ and $wv_j$ for each $j = 0, 1, \ldots, t$ and $uw$, see Figure 7. Note that $G^*$ is 3-node-connected.

In the graph $G^*$, consider the ‘path’ edges $v_i v_{i+1}$ for $i = 0, \ldots, t - 1$ together with the edge $uw$, altogether a set of $\frac{n_k}{2} + \frac{n_{k+1}}{2}$ edges. We construct the graph $G$ by subdividing $\frac{n_k}{2}$ of these edges into $k - 2$ parts, and the remaining $\frac{n_{k+1}}{2}$ into $k - 1$ parts. It is easily checked that $G$ is as desired. Furthermore, $G$ is a subdivision of the 3-node-connected graph $G^*$, and hence by a theorem of Whitney [5] any two embeddings of $G$ are equivalent. Now,
arguing as in the proof of Lemma 5, we may see that

$$|\text{add}(G)| = \frac{n_k (k - 3)(k + 2)}{2} + \frac{n_{k+1} (k - 2)(k + 3)}{2}.$$  

It now follows by the same computations as in (7) in the case of $k = \lfloor 2\alpha/(\alpha - 1) \rfloor$, and by (8), (9) and the fact that $m/n = \alpha_n = \alpha + O(1/n)$ in the case of $k = \lfloor 2\alpha/(\alpha - 1) \rfloor - 1$ that

$$\text{add}(n, m) \leq |\text{add}(G)| = f(\alpha)n + O(1).$$

\[\square\]

Observe that one can use (8) and (9) with $\alpha_n = \alpha$ to see that the function $f$ is continuous when $2\alpha/(\alpha - 1)$ is integral. For all other cases this statement is obvious.

5 Outerplanar graphs

We may apply a similar analysis to outerplanar graphs, which are graphs which have an embedding in the plane such that every node lies on the outer face. Given an outerplanar graph $G$, we call a non-edge $f$ outeraddable in $G$ if the graph $G + f$ obtained by adding $f$ as an edge is still outerplanar; and we let outadd($G$) denote the set of outeraddable non-edges of $G$. For $n \geq 2$ an outerplanar graph with $n$ nodes has at most $2n - 3$ edges. For such $n$ and $m$, let $\mathcal{OP}(n, m)$ denote the set of all outerplanar graphs with $n$ nodes $v_1, \ldots, v_n$ and $m$ edges, and let outadd($n, m$) denote the minimum value of $|\text{outadd}(G)|$ over all graphs $G \in \mathcal{OP}(n, m)$. We have:

**Theorem 11** Let $m^*(n)$ be the greatest $m$ such that outadd($n, m$) $\geq m$. Then

$$m^*(n) = \frac{7}{5}n + O(1) \quad \text{as} \quad n \to \infty.$$  

Let $R'_n$ denote a graph drawn uniformly at random from the set of all outerplanar graphs on the $n$ labelled nodes $v_1, \ldots, v_n$. Much as for Theorem 2, we may obtain the following Theorem.

**Theorem 12** As $n \to \infty$,

$$\mathbb{E}[m(R'_n)] \geq \frac{7}{5}n + o(n);$$
and indeed for any \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that

\[
\Pr \left( m(R_n^\varepsilon) < \left( \frac{T}{5} - \varepsilon \right) n \right) = o(e^{-\delta n}).
\]

There are more general results for outerplanar graphs corresponding to Theorem 1 and Theorem 2: we leave these to the reader. The main steps in proving Theorem 11 are given by the following two lemmas, which may be proved along the lines of the proofs of the corresponding earlier results. As before, let \( \hat{G} \) denote the graph obtained from \( G \) by deleting all isolated nodes.

**Lemma 13** There is a graph \( G \in \mathcal{OP}(n,m) \) with \( |\text{outadd}(G)| = \text{outadd}(n,m) \) such that \( \hat{G} \) is 2-node-connected.

