E3 : Mathematical Programming   H 2010

Problem Set 1   Introduction and Simplex Method

The simplex method should not be used in the first six questions. Attempted solutions to the questions marked with a star should be handed in at the lecture on Thursday of week 6, or reach my pigeonhole in Corpus by 2pm on that day (for Colin McDiarmid). These starred questions (1, 5 - 10) will be discussed in class. The other questions are for practice and private study, though you are welcome to ask me about any of them.

Q1* A manufacturer of ball bearings uses three types of machines in his operations, namely lathes, grinders and presses, of which he has 8, 16 and 13 respectively. The firm can make four types of bearings. Each bearing of type 1 requires 1 minute on a lathe, 3 minutes on a grinder and 2 minutes of the press's time, as shown by the first column in the table below. The times required on the different machines by the other bearings are shown in the remaining columns of the table. The unit profits of the various types of bearings are 9, 8, 11 and 6 pence each for types 1, 2, 3 and 4 respectively.

The manufacturer wishes to maximise his profits. Formulate this problem as an LP, stating the intended interpretation of the variables. (See also question 7 below.)

	bearings			
	1	2	3	4
lathe	1	3	2	2
grinder	3	2	3	1
press	2	3	3	2

Q2 A gambler plays a game (once) that requires dividing money between four possible gambles. The following table gives the corresponding gain (or loss) per dollar deposited in each of the four gambles for the three possible outcomes (states of nature).

		gamble			
		1	2	3	4
	1	-3	4	-7	15
outcome	2	5	-3	9	4
	3	3	-9	10	-8

The gambler has a total of \$500. In face of the uncertainty about the outcome, she decides to make an allocation that maximises the minimum return. Formulate the problem as an LP.
[Hint. Introduce an extra variable for the objective function value.]

Q3 A manufacturer has stocks of three factors A, B and C and also hires labour from outside at a fixed wage, to make a homogeneous product. The product sells at a fixed price of £10 per unit and the wage is £2 per hour. He wishes to use his factors in order to maximise his profits.

The four processes 1, 2, 3 and 4 which he has available show constant returns to scale and require 2, 2, 3, 3 hours of labour respectively when operated at a level which produces one unit of output. The required units of the factors A, B, C for one unit of output according to the following table.

		process			
		1	2	3	4
	A	6	4	5	8
factor	B	3	5	4	2
	C	2	1	1	0

The stocks available of the three factors are 25, 11, 2 respectively. Formulate this problem as an LP. (The optimal profit is £18.)

Q4 A company collects n types of solid waste materials and treats them to make saleable products. There are m grades of product produced and they are subject to the following specification : each pound of product i contains at least α_{ij} and not more than β_{ij} pounds of waste material j.

There are A_j pounds of waste material j available per week and it costs $\$t_j$ to treat each pound of it. It costs a further $\$c_i$ to produce a pound of product i and its sale price is $\$p_i$.

The problem facing the company is to determine just how much of each product grade to produce and the exact mix of materials in each grade, in order to maximise profit per week. Formulate this problem as an LP.

Q5* A gasoline refinery has two types of crude oil, A and B, which may be used to produce fuel oil in any mixture of the following three production patterns (which may be operated at fractional levels):

1. 1 unit of A + 2 units of B → 2 units of fuel oil + 3 units of gasoline
2. 2 units of A + 1 unit of B → 5 units of fuel oil + 1 unit of gasoline
3. 2 units of A + 2 units of B → 2 units of fuel oil + 1 unit of gasoline

Suppose that fuel oil sells for £1 a unit and gasoline sells for £10 a unit, and we start out with 10 units of A and 15 units of B.
(a) What is the most profitable mixture of production patterns?
(b) What would happen if gasoline sold at £7 a unit?
(c) What would happen if production patterns could not operate at fractional levels?

Q6* Consider the LP

$$\min 2x_1 + 5x_2$$

subject to

$$\begin{aligned} x_1 + 2x_2 &= 8 \\ -2x_1 + x_2 &\leq -4 \\ x_1 &\leq 7 \\ x_2 &\geq 0 \end{aligned}$$

[Note that x_1 is not restricted in sign.] Rewrite this LP in the canonical form

$$\min \mathbf{c}'\mathbf{x} \text{ subject to } A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0},$$

with the right-hand side vector **b** non-negative.

Q7* Use the simplex method to solve problem 1. (Optimal profit £53.)

