List Colouring Squares of Planar Graphs

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Abstract

In 1977, Wegner conjectured that the chromatic number of the square of every planar graph $G$ with maximum degree $\Delta \geq 8$ is at most $\lceil \frac{3}{2} \Delta \rceil + 1$. We show that it is at most $\frac{3}{2} \Delta (1 + o(1))$, and indeed this is true for the list chromatic number.

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1 Introduction

Most of our terminology and notation is standard. All graphs and multigraphs are finite. A multigraph can have multiple edges; a graph is supposed to be simple. The degree of a vertex is the number of edges incident with that vertex.
We require all colourings, whether we are discussing vertex, edge or list colouring, to be proper: neighbouring objects must receive different colours. We also always assume that colours are integers, which allows us to talk about the “distance” $|c_1 - c_2|$ between two colours $c_1, c_2$.

The chromatic number of a multigraph $G$, denoted $\chi(G)$, is the minimum number of colours required so that we can properly colour its vertices using those colours. If we colour the edges of $G$, we get the chromatic index, denoted $\chi'(G)$. The list chromatic number or choice number $\text{ch}(G)$ is the minimum value $k$, so that if we give each vertex $v$ of $G$ a list $L(v)$ of at least $k$ colours, then we can find a proper colouring in which each vertex gets assigned a colour from its own private list. The list chromatic index is defined analogously for edges.

### 1.1 Colouring the Square of Graphs

Given a graph $G$, the square of $G$, denoted $G^2$, is the graph with the same vertex set as $G$ and with an edge between any two different vertices that have distance at most two in $G$. If $G$ has maximum degree $\Delta$, then a vertex colouring of its square will need at least $\Delta + 1$ colours, but the greedy algorithm shows it is always possible with $\Delta^2 + 1$ colours. Diameter two cages and point-line incidence graphs of projective planes show that there exist graphs that in fact require $\Omega(\Delta^2)$ colours.

Regarding the chromatic number of the square of a planar graph, Wegner [5] posed the following conjecture, suggesting that for planar graphs far less than $\Delta^2 + 1$ colours suffice.

**Conjecture 1.1 (Wegner [5])** For a planar graph $G$ of maximum degree $\Delta \geq 8$: $\chi(G^2) \leq \left\lceil \frac{3}{2} \Delta \right\rceil + 1$.

Wegner also gave bounds for $\Delta \leq 7$ and examples showing that his conjectured bounds would be tight. For even $\Delta \geq 8$, these examples are sketched below. The graph in the picture has maximum degree $2k$ and yet all vertices except $z$
are adjacent in its square. Hence to colour these $3k + 1$ vertices, we need at least $3k + 1 = \frac{3}{2} \Delta + 1$ colours.

Many upper bounds on $\chi(G^2)$ for planar graphs in terms of $\Delta$ have been obtained in the last 15 years. The asymptotically best known upper bound so far has been found by Molloy and Salavatipour [4]: $\chi(G^2) \leq \lceil \frac{5}{3} \Delta \rceil + 78$.

Kostochka and Woodall [3] conjectured that for every square of a graph the list chromatic number equals the choice number. This leads directly to the following conjecture for planar graphs with $\Delta \geq 8$: $\text{ch}(G^2) \leq \lceil \frac{3}{2} \Delta \rceil + 1$.

In this extended abstract we announce the following theorem.

**Theorem 1.2** The square of every planar graph $G$ of maximum degree $\Delta$ has list chromatic number at most $(1 + o(1)) \frac{3}{2} \Delta$. Moreover, given lists of this size, there is a proper colouring in which the colours on every pair of adjacent vertices of $G$ differ by at least $\Delta^{1/4}$.

The $o(1)$ term here is as $\Delta \to \infty$. The first order term $\frac{3}{2} \Delta$ in is essentially best possible, as the examples above show. On the other hand, the term $\Delta^{1/4}$ is probably far from best possible; it was chosen to keep the proof simple.

## 2 Sketch of the proof of Theorem 1.2

To prove Theorem 1.2, we need to show that for every $\epsilon > 0$, there is a $\Delta_\epsilon$ such that for every $\Delta \geq \Delta_\epsilon$ we have: for every planar graph $G$ of maximum degree at most $\Delta$, given lists of size $c_\epsilon = \lfloor (\frac{3}{2} + \epsilon) \Delta \rfloor$ for each vertex $v$ of $G$, we can find the desired colouring.

We proceed by induction on the number of vertices of $G$, using a recursive algorithm. In each iteration, we split off a set $R$ of vertices which are easy to handle, recursively colour $G^2 - R$ (which we can do by the induction hypothesis), and then extend this colouring to the vertices of $R$. In extending the colouring, we must ensure that no vertex $v$ of $R$ receives a colour used on a vertex of $V - R$ which is adjacent to $v$ in $G^2$. Thus, we modify the list $L(v)$ of colours available for $v$ by deleting those which appear on such neighbours.

