

3. The random assignment problem

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Brief Description: Suppose you have n jobs and n machines which can do them. For each pair (i,j) there is a cost $c(i,j)$ for performing job i on machine j . The assignment problem is to find an assignment of jobs to machines (i.e. a permutation π) which minimizes the total cost $c(1,\pi(1)) + c(2,\pi(2)) + \dots + c(n,\pi(n))$. The random version of the problem supposes that the costs are i.i.d. random variables. Assume that these have an Exponential(1) distribution and consider $A_n = n^{-1}$ [minimal total cost]. In 1985, it was conjectured by Mezard and Parisi, using the non-rigorous replica method from statistical physics, that A_n converges in probability to $\zeta(2) = \pi^2/6$. This striking conjecture was finally rigorously proved by Aldous in 2001 using the so-called objective method. The idea is that, locally, the structure of the problem is "tree-like". In particular, the limit of the finite- n problem can be expressed as an optimal assignment on a certain infinite tree.

The structure of random combinatorial optimization problems, such as the random assignment problem, is a subject of much interest in several different disciplines: computer science, statistical physics, combinatorics and probability. The mathematical justification of the powerful, but non-rigorous, tools used by the statistical physics community (e.g. the replica and cavity methods) to arrive at deep results is a very significant open problem.

The purpose of this project is to give an exposition of the random assignment problem and the solution of the Mezard-Parisi conjecture. Simulation might easily form part of this exposition if the student is so inclined. There is a rich probabilistic literature: a starting point might be the short, straightforward proof given in 2009 by Wästlund. Aldous' papers on the subject are very readable and there is also a beautiful survey of the wider issues by Aldous and Steele.

Prerequisite courses/knowledge: Part A *Probability*, BS3a *Applied Probability*

Level of computing required? None required; good level desirable if simulation to be undertaken.

Data available? (if relevant) N/A