**Proof.** Let \( G \in \mathcal{OP}(n,m) \) be such that \( |\text{outadd}(G)| = \text{outadd}(n,m) \). If \( \hat{G} \) has a component with a cut edge, then we can apply the operation described in the proof of Lemma 6 to obtain a graph \( G' \) which has only 2-edge-connected components, see also Figure 2. Observe that this operation keeps the graph outerplanar, and that the same analysis as in the proof of Lemma 6 shows that \( |\text{outadd}(G')| = |\text{outadd}(\hat{G})| \) and \( m(G') = m(\hat{G}) \). So we can assume that \( \hat{G} \) consists of 2-edge-connected components. Now we show as in Claim 8.1 and Claim 8.2 that \( \hat{G} \) is 2-node-connected. \( \blacksquare \)

Note that we cannot insist on \( G \) being 2\(^*\)-connected: for the graph \( G = K_4 - e \) shown in Figure 8 is the unique graph \( G \in \mathcal{OP}(4,5) \) (up to isomorphism) with a non-addable edge. Note also that for any 2-node-connected outerplanar graph, and any two nodes \( u \) and \( v \), the graph \( G - u - v \) has at most 2-components.
Lemma 14 Let the graph $G \in \mathcal{OP}(n, m)$ have $i$ isolated nodes and be such that $\hat{G}$ is 2-node-connected. Suppose that $\hat{G}$ has an outerplanar embedding with $n_k$ $k$-faces for each $k = 3, 4, \ldots, n$ not counting the outer face. Then we have

$$|\text{outadd}(G)| = \frac{1}{2} \sum_{k=3}^{n} k(k-3)n_k + i(n - \frac{i+1}{2}).$$

Proof. Observe that any non-edge $f \in \text{outadd}(\hat{G})$ can be drawn in this embedding into exactly one bounded face to obtain an outerplanar embedding of $\hat{G} + f$. Now there are $\binom{k}{2} - k = \frac{1}{2}k(k-3)$ edges we can add to a $k$-face. Hence we have

$$|\text{outadd}(\hat{G})| = \frac{1}{2} \sum_{k=3}^{n} k(k-3)n_k.$$

In addition, we can add all the $\binom{i}{2}$ non-edges between isolated nodes, and the $i(n - i)$ non-edges between an isolated node and a node of $\hat{G}$. ■

Proof of Theorem 11. We first give a lower bound for $\text{outadd}(n, m)$. We are following the pattern of the proof of Theorem 2, though of course the definition of the function $h$ has to be changed. Suppose that

$$\frac{4}{3}n \leq m < \frac{3}{2}n - \frac{7}{2}. \quad (10)$$

Let $G \in \mathcal{OP}(n, m)$ be such that $|\text{outadd}(G)| = \text{outadd}(n, m)$ and $\hat{G}$ is 2-node connected. Such a graph $G$ exists by Lemma 13. Suppose that $G$ has $i$ isolated nodes. Fix an outerplanar embedding of $\hat{G}$, with $n_k$ $k$-faces for $k = 3, \ldots, n$ not counting the outer face. By Lemma 14,

$$\text{outadd}(n, m) = |\text{outadd}(G)| = h(i, n_3, \ldots, n_n),$$

where

$$h(i, n_3, \ldots, n_n) = \frac{1}{2} \sum_{k=3}^{n} k(k-3)n_k + i \left( n - \frac{i+1}{2} \right).$$

We know that $i$ and all the $n_k$ are non-negative integers, that

$$\sum_{k=3}^{n} n_k = m - (n - i) + 1$$

and that

$$\sum_{k=3}^{n} kn_k = 2m - (n - i).$$
Suppose that \( i, n_3, \ldots, n_n \) are chosen to minimise \( h(i, n_3, \ldots, n_n) \) subject to these constraints. Assume that this minimum value \( h^* \) satisfies \( h^* < 2n - 3 \) (for otherwise \( \text{add}(n, m) \geq h^* \geq 2n - 3 > m \)). Since \( k(k - 3) \) gives a convex function, it must be the case that either just one value \( n_k \) is non-zero or two adjacent values \( n_k \) and \( n_{k+1} \) are non-zero. Also observe that \( h^* \geq 2n - 3 \) if \( i \geq 2 \), and so \( i = 0 \) or 1.

