e3 : Mathematical Programming H 2010
Problem Set 3 Transportation Problem, Assignment Problem
Shortest paths, Dynamic Programming, Zero-Sum Games

Attempted solutions to the questions marked with a star should be handed in at the lecture on Thursday of week 8, or reach my pigeonhole in Corpus by 2pm on that day (for Colin McDiarmid). These starred questions (3,4,7,9,10,11,14,16) will be discussed in class. The other questions are for practice and private study, though you are welcome to ask me about any of them.

Q1 A caterer has to provide napkins for dinners on a number of successive days. The napkins can either be bought at $b = 15p$ each, laundered by the following day at $c = 6p$ each, or laundered in two days at $d = 3p$ each. The caterer has to decide how many napkins should be bought and how many should be sent to the different laundry services so as to minimise the cost of providing the napkins.

Suppose that the caterer requires 30 napkins on Monday, 40 on Tuesday, 60 on Wednesday and 40 on Thursday. Formulate the problem as a transportation problem, and solve it. [The optimal cost is $1500p$.]

Q2 A jam manufacturer has to supply d_1, d_2, d_3, d_4 tons of jam respectively during the next four months. His plant can produce up to s tons per month. The cost of producing one ton of jam will be £c_1, c_2, c_3, c_4 during these months, and there is a storage cost of £k per ton per month if the jam is not delivered in the same month that it is produced.

Formulate this problem as a transportation problem. Find an optimal production schedule when $(d_1, d_2, d_3, d_4) = (4, 7, 5, 7)$, $s = 6$, $k = 1$ and $(c_1, c_2, c_3, c_4) = (3, 5, 4, 6)$. [The optimal cost is £106.]

Q3* Four empty lorries each stand at Birmingham, Bristol and Cambridge. Three each are needed at Manchester, Northampton, Nottingham and Oxford. The costs of sending them are directly proportional to the distances in miles, which are given in the table below. What lorries should be sent where? [The optimal cost is 860.]

	Manchester	Northampton	Nottingham	Oxford
Birmingham	80	50	50	60
Bristol	160	100	140	70
Cambridge	150	50	80	80

Why should you know ahead of time that there will be an optimal solution using just six routes?

Calculate possible shadow prices. What do they stand for?

Q4* A company has three factories which ship to four markets. The table below gives the transport costs in £ per unit sent. The factory capacities are 6, 3, 10 units respectively. Variable production costs per unit are £1 higher in the first factory compared with the variable production costs in the second and third factories. The demands at the markets are estimated to be 7, 5, 3, 2 units respectively.

		Market			
	1	1	2	10	7
Factory	2	1	0	6	1
	3	5	8	15	9

Find a minimum cost production-shipping schedule, and discuss whether your solution is unique. [If you take variable production costs as £1,0,0 then the optimal cost is £83.] Suppose that the first market wants one more unit. Find a new optimal schedule and the change in cost. How can shadow costs help to find such a change in total cost?

Q5 The unit costs of shipping from factories F_1, F_2, F_3 to markets M_1, M_2, M_3 are shown in the table below.

	M_1	M_2	M_3
F_1	5	1	7
F_2	6	4	6
F_3	3	2	5

Supplies at factories F_1, F_2, F_3 are 10, 80, 15 respectively, and demands at markets M_1, M_2, M_3 are 75, 20, 50 respectively; thus total demand exceeds total supply.

(a) Suppose that penalty costs per unit of unsatisfied demand are 7, 5, 6 at markets M_1, M_2, M_3 respectively. Find an optimal solution. Is it unique? [The optimal cost is 755.]

(b) Suppose now that there are no penalty costs, but that the demand at market M_3 must be satisfied. Find an optimal solution. [The optimal cost is 515.]

Q6 The transportation problem represented by the following table is complicated by the fact that source X can supply the first 20 units at a cheaper rate than the remaining 60 units which are within its capacity. Moreover, source Z must supply at least 40 units or none at all. Find an optimal solution.

		Destination (demands)		
		A (30)	B (40)	C (30)
	X (first 20)	9	11	13
Sources	X (next 60)	10	12	14
(capacities)	Y (30)	13	14	16
	Z (50)	12	15	15

[The optimal cost is 1220.]

