E3 : Mathematical Programming H 2010
Problem Set 2 Duality and Sensitivity

Attempted solutions to the questions marked with a star should be handed in at the lecture on Thursday of week 7, or reach my pigeonhole in Corpus by 2pm on that day (for Colin McDiarmid). These starred questions (1,2,4,6,7) will be discussed in class. The other questions are for practice and private study, though you are welcome to ask me about any of them.

Q1* Investigate graphically the following linear programmes and their duals.

(a)

$$\max\ x_1 + \frac{3}{2}x_2$$

subject to

$$\begin{aligned} 2x_1 + 3x_2 &\leq 6 \\ x_1 + 4x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(b)

$$\min\ x_1 + x_2$$

subject to

$$\begin{aligned} x_1 - 2x_2 &\geq 2 \\ -x_1 + x_2 &\geq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Q2* Find an optimal solution to the LP below, by investigating the dual programme graphically and using complementary slackness.

$$\min\ 2x_1 + x_2 + 5x_3$$

subject to

$$\begin{aligned} -4x_1 + x_2 + 2x_3 &\geq 2 \\ x_1 - x_2 + x_3 &\geq -1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Q3 A farmer wishes to choose the least cost diet that will meet the nutritional requirements of his pigs. Pigs require 4, 8, 9 pounds respectively of nutrients A, B, C per month. There are four varieties of pig food available. The nutritional contents of the foods per kg are shown in the following table.

		food			
		1	2	3	4
	A	1	2	1	4
nutrient	B	1	3	0	2
	C	4	2	6	1

If the foods cost £5, £7, £7, £9 per kg respectively, find an optimal diet for the pigs (from the point of view of the farmer!). Interpret the dual to this problem. [Optimal cost £21.60.]

Q4* (You do not need to use the simplex method here.) There are three available processes for producing a certain output. To produce 100 units of output:

- process A requires 20 man-days of labour and 10 days of machine time
- process B requires 10 man-days of labour and 20 days of machine time
- process C requires 17.5 man-days of labour and 16 days of machine time

Each process is divisible and operates under constant returns to scale, independently of the other processes.

(a) Assume that 20 man-days and 20 machine-days are available. Set up and solve the output maximisation primal linear programme and do the same for the dual. Interpret the dual variables. What is the marginal rate of substitution between man-days and machine-days at the output maximising point?

(b) Construct the marginal productivity of labour curve given that machine time is fixed at 20 days.

(c) What is the marginal productivity of machine-days when the factor input levels are 20 man-days and 20 machine-days? If machines are rented by the day, and if both factors are paid their marginal product in kind, how much of the product will be left over for management? Why?

Q5 Ricardia is a country producing only wheat and is endowed with two types of agricultural land: 60 acres of low land and 40 acres of high land. On an acre of low land, 2 man-years are required to produce a gross output of 50 bushels of wheat, whilst on half an acre of high land 2 man-years are required to produce 30 bushels. (It is assumed that the indicated input proportions for each type of land are fixed.) There are 200 labourers in Ricardia.

(a) What is the social marginal product of labour?

(b) Indicate the rent (in bushels per acre per season) imputed to the two types of land.

(c) If the wage paid to labour is 10 bushels per season, and assuming that the wheat required for wages must be set aside from the previous seasons crop, indicate the functional distribution of income between rent, profit and labour in Ricardia.

Q6* In problem 1.7 the simplex method was used to solve the ball bearing manufacturer's problem.

(a) The optimal production plan was to produce 4 bearings of type 1, 1 of type 3 and 1 of type 4 per minute. What would be the effect if the selling price of the second type of bearing were increased by $1p$, $2p$, $3p$? [With $3p$ increase, the new optimal solution is $(\frac{32}{7}, \frac{8}{7}, 0, 0)$, with profit $53\frac{5}{7}p$.]

(b) By how much could the selling price of type 4 bearings be increased before we should alter the production plan?

(c) Suppose that a new type of bearing required 3 minutes on the lathe, 2 minutes on the grinder and 1 minute on the press. What is the least selling price at which it would be worth making?

Q7* A manufacturer uses materials A, B, C to produce products $P_1, \ldots, P_4$. The table below gives the amount of material required to make one unit of each product, and the profit per unit.

	P_1	P_2	P_3	P_4
A	5	1	9	11
B	4	3	4	1
C	3	2	5	10
profit	12	5	15	10

The available quantities are 1500 units of A, 1000 units of B and 800 units of C. What quantities of each product should be made to maximise profit?

Suppose that the amounts of materials A, B, C are simultaneously increased by $t, 2t, 3t$ respectively. How large can t be without changing the optimal basis? What is the profit $z(t)$ for t in this range?
[The optimal solution is $(225, 0, 25, 0)$. For $0 \leq t \leq \frac{100}{3}$, $z(t) = 3075 + \frac{33}{4}t$.]

Q8 A petroleum refinery can make gasoline and diesel fuel from two different raw stocks, A and B, by three different processes, as follows (all figures in gallons).

process	1	2	3
yield of gasoline	4	2	2
yield of diesel fuel	2	2	4
input of stock A	5	3	5
input of stock B	3	3	3

There are available 5 and 4 million gallons per day of stock A and stock B respectively.

(a) What process or combination of processes will be used at each ratio of diesel and gasoline prices (π_d/π_g) between 0 and ∞ ?
[Let $\pi_g = 1$, and write the objective function as $(2+\pi)x_1 + (1+\pi)x_2 + (1+2\pi)x_3$. Maximise first with $\pi = 0$, then study the effects of increasing π.]

(b) Graph the production possibility curve (or transformation curve between gasoline and diesel fuel) for this firm.