Q8* (Unbounded solutions) (Here we investigate what happens in the simplex method if none of the basic variables yields an upper bound on the entering variable.) Suppose that in the chemical manufacturer example from lectures the coefficients of x_2 in the three constraints were -1, -3, 0. Then the initial tableau is

z	1	-1	-1	0	0	0	0
x_3	0	2	-1	1	0	0	11
x_4	0	1	-3	0	1	0	18
x_5	0	1	0	0	0	1	4

Show that if x_2 is the entering variable then we can make x_2 as big as we like, and thus the profit z as big as we like.
[In general, if there is no candidate for leaving variable then the problem has an *unbounded optimum*; that is, for any constant M there is a feasible solution with objective function value $> M$.]

Q9* (Degeneracy) (Here we investigate what happens if more than one variable yields the tightest upper bound on the entering variable, and is thus a candidate for leaving variable.)
Suppose that in the chemical manufacturer example from lectures the amount of ingredient X available is reduced to 6 units, so that we obtain an initial tableau

z	1	-1	-1	0	0	0	0
x_3	0	2	1	1	0	0	6
x_4	0	1	3	0	1	0	18
x_5	0	1	0	0	0	1	4

Let x_2 be the entering variable. Note that both x_3 and x_4 limit the increase of x_2 to 6, and so either may be the leaving variable. Let us arbitrarily select x_4 as leaving variable. Check that the next tableau is

z	1	$-\frac{2}{3}$	0	0	$\frac{1}{3}$	0	6
x_3	0	$\frac{5}{3}$	0	1	$-\frac{1}{3}$	0	0
x_2	0	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0	6
x_5	0	1	0	0	0	1	4

Observe that this tableau corresponds to the basic feasible solution $\mathbf{x} = (0, 6, 0, 0, 4)'$ with objective function value $z = 6$, in which as well as the non-basic variables the basic variable x_3 is zero. Such a solution is called *degenerate*.

There is an unfortunate consequence at the next iteration. Make x_1 the entering variable and calculate the next tableau. What is the corresponding solution, and in particular what is the value of the basic variable x_1 ? Sketch the feasible region for the original problem and for the new version considered here, and comment.
[In general, a problem is call *degenerate* if some basic feasible solution is degenerate, and otherwise is *non-degenerate*. A degenerate basic feasible solution corresponds to an extreme point of the feasible region through which "too many" bounding hyperplanes pass.]

Q10* Use phase I of the two-phase method to construct an initial tableau for phase II, for the LP

$$\max\ x_1 - 3x_2 + 2x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + x_4 &\geq 7 \\ -2x_1 + x_2 + x_3 \quad &\leq 13 \\ x_1 - 2x_2 \qquad - 4x_4 &\leq 10 \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

[Hint: how many artificial variables need you introduce?]

Q11 For an activity analysis it is clear that if we increase the profit contribution of some good then we should not start to make less of it (?). Consider the following more general problem about LP's.

Let $\mathbf{x}^*$ be an optimal solution to a standard maximum problem

$$\max\ \mathbf{c}'\mathbf{x} \text{ subject to } A\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0}.$$

Now change the objective function as follows: increase some coordinate c_k of $\mathbf{c}$ to $\hat{c}_k > c_k$, and leave all other coordinates unchanged. Let $\hat{\mathbf{c}}$ be the new objective function vector. Let $\hat{\mathbf{x}}^*$ be an optimal solution to the new problem with $\mathbf{c}$ replaced by $\hat{\mathbf{c}}$. Show that $\hat{x}^*_k \geq x^*_k$.

Q12 A subset C of R^n is called *convex* if whenever $\mathbf{x}$ and $\mathbf{y}$ are in C then so is $t\mathbf{x} + (1-t)\mathbf{y}$ for any $0 \leq t \leq 1$.

(a) Let $\mathbf{a} \epsilon R^n$ with $\mathbf{a}$ not $\mathbf{0}$, and let $c \epsilon R$. The *hyperplane* $H = \{\mathbf{x} \epsilon R^n : \mathbf{a}'\mathbf{x} = c\}$ divides the whole space R^n into two (closed) *half-spaces* $A = \{\mathbf{x} \epsilon R^n : \mathbf{a}'\mathbf{x} \leq c\}$ and $B = \{\mathbf{x} \epsilon R^n : \mathbf{a}'\mathbf{x} \geq c\}$. Show that H, A and B are each convex.

(b) Show that the intersection of any collection of convex sets is convex.

(c) Deduce that the feasible region of any linear programme is a convex set. What about the set of optimal solutions?

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