1 Linear Models

1.1 A first example

We start with a toy example. A chemical firm sells two products Acidic and Caustic, whose production requires the use of two scarce resources R and S. Each ton of Acidic requires 2 units of R and 1 unit of S, while each ton of Caustic requires 1 unit of R and 3 units of S. Each week the supplies of R and S are limited to 11 and 18 units respectively. The sale of each ton of Acidic or Caustic yields a profit of £1000. Market research indicates that no more than 4 tons of Acidic can be sold per week. How can the firm maximise its weekly profit?

What pairs of numbers (x_1, x_2) represent a feasible weekly production of x_1 tons of Acidic and x_2 tons of Caustic? The amount of each product produced certainly is non-negative, so we have the inequalities $x_1 \geq 0$ and $x_2 \geq 0$. The amount of ingredient R required to produce x_1 tons of Acidic and x_2 tons of Caustic is $2x_1 + x_2$. Thus

$$2x_1 + x_2 \leq 11.$$

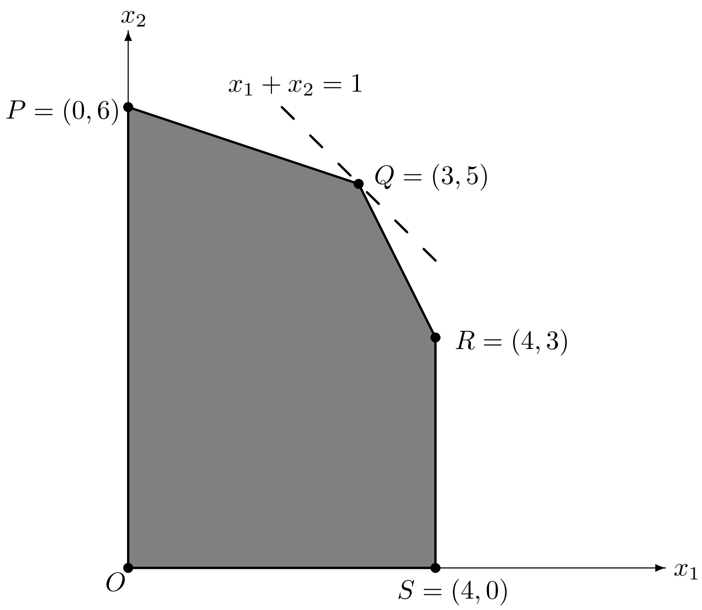
Similarly by considering ingredient S we have

$$x_1 + 3x_2 \leq 18,$$

and finally

$$x_1 \leq 4.$$

Thus any pair (x_1, x_2) of production figures satisfies these constraints, and conversely any pair of numbers satisfying these constraints represents a feasible production program. We call such a vector (x_1, x_2) a *feasible solution*. The shaded polygon OPQRS below represents the *feasible region*, that is, the set of all feasible solutions.



Associated with each feasible solution (x_1, x_2) is the profit £1000 $(x_1 + x_2)$ which we wish to maximise. Thus the problem may be stated mathematically as follows:

$$\max \; x_1 + x_2$$

subject to

$$\begin{aligned} 2x_1 + x_2 &\leq 11 \\ x_1 + 3x_2 &\leq 18 \\ x_1 &\leq 4 \\ x_1, x_2 &\geq 0. \end{aligned}$$

This is a *linear program* — a problem of optimising (i.e. maximising or minimising) a linear function, the *objective function*, subject to linear constraints. A feasible solution which optimises the objective function is called an *optimal solution*, and its value is called the *value* of the program. Note that in formulating this model we have made assumptions of divisibility, proportionality and additivity.

The vertices or extreme points O, P, Q, R, S of the feasible region OPQRS have a special significance. The following observation is the key to a way to solve the problem: **amongst the vertices is an optimal solution.** Thus all we need do is evaluate the objective function x_1+x_2 at these points, to obtain 0, 6, 8, 7, 4 and thus see that $Q = (3, 5)$ or $x_1 = 3$, $x_2 = 5$ is the optimal solution, and the value of the program is 8. This may be verified from the figure by observing that the entire feasible region lies in the half-plane $x_1 + x_2 \leq 8$. The simplex method to be described later is an efficient method for finding an optimal vertex whilst typically examining only a small proportion of the vertices.

