Comparison of parallel solution techniques for the Eikonal equation

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1 Motivation

Modelling the dynamics of propagation of the electrical signal through the heart has many important applications such as predicting heart attacks, treating coronary heart disease and designing safer drugs. Thus it is important to provide effective models of signal propagation which can be computed in efficient ways.

There have been many approaches to this end. An electrical wave travelling through heterogeneous heart tissue is usually simulated using a sophisticated system of coupled ODEs and PDEs. These equations are solved on a realistic geometric mesh with numerical schemes (an example using finite element techniques is shown in §1.1). In cases where a fast approximation to the wave dynamics is needed, without so much physiological fidelity, the dynamics can be approximated by the *Eikonal equation* [1,2]. In its most simple form, the Eikonal equation is

$$|\nabla d| = 1 \text{ in } \Omega, \tag{1}$$

together with a Dirichlet boundary condition, such as $d = 0$ on some part of the boundary of the domain, Ω. In this simple form of Equation 1, the solution, d, can be viewed as the straight-line distance from the boundary to each point inside the domain. This would correspond to a signal which moved at a constant rate through the domain, thus, if you know the distance between any two points, you can infer the signal propagation time between them. A simple example of the solution to the Eikonal equation is shown in Figure 1.

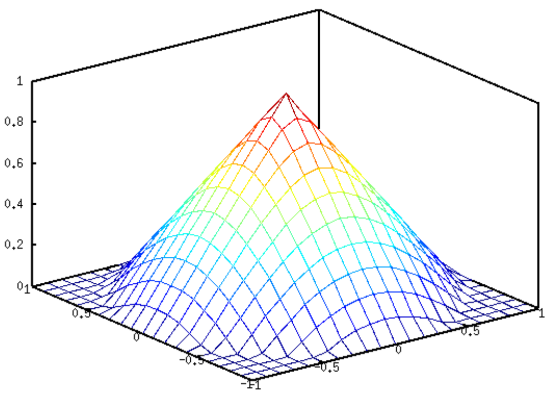

Figure 1: An Eikonal solution. The distance from a unit circle in 2D ($d = 1 - \sqrt{x^2 - y^2}$).

With this distance calculation one can do useful things, such as calculate the distance to one or more boundaries so that one can approximate a material property. In heart modelling the muscle fibre directions vary across the width of the heart wall in a predictable way so, if one is given a heart geometry with no material properties, it is possible to construct an approximation to the muscle fibres by measuring distances.

When one has a heterogeneous model with muscle fibre direction given, one would not expect a signal to travel uniformly quickly through heart tissue. The signal travels faster along fibres than between them, so one has to adapt the Eikonal approximation to wave propagation. A more general form of the Eikonal equation has some function of space f on the right-hand side, so that the solution no longer measures straight-line distance, but some weighted distance:

$$|\nabla d| = f(x, y, z) \text{ in } \Omega. \tag{2}$$

(This is still a simplication, because the f here is a scalar at each point in space and not a tensor, see [1]). One can then approximate, using this weighted distance, what the response of tissue would be to some electrical stimulus.

1.1 Current progress

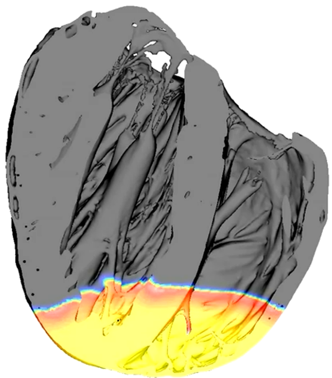

Figure 2: A snapshot from a numerical simulation of the solution to the full finite element model in a rabbit's heart. The entire video can be viewed at `http://www.youtube.com/user/ChasteProject#p/u/8/-2A--fdaxFo`.

The Eikonal approximation methods tend to take place on the same geometric domain that is used in the finite-element solver: an unstructured conforming grid of tetrahedra. The state-of-the-art computational meshes have such a high-level of detail that it is not feasible to fit the entire mesh on one computer. Instead, the geometry is spatially partitioned between several processes. As our software has developed we have been able to make the solvers faster and more efficient in parallel. The following shows the compute time for accurately simulating 1 second of normal heart rhythm on a large supercomputer:

Jan 2009: 1 month (32 procs)
May 2010: 2 hours (2048 procs)
May 2011: 45 secs (4224 procs).

The aim of this project is to provide faster approximations to wave dynamics using the weighted Eikonal equation (Equation 2). If the distance function is weighted (typically by having a right-hand side f associated with each tetrahedral element) then it becomes necessary to explore viable routes between the source boundary and each point in space. This can be approximated (over-estimated) by travelling only on the edges of the mesh, turning the problem into a more sophisticated (multiple source) version of Dijkstra's algorithm [3]. Dijkstra's algorithm can be parallelized by using spatially partitioned queues and a synchronisation method [4]. A parallel Dijkstra's algorithm for Equation 1 has been implemented within our software framework [5,6], using an open-source

4 million node rabbit mesh, which has been tagged with fibre directions/tensors on elements. A Dijkstra implementation for Equation 2 requires an interpolation of f from the elements onto the edges of the mesh.

One of the main weaknesses with the method, though, is the over approximation in the solution. By restricting the travel route to edges in the graph it allows for efficient computation and parallelization via Dijkstra's algorithm, but does not compute the true shortest path which may travel through tetrahedral elements. The project would look to address this problem as one of its main aims.

2 The project

The project will work mainly with the Chaste code base [5] to construct additional solution methods to the heart model with the aim of providing both more accurate solutions, but also explore methods which will improve efficiency. Two solution methods will be explored here, and these will be compared with the graph-based solution method to provide details on both model accuracy and computational efficiency. The project would involve implementation of the below solution methods as part of the Chaste code base and analysis of the results.

2.1 Fast marching

A graph-based wavefront algorithm can be made less approximate by allowing the solution to travel across elements (rather than on their edges, as in the Dijkstra-based method). This technique means that the Eikonal equation is solved element-wise – usually giving rise to a quadratic equation in the unknown d at the vertices of each element. Fast marching was first using in structured finite element grids but has recently been used to good effect in unstructured meshes [7,8]. Parallelization of the fast-marching will follow the same structure as used in the parallelization of the graph-based technique.

2.2 Nonlinear PDE with smoothing

Parallel nonlinear PDE solvers exist in our software framework [5], so it is feasible to solve the Eikonal equation as a single PDE. Complications arise because the solution may not be smooth (as at the origin in Figure 1). In order for a finite-element scheme to converge, it becomes necessary to add a smoothing Laplacian term to the Eikonal equation [9]. This smoothing term can be reduced during the process of nonlinear iteration until the scheme converges.

2.3 Project plan

- Week 1: Familiarise with Chaste code base, read through tutorials [10]. Implement solution methods on toy problems.
- Weeks 2-4: Extend the graph-based (Dijkstra) implementation to handle heterogeneity by interpolating for elements to edges. Code up the above solution methods as part of the Chaste code base.
- Weeks 5-6: Calculate results, compare computational times, write up.

3 References

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