If \( n_5 = n_6 = \ldots = 0 \), then

\[
4(m - (n - i) + 1) = 4(n_3 + n_4) \geq 2m - (n - i),
\]
and it follows that \( 1 \geq i \geq n - \frac{4}{3}m - \frac{4}{3} \), and so \( m \geq \frac{2}{3}n - \frac{4}{3} \), contradicting (10). Thus at least one of \( n_5, n_6, \ldots \) is non-zero. If \( n_3 = n_4 = 0 \) then \( 2m - n + i \geq 5(m - n + i + 1) \), and so \( m < \frac{2}{3}n \), again contradicting (10): thus at least one of \( n_3, n_4 \) is non-zero. But now \( n_3 = n_5 = n_7 = \cdots = 0 \).

We obtain

\[
4n_4 + 5n_5 = 2m - (n - i) \quad \text{and} \quad n_4 + n_5 = m - (n - i) + 1.
\]

Hence

\[
n_5 = 3(n - i) - 2m - 4 \quad \text{and} \quad n_4 = 3m - 4(n - i) + 5.
\]

Therefore

\[
\text{outadd}(n, m) \geq \frac{4}{2}(3m - 4(n - i) + 5) + \frac{10}{2}(3(n - i) - 2m - 4)
\]
\[
= \frac{7}{2}(n - i) - 4m - 10 \geq \frac{7}{2}n - 4m - 17,
\]

since \( i \leq 1 \). If we set \( m = \lfloor \alpha n \rfloor \), where \( \frac{1}{3} < \alpha < \frac{4}{2} \) and \( n \) is sufficiently large that (10) holds, then we have

\[
\text{outadd}(n, m) \geq \left(7 - 4\alpha\right)n - 17,
\]

and in particular \( \text{outadd}(n, \lfloor \frac{7}{6}n \rfloor) \geq \frac{7}{6}n - 17 \). Hence \( \text{outadd}(n, \lfloor \frac{7}{6}n \rfloor - 9) \geq \frac{7}{6}n - 8 \). Thus, if \( m \leq \frac{7}{6}n - 9 \) then \( \text{outadd}(n, m) > m \).

We now give a corresponding upper bound for \( \text{outadd}(n, m) \). Consider the following sequence of graphs. Let \( G_0 \) consist of a single edge \( a_0b_0 \). For \( k \geq 1 \), \( G_k \) is obtained from \( G_{k-1} \) by adding the five nodes \( a_k, b_k, c_k, d_k, e_k \), and the seven edges \( b_0a_k, a_kb_k, b_kc_k, c_kd_k, d_ke_k, a_{k-1}e_k \) and \( b_0d_k \), see Figure 9. It is easy to verify that for each \( k \geq 1 \), \( G_k \) is outerplanar, and has \( k \) 4-faces and \( k \) 5-faces (not counting the outer face). Hence \( \text{outadd}(5k + 2, 7k + 1) \leq \text{outadd}(G_k) = 7k \).

By making minor adjustments to this construction, we may see that for each \( n \geq 7 \), if \( m = \lfloor \frac{7}{6}n \rfloor \) then \( \text{outadd}(n, m) < m \). \qed
6 Concluding Remarks

If we know more about the random planar graph, then we can improve at least a little on Theorem 1, using essentially the same method of proof. Throughout the proof of that theorem we may replace \( \text{add}(n, m) \) by the average value of \( \text{add}(G) \) over all graphs \( G \in \mathcal{P}(n, m) \); and if we can show that the average is substantially bigger than \( \text{add}(n, m) \) then we obtain a better lower bound on the number of edges. Using results of [3] together with some calculations similar to those above but more laborious, we may show for example that there exists a \( \delta > 0 \) such that \( \Pr(m(R_n) \leq \frac{11}{9}n + \delta n) = o(e^{-\delta n}) \). Thus in particular \( \mathbb{E}[m(R_n)] \geq \frac{10}{9}n \) for all \( n \) sufficiently large, and we have managed to lose the untidy \( o(n) \) term in the abstract.

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References