Q7* You are advising the manager of a racing car team with five different drivers D1,..,D5 and five different cars C1,..,C5. The team is entered for a race where the winning team is the one with the lowest total time for all five cars. The time in which a driver can complete the course when paired with each car is shown in the table below.

		cars				
		C1	C2	C3	C4	C5
	D1	3	5	6	9	10
	D2	4	8	8	11	13
drivers	D3	6	9	10	12	14
	D4	8	10	10	15	16
	D5	13	13	17	18	20

Find the lowest possible aggregate time the team can produce, and the pairing by which it is achieved. How could you demonstrate to the (moderately numerate) manager that this is the optimal solution? [The optimal total time is 49.]

Q8 A short haul airline has two types of aircraft, X and Y, with empty operating costs per mile of £10 and £15 respectively, and average scheduled operating costs per mile of £20 and £30 respectively. Among the airline's daily routes are four of 300,150,500 and 250 miles beginning at terminus A,B,C and D respectively. Current aircraft availability for these routes is an aircraft of type X at each of termini E and G and an aircraft of type Y at each of termini B and F. Distances in miles between the termini are given in the following table.

	A	B	C	D
B	33	0	200	400
E	300	100	300	200
F	400	100	100	500
G	200	200	400	200

Determine a minimum cost aircraft assignment for the four flights under consideration for the next day's schedule. [The optimal cost is £36000.]

Q9* (from finals 1982)
(a) Consider the linear programme

$$\text{(P)} \qquad \max \mathbf{c}'\mathbf{x} \text{ subject to } A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}.$$

State the dual programme (D). Let $\mathbf{x}$ and $\mathbf{y}$ be feasible vectors for the programmes (P) and (D) respectively. State the complementary slackness theorem which provides conditions for $\mathbf{x}$ and $\mathbf{y}$ to be optimal in their respective programmes.

Suppose that the vector $\mathbf{x} = (1, 2, 0, 3, 0)'$ is an optimal solution to a programme (P) as above. Change the right hand side vector $\mathbf{b}$ to a new vector $\hat{\mathbf{b}}$ and suppose that the vector $\hat{\mathbf{x}} = (3, 2, 0, 1, 0)'$ is feasible for the new programme. Why must $\hat{\mathbf{x}}$ be optimal for the new programme?

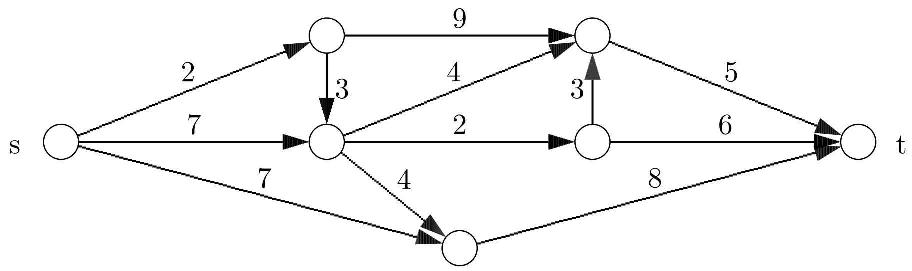
[Hint. If $\mathbf{y}$ is optimal for the dual of the original problem then $\mathbf{y}$ satisfies constraints 1,2 and 4 of the dual at equality; and further $\mathbf{y}$ will still be feasible for the dual of the new problem.]

(b) We wish to assign four jobs to four machines so as to minimise the total costs. The cost of assigning job i to machine j is given in the table below.

	machines			
	5	7	3	3
jobs	2	5	7	5
	3	3	6	8
	4	2	6	7

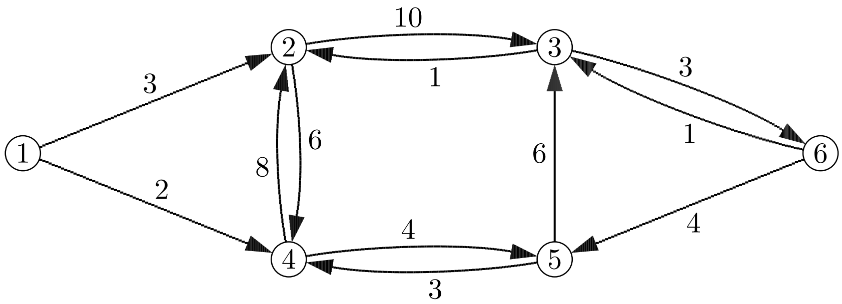
Find an optimal solution. How is the idea of complementary slackness involved in your solution method? [The optimal cost is 13.]

Q10* In the acyclic network shown below, number the nodes so that arcs go from small numbers to large numbers, and find both a shortest and a longest s-t path. If the arcs correspond to activities (tasks) in a project, which are the critical activities?



[Minimum length 13, maximum length 19.]