Finally note that points other than vertices may be optimal. For example, if we wish to maximise $2x_1 + x_2$ then any point on the line segment QR is optimal.

1.2 Activity analysis

This is an extension of the introductory example. A firm uses m resources (whose supply is limited) to produce n goods (or operate n activities). It requires a_{ij} units of resource i to produce one unit of good j. There are b_i units of resource i available, the profit per unit of good j produced is c_j, and we wish to maximise profit.

We must then choose a program $x_1, \ldots, x_n$ of non-negative 'activity levels' to maximise $c_1x_1 + \ldots + c_nx_n$ subject to the limited availability of each resource i, that is subject to

$$a_{i1}x_1 + \ldots + a_{in}x_n \leq b_i.$$

Note that we are assuming again divisibility, proportionality and additivity. This is an example of a *standard maximum problem*

$$\max \; \mathbf{c}'\mathbf{x} \text{ subject to } A\mathbf{x} \leq \mathbf{b}, \; \mathbf{x} \geq \mathbf{0},$$

where $\mathbf{c}$ is a given n-vector, $\mathbf{b}$ is a given m-vector, $A = (a_{ij})$ is a given $(m \times n)$ matrix and $\mathbf{x} = (x_1, \ldots, x_n)'$ is a variable n-vector. It is convenient to talk in terms of maximising profit but the c_j could be some measure of public welfare.

1.3 Diet problem

Example A farmer wishes to choose the least cost diet that will meet the nutritional requirement of his pigs. Pigs require 4, 8, 9 units respectively of nutrients A, B, C per week. There are four varieties of food available. The nutritional contents of the foods per kg are shown in the following matrix.

		food			
		1	2	3	4
nutrient	A	1	2	1	4
	B	1	3	0	2
	C	4	2	6	1

If the foods cost £5, 7, 7, 9 per kg respectively, find an optimal weekly diet for the pigs (from the point of view of the farmer!).

Suppose that we use x_j units of food j. Then we must solve the LP

$$\min\ 5x_1 + 7x_2 + 7x_3 + 9x_4$$

subject to

$$\begin{array}{ccccccccc} 1x_1 & + & 2x_2 & + & 1x_3 & + & 4x_4 & \geq & 4 \\ 1x_1 & + & 3x_2 & & & + & 2x_4 & \geq & 8 \\ 4x_1 & + & 2x_2 & + & 6x_3 & + & 1x_4 & \geq & 9 \end{array}$$

$$x_1, x_2, x_3, x_4 \geq 0.$$

General Model A hospital dietician has to plan a week's diet for a patient, which must include at least a prescribed amount of each of the nutrients $N_1, \ldots, N_m$. She has n different foods F_1, ..., F_n at her disposal, of which she knows the nutritional value and cost. Financial restrictions require her to minimise the cost of the food used in the diet.

Let a_{ij} be the number of units of nutrient N_i in one unit of food F_j; let b_i be the minimum number of units of nutrient N_i to be included in the diet; and let c_j be the cost of one unit of food F_j. Then the dietician's problem is to choose $x_j \geq 0$ units of food F_j so that the cost of the diet $c_1x_1 + c_2x_2 + \ldots + c_nx_n$ is a minimum, subject to the diet containing at least b_i units of nutrient N_i for each i, that is the x_j must satisfy

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \geq b_i$$

for each $i = 1, 2, \ldots, m$.
This is an example of a *standard minimum problem*

$$\min\ \mathbf{c}'\mathbf{x} \text{ subject to } A\mathbf{x} \geq \mathbf{b},\ \mathbf{x} \geq \mathbf{0}.$$

A third important class of LP models are transportation models, which we shall discuss later.

1.4 Reformulations

We have now met the standard maximum and minimum forms of an LP. Another convenient form is the 'equality form' maximum problem

$$\max \mathbf{c}'\mathbf{x} \text{ subject to } A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0},$$

(Be warned that these names are not standard.) Any LP can be put into this form (or say into the standard minimum form) by straightforward manipulations. For example, consider the LP

$$\min -x_1 + 2x_2$$

subject to

$$\begin{aligned} x_1 + 3x_2 &= 4 \\ 2x_1 - x_2 &\leq 5 \\ x_1 + 2x_2 &\geq 3 \\ x_2 &\geq 0. \end{aligned}$$

