Comparison of parallel solution techniques for the Eikonal equation

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1 The Eikonal equation

In its most simple form, the Eikonal equation is that solution to

$$|\nabla d| = 1 \text{ in } \Omega, \tag{1}$$

together with a Dirichlet boundary condition, such as $d = 0$ on some part of the boundary of the domain Ω. The solution, d, can be viewed as the straight-line distance from the boundary to each point inside the domain. This is illustrated for the interior of a unit disc in Figure 1. More generally the Eikonal equation has some function of space f on the right-hand side so that the solution no longer measures straight-line distance but some weighted distance.

$$|\nabla d| = f(x, y, z) \text{ in } \Omega. \tag{2}$$

1.1 Biological application

The Eikonal equation is used in computational biology in two main ways

1. To calculate the distance to one or more boundaries so that we can approximate a material property. (In heart modelling the muscle fibre directions vary across the width of the heart wall in a predictable way.)
2. To approximate, using a weighted distance, what the response of tissue would be to some stimulus. An electrical wave travelling through heterogeneous heart tissue can be simulated using a sophisticated system of ODEs and PDEs, or approximated by the Eikonal equation [1,2].

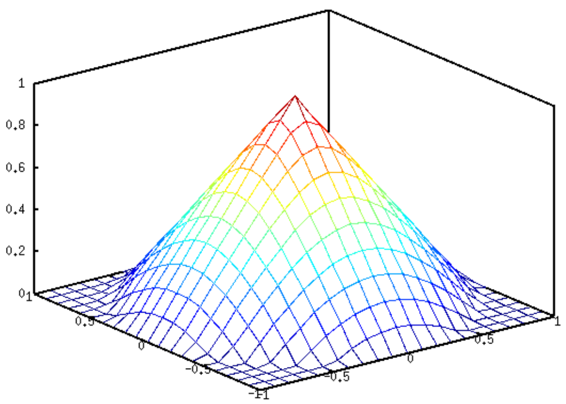

Figure 1: An Eikonal solution. The distance from a unit circle in 2D ($d = 1 - \sqrt{x^2 - y^2}$).

Both of these use-cases tend to take place on the same geometric domain that is used in the finite-element solver: an unstructured conforming grid of tetrahedra. The state-of-the-art computational meshes have such a high-level of detail that it is not feasible to fit the entire mesh on one computer. Instead, the geometry is spatially partitioned between several processes.

2 Solution methods

2.1 Brute force

In the case where the right-hand side of the equation is 1 (as above) and where the boundary where the Dirichlet condition is applied is small then it is feasible to calculate the Euclidean distance from each boundary point to each non-boundary point. This assumes that, in a parallel setting, the location of the boundary points are known to all processes. It also assumes that the mesh is convex; if the mesh contains cavities, then the straight line distance is not viable.

2.2 Graph-based wavefront methods

If the distance function is weighted then it becomes necessary to explore viable routes between the source boundary and each point in space. This can be approximated (over-estimated) by travelling only on the edges of the mesh, turning the problem into a more sophisticated (multiple source) version of Dijkstra's algorithm. Dijkstra's algorithm can be parallelized by using spatially partitioned queues and a synchronisation method. This has been implemented within our software framework [4].

2.3 Fast marching

A graph-based wavefront algorithm can be made less approximate by allowing the solution to travel across element (rather than on their edges). This technique means that the Eikonal equation is solved on an element-wise base – usually giving rise to a quadratic equation in unknown. Fast marching was first using in structured finite element grids but has recently been used to good effect in unstructured meshes [5]. Parallelization of the fast-marching will follow the same structure as used in the parallelization of the graph-based technique.

2.4 Nonlinear PDE with smoothing

Parallel non-liner PDE solvers exist in our software framework [4], so it is feasible to solve the Eikonal equation as a single PDE. Complications arise because the solution may not be smooth (as at the origin in Figure 1. In order for a finite-element scheme to converge, it becomes necessary to add a smoothing Laplacian term to the Eikonal equation [6]. This smoothing term can be reduced during the process of nonlinear iteration until the scheme converges.

3 Skill set

There is opportunity for a numerical analysis to work on the PDE solution techniques and for a computer scientist to work on the marching techniques.

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