# Increasing Interdependence in Multivariate Distributions *

Margaret Meyer     Bruno Strulovici

– Extended Abstract –

April 18, 2008

This paper compares $n$-dimensional random vectors in terms of their interdependence. We adopt the stochastic dominance approach, relating orderings of interdependence expressed directly in terms of joint probability distributions to orderings expressed indirectly through properties of objective functions whose expectations are used to evaluate distributions. Since the expected values of additively separable objective functions depend only on marginal distributions, attitudes towards interdependence must be represented through non-separability properties. We argue that the property of supermodularity (Topkis, 1978) of an objective function is a natural property with which to capture a preference for greater interdependence. Accordingly, we seek to characterize a partial ordering on joint distributions, with identical marginals, which is equivalent to one distribution's yielding a higher expectation than another for all supermodular objective functions. Following the statistics literature, we refer to this partial ordering as the "supermodular stochastic ordering" (Shaked and Shanthikumar, 1997).

For the special case of two-dimensional random vectors, the economics and statistics literatures have provided a complete characterization of the supermodular ordering. Specifically, Epstein and Tanny (1980) and Tchen (1980), among others, have shown that one bivariate distribution dominates another according to the supermodular ordering if and

*Nuffield College and Department of Economics, Oxford University

only if the first distribution dominates the second in the sense of both upper-orthant and lower-orthant dominance. Hu, Xie, and Ruan (2005) have shown that this equivalence continues to hold in three dimensions in the special case of Bernoulli random vectors, but the equivalence breaks down for more than three dimensions (Joe, 1990) and even in three dimensions for larger supports (Muller and Scarsini, 2000). In general, the supermodular ordering is strictly stronger than the combination of upper-orthant and lower-orthant dominance.

Focusing on the case of discrete supports, we are able to make substantial progress in characterizing the supermodular ordering for more than two dimensions.

Comparing two $n$-dimensional distributions with identical marginals, we first prove that one distribution is preferred to the other by every supermodular objective function if and only if the first distribution can be derived from the other by a sequence of non-negative "elementary transformations". Intuitively, our elementary transformations play a similar role to the Rothschild-Stiglitz (1970) elementary transformations that define mean-preserving spreads for univariate distributions. For multivariate distributions, our elementary transformations provide a local characterization of the notion of "greater interdependence", and they are a natural multivariate generalization of the bivariate "correlation-increasing transformations" defined by Epstein and Tanny (1980). By duality, these transformations can also be interpreted as local test functions for the supermodularity of objective functions. We then develop an algorithm, based on the "double description method" conceptualized by Motzkin et al. (1953) and developed by Avis and Fukuda (1992) to generate, for any $n$-dimensional discrete support, a set of inequalities that are equivalent to preference by all supermodular functions. These inequalities can be easily checked and can thus be straightforwardly used to determine whether different policies, mechanisms, portfolios, etc., can be ranked according to the supermodular ordering. To generate this inequality-based comparison, we exploit the geometric properties of the set of supermodular functions. Precisely, this set is a cone, which can be described either through its positive dual or through its extreme rays, each of which determines exactly one of the inequalities defining the supermodular ordering. We complement the algorithm with constructive proofs of the results for many of the cases considered.

Our methods and results are applicable to a wide range of questions in economics and related fields. Consider first some applications in welfare economics. In many group settings where individual outcomes (e.g. rewards) are uncertain, members of the group may be concerned, ex ante, about how unequal their ex post rewards will be (Meyer and Mookherjee, 1987, and Kroll and Davidovitz, 2003). (This concern is distinct from concerns about the mean level of rewards and about their riskiness.) An aversion to ex post inequality can be formalized by adopting an ex post welfare function that is supermodular in the realized utilities of the different individuals. We then want to know: Given two mechanisms for allocating rewards (formally, two joint distributions of random utilities), when can we be sure that one mechanism generates higher expected welfare than the other, for all supermodular ex post welfare functions? Our stochastic dominance theorems allow us to answer this question.

Consider a specific illustration. Intuitively, when groups dislike ex post inequality, tournament reward schemes, which distribute a fixed set of rewards among individuals, one to each person, should be particularly unappealing, since they generate a form of negative correlation among rewards: if one person receives a higher reward, this must be accompanied by another person's receiving a lower reward. This intuitive reasoning suggests the conjecture that tournaments should be dominated, in the sense of the supermodular ordering, by reward schemes that provide each individual with the same marginal distribution over rewards but determine rewards independently. Meyer and Mookherjee (1987) proved this conjecture, but only for the special case of a symmetric tournament (one in which each individual has an equal chance of winning each of the rewards), and their method of proof was laborious. Here, we allow tournaments to be arbitrarily asymmetric across individuals, and we apply our three-dimensional characterization result to show that the conjecture is true for the case of three individuals.