Q11* Use Dijkstra's algorithm to find a shortest 1-6 path in the network shown.



[Minimum length 15.]

Q12 (knapsack problem)

A wreck on the sea-bed contains large numbers of gold bars varying in weight and value. Bars of type i weigh a_i pounds each and are worth £c_i. A (greedy) diver is loading some of the gold bars into a box attached to a hoist, and wants to maximise the value of the load, which must not weigh more than q pounds. There are n different types of bar.

Find a recurrence relation for $F_n(q)$, the required maximum value. Hence solve the following instance, where $n = 3$ and $q = 12$.

i	1	2	3
a_i	2	3	5
c_i	7	10	18

[$F_3(12) = 43$.]

Q13 A sales organisation is considering how to allocate their four representatives among four geographical regions. All salesmen are taken to be equally effective. If i salesmen are allocated to region j, the estimate $c_j(i)$ of the corresponding profit from that region is given in the table below.

		region j			
		1	2	3	4
	0	0	0	0	0
	1	33	41	25	36
men i	2	78	65	50	48
	3	102	80	73	56
	4	123	88	90	60

(a) Use dynamic programming to find how many men should be allocated to each region in order to maximise total profits. [The optimal profit is 155.]
(b) Solve this problem also by the different approach of considering incremental profits. Why does this second method work here?

Q14* Let $\mathbf{c}_1, \ldots, \mathbf{c}_k$ be k given n-vectors and let the k numbers $g_1, \ldots, g_k$ be given 'goals'. Consider the non-linear programme

$$\max f(\mathbf{x}) \text{ subject to } A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0},$$

where the objective function $f(\mathbf{x})$ is given by

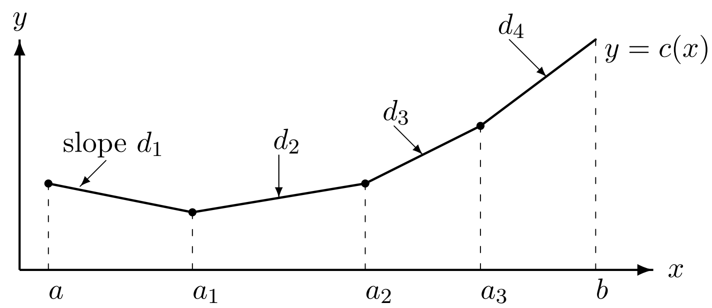
$$f(\mathbf{x}) = \min_i (\mathbf{c}_i'\mathbf{x} - g_i).$$

(i) How could you solve this as an LP? [Hint: introduce a new variable x_0 and k new constraints $x_0 \leq \mathbf{c}_i'\mathbf{x} - g_i$.]

How would you minimise $f(\mathbf{x})$ where
(ii) $f(\mathbf{x}) = \max_i \left(|\mathbf{c}_i'\mathbf{x} - g_i|\right)$,
(iii) $F(\mathbf{x}) = \sum_{i=1}^{k} |\mathbf{c}_i'\mathbf{x} - g_i|$?

Q15 A firm sets $n+1$ activity levels $x, x_1, \ldots, x_n$ subject to certain linear constraints, and wants to minimise costs $c(x) + \sum_{j=1}^{n} c_j x_j$. Here the costs have a non-linear component $c(x)$. For feasible activity levels we always have $a \leq x \leq b$ where $0 < a < b$ are given. Over this range the function $c(x)$ is convex and piecewise linear (or at least it may be approximated satisfactorily by such a function) - see the figure below.

3 break points
at $x = a_1, a_2, a_3$



Show that the firm's problem can be formulated as a linear programme, by replacing the variable x by $k+1$ new non-negative variables $z_1, \ldots, z_{k+1}$, where k is the number of break points (=3 in the figure). Why is this important? [Hint. Set $a_0 = a, a_{k+1} = b$. Consider the constraints $0 \leq z_i \leq a_i - a_{i-1}$ for $i = 1, \ldots, k+1$ and $x = \sum_{i=1}^{k+1} z_i$, and the function $c(a) + \sum_{i=1}^{k+1} d_i z_i$.]

Q16* Find the value and optimal strategies in the following two-person zero-sum game

$$\begin{bmatrix} 3 & 1 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

where the payoffs shown are those of the row player.

Q17 Solve the following zero-sum game:

$$\begin{bmatrix} 3 & 3 & 5 & 5 \\ 5 & 4 & 3 & 3 \\ 5 & 4 & 5 & -1 \end{bmatrix}$$

[Value 11/3.]