We replace the unrestricted variable x_1 by $x_1' - x_1''$ and introduce a slack variable x_3 and a surplus variable x_4 to obtain

$$\max x_1' - x_1'' - 2x_2$$

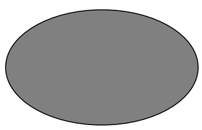
subject to

$$\begin{aligned} x_1' - x_1'' + 3x_2 &= 4 \\ 2x_1' - 2x_1'' - x_2 + x_3 &= 5 \\ x_1' - x_1'' + 2x_2 - x_4 &= 3 \\ x_1', x_1'', x_2, x_3, x_4 &\geq 0. \end{aligned}$$

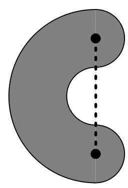
1.5 Convex sets and vertices

We now discuss further the notion of a vertex of the feasible region of an LP. A set is said to be convex if whenever two points lie in the set then so does the entire line segment joining them. Formally, a subset S of R^n is *convex* if for all points $\mathbf{x}$, $\mathbf{y}$ in S and $0 \leq t \leq 1$ the point $t\mathbf{x} + (1-t)\mathbf{y}$ is in S.

convex



not convex



The feasible region of an LP is convex. To see this, consider for example a feasible region of the form

$$F = \{\mathbf{x} \epsilon R^n : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\},$$

where A is an $m \times n$ matrix and $\mathbf{b} \in R^m$. Let $\mathbf{x}$ and $\mathbf{y}$ be points (vectors) in F and let $0 \leq t \leq 1$. To show that F is convex we must prove that the point $\mathbf{z} = t\mathbf{x} + (1-t)\mathbf{y}$ is in F. But certainly $\mathbf{z} \geq \mathbf{0}$, and by the linearity of matrix multiplication

$$A\mathbf{z} = tA\mathbf{x} + (1-t)A\mathbf{y} \leq t\mathbf{b} + (1-t)\mathbf{b} = \mathbf{b}.$$

Hence indeed $\mathbf{z}$ is in F, and thus F is convex. (See also exercise 1.9.)

A *vertex* or *extreme point* of a convex set S is a point $\mathbf{v}$ in S which does not lie strictly between any two points of S; that is, if $\mathbf{v} = t\mathbf{x} + (1-t)\mathbf{y}$ where $0 < t < 1$ and $\mathbf{x}$ and $\mathbf{y}$ are in S then in fact $\mathbf{x} = \mathbf{y} = \mathbf{v}$.

Consider again the chemist problem. It may be seen that the extreme points of the feasible region F are indeed the five 'corner' points O, P, Q, R, S. Now add slack variables so that the problem becomes

$$\max\ x_1 + x_2$$

subject to

$$\begin{array}{rcrcrcrcrcr} 2x_1 & + & x_2 & + & x_3 & & & & & = & 11 \\ x_1 & + & 3x_2 & & & + & x_4 & & & = & 18 \\ x_1 & & & & & & & + & x_5 & = & 4 \end{array}$$
$$x_1, \ldots, x_5 \geq 0.$$

The five vertices O, P, Q, R, S correspond to the feasible points in R^5 with as many (2) zero co-ordinates as possible, that is lying on as many (2) as possible of the boundary lines $x_j = 0$. We shall see that these points are the 'basic feasible solutions', as used in the simplex method.

1.6 Basic solutions

Consider the chemist example, when we have added the slack variables. In matrix form the constraints are $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, where

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 11 \\ 18 \\ 4 \end{pmatrix}.$$

We may choose the three variables x_2, x_1, x_5 as 'basic variables', leaving x_3 and x_4 as 'non-basic variables'. Set the non-basic variables to 0, that is set $x_3 = x_4 = 0$, and solve $A\mathbf{x} = \mathbf{b}$. Note that this reduces to solving $B\mathbf{x}_B = \mathbf{b}$, where $\mathbf{x}_B = (x_2, x_1, x_5)'$ and

$$B = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

This system of equations has a unique solution since the matrix B is non-singular (that is, has an inverse B^{-1}). We obtain the 'basic feasible solution' $(3, 5, 0, 0, 1)$, which corresponds to the vertex Q in figure 1. There are four other basic feasible solutions, corresponding to the four other extreme points in the figure.

