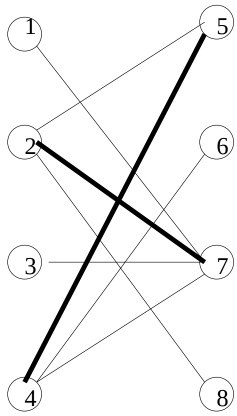
Combinatorial Optimisation HT 2010
Problem set 2, MSc in Applied Statistics, for week 8

1 Use our maximum matching algorithm to obtain a maximum cardinality matching and a minimum cardinality cover (of edges by nodes) in the bipartite graph shown, starting from the matching $(2, 7)$, $(4, 5)$ shown.



2 Suppose that we have a set of workmen and a set of jobs, where each workman is qualified to do exactly k jobs and there are exactly k workmen qualified to do each job. Show that there is a complete assignment of workmen to jobs for which they are qualified, one workman to each job.

3 You are the manager of a racing car team with five drivers I - V and five cars A - E. You are entered for a race where the winning team is the one with the lowest total time for all five cars. The table below gives the expected time in which a driver will complete the course when paired with each car.

	A	B	C	D	E
I	3	5	6	9	10
II	4	8	8	11	13
III	6	9	10	12	14
IV	8	10	10	15	16
V	13	13	17	18	20

Find an optimal assignment of drivers to cars. Is it unique?

4 Use the general procedure given in lectures for solving an assignment problem, showing all the steps, to find all the optimal assignments for the problem with cost matrix:

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 4 & 3 & 2 & 1 \\ 1 & 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 2 & 1 \\ 3 & 2 & 1 & 0 & 0 \end{bmatrix}$$

How can you be sure that there are no other optimal assignments?

5 Recall that the matching M in a (finite) graph G is of maximum size if and only if there is no augmenting path.

Two players play a game on G as follows. They alternately pick distinct nodes $N_1, N_2, \ldots$ such that, for each $i = 1, 2, \ldots$, N_{i+1} is adjacent to N_i. Thus the first player can pick any node for N_1, the second must pick a node adjacent to this node for N_2, the first player must pick a node adjacent to N_2 (but cannot pick N_2) for N_3, and so on. The last player able to pick a node wins.

Show that: if G has a complete matching then the second player has a winning strategy; and if not then the first player has.

6 Illustrate an algorithm for finding the maximum flow in a network by finding the maximum flow between the vertices 1 and 5 in the network whose vertex set is $\{1, 2, ..., 5\}$, and where the capacity c_{ij} of the directed edge joining vertex i to vertex j is given by the (i, j)-entry in the matrix

$$(c_{ij}) = \begin{pmatrix} - & 4 & 11 & 8 & 0 \\ 0 & - & 2 & 0 & 4 \\ 0 & 0 & - & 0 & 11 \\ 0 & 0 & 3 & - & 7 \\ 0 & 0 & 0 & 0 & - \end{pmatrix}.$$

Start from the initial flow

$$(x_{ij}) = \begin{pmatrix} - & 4 & 7 & 8 & 0 \\ 0 & - & 1 & 0 & 3 \\ 0 & 0 & - & 0 & 11 \\ 0 & 0 & 3 & - & 5 \\ 0 & 0 & 0 & 0 & - \end{pmatrix}.$$