Note that $(G - R)^2$ may differ from $G^2 - R$, as there may be non-adjacent vertices of $G$ with a common neighbour in $R$ but no common neighbour in $G - R$. When choosing $R$ we must ensure that we can construct a planar graph $G_1$ on $V - R$ such that $G^2 - R \subseteq G_1^2$. We also must ensure that the connections between $R$ and $V - R$ are limited, so that the modified lists used when colouring $G^2[R]$ are still reasonably large. Finally, we want $G^2[R]$ to have a simple structure so that we can prove we can list colour it as desired.

We begin with a simple example. Call a vertex $v$ removable if it has at
most $\Delta^{1/4}$ neighbours and at most two neighbours whose degree exceeds $\Delta^{1/4}$. Note that if $v$ is a removable vertex with exactly two neighbours $x$ and $y$, then setting $G_1 = G - v + e$, where $e$ is an edge between $x$ and $y$, we have that $G_1$ is a planar graph with $G^2 - v \subseteq G^2_1$. On the other hand, if $v$ is a removable vertex with at least three neighbours, then it must have a neighbour $w$ of degree less than $\Delta^{1/4}$. Then the graph $G_2$ obtained by contracting the edge $wv$ is a planar graph of maximum degree $\Delta$ such that $G^2 - v \subseteq G^2_2$.

Thus, for any removable vertex $v$, we can recursively list colour $G^2 - v$. If, in addition, $v$ has at most $c_e - 1 - 2\Delta^{1/2}$ neighbours in $G^2$, then our bound on $d_G(v)$ ensures that there will be a colour in $L(v)$ which appears on no vertex adjacent to $v$ in $G^2$ and is not within $\Delta^{1/4}$ of any colour assigned to a neighbour of $v$ in $G$. To complete the colouring we give $v$ any such colour.

We can easily extend these ideas to sets with more than one vertex.

**Lemma 2.1** If $R$ is a set of removable vertices of $G$, then there is a planar graph $G_1$ with vertex set $V - R$ and maximum degree $\Delta$ such that $G^2 - R \subseteq G^2_1$.

The above remarks show that no minimal counterexample to our theorem can contain a removable vertex of low degree in $G^2$. We are about to describe another, more complicated, reduction.

For a multigraph $H$, let $H^*$ be the graph obtained from $H$ by subdividing each edge exactly once. For each edge $e$ of $H$, let $e^*$ be the vertex of $H^*$ which we placed in the middle of $e$, and let $E^*$ be the set of all such vertices. We call this set of vertices corresponding to the vertices of $E^*$ the core of $H^*$. A removable copy of $H$ is a subgraph of $G$ isomorphic to $H^*$ such that the vertices of $G$ corresponding to the vertices of the core of $H^*$ are removable.

The subgraph $J$ of $G^2$ induced by the core of some copy of $H^*$ in $G$ contains a subgraph isomorphic to $L(H)$, the line graph of $H$. So the list chromatic number of $J$ is at least the list chromatic number of $L(H)$. If the copy is removable, then removing the edges of this copy of $L(H)$ from $J$ yields a graph in which vertices in the core have degree at most $\Delta^{1/2}$. Thus, the key to list colouring $J$ will be to list colour $L(H)$. Fortunately, list colouring line graphs is much easier than list colouring general graphs. In particular, using a sophisticated argument due to Kahn [2], we prove the following lemma, describing certain sets of removable vertices which we use to perform reductions.

**Lemma 2.2** Suppose $R$ is the core of a removable copy of $H^*$ in $G$, for some multigraph $H$, such that for any set $X$ of vertices of $H$ and corresponding set $X^*$ of vertices of the copy of $H^*$, we have that the sum of the degrees in $G - R$ of the vertices in $X^*$ exceeds the number of edges of $H$ out of $X$. 
by at most \( \epsilon \frac{|X| \Delta}{10} \). Then, any \( c_{\epsilon} \)-colouring of \( G^2 - R \) can be extended to a \( c_{\epsilon} \)-colouring of \( G^2 \).

The following lemma shows that we will indeed be able to find a removable set of vertices which we can use to perform a reduction.

**Lemma 2.3** There is some \( \Delta_0 \) such that any planar graph \( G \) of maximum degree \( \Delta \geq \Delta_0 \) contains one of the following:

(A) a removable vertex \( v \) which has degree less than \( \frac{3}{2} \Delta + \Delta^{1/2} \) in \( G^2 \), or

(B) a removable copy of \( H^* \) with core \( R \), for some multigraph \( H \) which contains an edge and is such that for any set \( X \) of vertices of \( H \) we have:

the sum of the degrees in \( G - R \) of the vertices in \( X \) exceeds the number of edges of \( H \) out of \( X \) by at most \( |X| \Delta^{9/10} \).

Combining Lemmas 2.1, 2.2, and 2.3 yields Theorem 1.2. Thus, it remains to prove the last two of these lemmas. The proof of Lemma 2.3 uses not much more than the fact that planar graphs have low edge-density. The proof of Lemma 2.2 is much more complicated. We follow the approach developed by Kahn [2] for his proof that the list chromatic index of a multigraph is asymptotically equal to its fractional chromatic number. We need to modify the proof so it can handle our situation in which we have a graph which is slightly more than a line graph and in which we have lists with fewer colours than he permitted.

Full details will be given in [1].

**References**