In general, consider a system of equations $A\mathbf{x} = \mathbf{b}$, where A is an $(m \times n)$ matrix, $\mathbf{b}$ is an m-vector and $\mathbf{x} = (x_1, \ldots, x_n)'$ is a variable n-vector. [We assume for simplicity that the m linear equations comprising the system are independent, so that A has rank m.] A *basic solution* to the system $A\mathbf{x} = \mathbf{b}$ is obtained as follows. Choose m variables x_j (in some order) such that the $(m \times m)$ matrix B formed from the corresponding columns of A is non-singular. Call these variables *basic* and the other variables *non-basic*. Set each non-basic variable to zero. Solving the system $A\mathbf{x} = \mathbf{b}$ now reduces to solving the system $B\mathbf{x}_B = \mathbf{b}$, where $\mathbf{x}_B$ is the vector of basic variables. If it turns out that $\mathbf{x}_B \geq \mathbf{0}$, we say that we have a *basic feasible solution.*

Now consider an LP with constraints $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$. We have already seen that the feasible region F is convex, and that the extreme points of F are of particular importance.

It may be shown that a point $\mathbf{x}$ is an extreme point of F if and only if the corresponding point $(\mathbf{x}, \mathbf{s})'$ is a basic feasible solution to the system $A\mathbf{x} + \mathbf{s} = \mathbf{b}$, $(\mathbf{x}, \mathbf{s})' \geq \mathbf{0}$, where we have added the vector $\mathbf{s}$ of slack variables. The simplex method works only with basic feasible solutions.

Clearly a basic solution can have at most m non-zero co-ordinates. Often we meet problems such that a non-negative solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$ is a basic feasible solution if and only if it has exactly m non-zero co-ordinates. Such problems are called *non-degenerate*. Now let us ignore the algebra and get started on the simplex method.

2 Simplex Method

2.1 Solving the Chemist example by the simplex method

The example (in equality form) may be written as the problem of choosing activity levels $x_j \geq 0$ in order to maximise the profit z where

$$\begin{array}{ccccccccccccc} z & - & x_1 & - & x_2 & & & & & & & = & 0 \\ & & 2x_1 & + & x_2 & + & x_3 & & & & & = & 11 \\ & & x_1 & + & 3x_2 & & & + & x_4 & & & = & 18 \\ & & x_1 & & & & & & & + & x_5 & = & 4 \end{array}$$

Step 1 Find an initial basic feasible solution. We take the obvious one $\mathbf{x} = (0,\ 0,\ 11,\ 18,\ 4)'$, with basic variables the slack variables x_3, x_4, x_5, and non-basic variables x_1, x_2 set to 0. (This corresponds to throwing away all resources, for no profit.)

Step 2 Since the profit $z = x_1 + x_2$ the current solution may be improved by increasing x_1 or x_2. Let us increase x_2, whilst leaving x_1 fixed at 0.

Step 3 When x_2 increases the basic variables x_3, x_4, x_5 must adjust in order to maintain a feasible solution, and we can continue to increase x_2 until one of these basic variables is pushed down to 0. From the first constraint we must keep $x_2 \leq 11$, and from the second we need $x_2 \leq 18/3 = 6$. The third constraint gives no upper bound on x_2 since the coefficient of x_2 is ≤ 0.

Thus x_2 can be increased up to at most 6. We now increase x_2 to 6 and obtain a new basic feasible solution $(0, 6, 5, 0, 4)'$ with z-value 6, in which the entering variable x_2 has replaced the leaving variable x_4 in the basis. In the figure we started at vertex O of the feasible region and moved along an edge (the x_2-axis) until we were stopped by the inequality corresponding to the slack variable x_4, at the new vertex P.

The above calculation was easy since
(a) The z-equation expresses the objective function in terms of the non-basic variables only. Thus when we increase x_2 (and adjust the basic variables) we know that the objective function z increases, at a rate here of 1 per unit increase in x_2.
(b) Each of the constraint equations involves exactly one basic variable, with coefficient 1.

Step 4 We now re-arrange the z-equation and constraint equations so that the conditions (a) and (b) above hold for the new basic feasible solution. The second constraint equation gave the new value 6 for the entering variable x_2 and determined the leaving variable x_4. Divide this equation by 3 to obtain the *pivot equation*

$$\frac{1}{3}x_1 + x_2 \quad + \frac{1}{3}x_4 \quad = 6,$$