A second application in welfare economics concerns comparisons of inequality or poverty when separate data are available on different dimensions of economic status, for example, income, health, and education (Atkinson and Bourguignon, 1982, and Bourguignon and Chakravarty, 2002). Depending on whether the different attributes are regarded as complements or substitutes at the individual level, the function aggregating the attributes into an individual welfare measure will be supermodular or submodular, and our stochastic dominance theorems provide the conditions under which one multidimensional distribution can be ranked above another for all welfare measures in the given class.

Another set of economic applications concerns comparisons of the efficiency of two-sided or many-sided matching mechanisms when the outcomes of the matching process are subject to informational or search frictions. Consider, for example, settings where different categories of workers (e.g. newly-qualified and experienced, or technical and managerial) are matched with firms. Suppose that workers within each category, as well as firms, are heterogeneous and that the production function giving the output of a matched set of workers at a given firm, as a function of the workers' types and the firm's type, is supermodular. In the absence of any frictions, the efficient matching would be perfectly assortative, matching the highest-quality worker in each category with the highest-quality firm, the next-highest-quality workers with the next-highest-quality firm, etc. Such a matching would correspond to a "perfectly correlated" joint distribution of the random variables representing quality in each category (dimension). When, however, matches are formed based only on noisy information, or when search is costly, or when signaling is constrained by market imperfections such as borrowing constraints, perfectly assortative matching will generally not arise. In these settings, our stochastic dominance theorems can be used to assess when one matching mechanism will generate higher expected output than another, for all supermodular production functions. Fernandez and Gali (1999) and Meyer and Rothschild (2003) apply existing two-dimensional results to compare matching institutions, but multi-dimensional applications remain largely unexplored. One exception is Prat (2002), but he compares only a perfectly correlated joint distribution with an independent one, and Lorentz (1953) has shown that the former is preferred to the latter for all supermodular objective functions.

In finance, characterizations of the supermodular ordering can be applied to the comparison of the dependence among assets in a portfolio, and in insurance, to the comparison of the dependence among claim streams (Muller and Stoyan, 2002, and Denuit, Dhaene, Goovaerts, and Kaas, 2005).

**References**

Atkinson, A. B., and Bourguignon, F., "The Comparison of Multi-Dimensioned Distributions of Economic Status", Review of Economic Studies, 1982, 49, 183-201.

Avis, D., and Fukuda, K., "A Pivoting Algorithm for Convex Hulls and Vertex Enumeration of Arrangements and Polyhedra", Discrete and Computational Geometry, 1992, 8, 295-313.

Bourguignon, F., and Chakravarty, S.R., "Multi-dimensional poverty orderings", DELTA W.P. 2002-22.

Denuit, M., Dhaene, J., Goovaerts, M.J., and Kaas, R., Actuarial Theory for Dependent Risks: Measures, Orders, and Models, 2005, Wiley.

Epstein. L., and Tanny, S., “Increasing Generalized Correlation: A Definition and Some Economic Consequences”, Canadian Journal of Economics, 1980, 13, 16-34.

Fernandez, R., and Gali, J., “To Each According to? Markets, Tournaments, and the Matching Problem with Borrowing Constraints”, Review of Economic Studies, 1999, 66, 799-824.

Hu, T., Xie, C., and Ruan, L, “Dependence Structures of Multivariate Bernoulli Random Vectors”, Journal of Multivariate Analysis, 2005, 94, 172-195.

Joe, H., “Multivariate Concordance”, Journal of Multivariate Analysis, 1990, 35, 12-30.

Kroll, Y., and Davidovitz, L., “Inequality Aversion versus Risk Aversion”, Economica, 2003, 70, 19-29.

Lorentz, G.G., “An Inequality for Rearrangements”, American Mathematical Monthly, 1953, vol. 60, 176-179.

Meyer, M.A., and Mookherjee, D., “Incentives, Compensation and Social Welfare”, Review of Economic Studies, 1987, 45, 209-226.

Meyer, M.A., and Rothschild, M., “(In)efficiency in Sorting and Matching”, mimeo, Oxford, 2004.

Motzkin, T. S., Raiffa, H., Thompson, G. L. , and Thrall, R. M.. “The Double Description Method”. In H. W. Kuhn and A. W. Tucker Ed., Contributions to the Theory of Games II, 1953, 51-73.

Muller, A., and Scarsini, M., “Some Remarks on the Supermodular Order”, Journal of Multivariate Analysis, 2000, 73, 107-119.

Muller, A., and Stoyan, D., Comparison Methods for Stochastic Models and Risks, 2002, Wiley.

Prat, A., “Should a Team Be Homogenous?”, European Economic Review, 2002, 46, 1187-1207.

Rothschild, M. and Stiglitz, J. E. “Increasing risk: I. A definition”, Journal of Economic Theory, 1970, 2, 225-243.

Shaked, M., and Shanthikumar, J.G., “Supermodular Stochastic Orders and Positive

Dependence of Random Vectors", Journal of Multivariate Analysis, 1997, 61, 86-101.

Tchen, A.H., "Inequalities for Distributions with Given Marginals", Annals of Probability, 1980, 8, 814-827.

Topkis, D.M., "Minimizing a Submodular function on a Lattice", Operations Research, 1978, 26, 305-331.