3 Duality and sensitivity

3.1 Dual of standard LP's

Consider again our favourite LP

$$\max \quad x_1 + x_2$$

subject to

$$\begin{array}{rcrcr} 2x_1 & + & x_2 & \leq & 11 \\ x_1 & + & 3x_2 & \leq & 18 \\ x_1 & & & \leq & 4 \\ & x_1, & x_2 & \geq & 0 \end{array}$$

Let us forget that we know that the optimum value z^* is 8, and consider how we might obtain upper bounds on z^*. From the first constraint, for any feasible (x_1, x_2)

$$z = x_1 + x_2 \leq 2x_1 + x_2 \leq 11$$

and hence $z^* \leq 11$. We can improve on this by considering the first two constraints. We see that for any feasible (x_1, x_2)

$$z = x_1 + x_2 \leq 1/3\ (2x_1 + x_2) + 1/3\ (x_1 + 3x_2) \leq 11/3 + 18/3 = 29/3,$$

and so $z^* \leq 29/3$.

In general we can obtain an upper bound on z^* by taking non-negative linear combinations of the constraints; that is, we multiply the first constraint by some $y_1 \geq 0$, the second by $y_2 \geq 0$, and the third by $y_3 \geq 0$, and then add them up. (In the first case above we had $y_1 = 1$, $y_2 = y_3 = 0$, and in the second case we had $y_1 = y_2 = \frac{1}{3}$, $y_3 = 0$.) We find

$$(2y_1 + y_2 + y_3)x_1 + (y_1 + 3y_2)x_2 \leq 11y_1 + 18y_2 + 4y_3.$$

We wish to use the quantity on the right as an upper bound for z^*. This is permissible as long as the coefficients of x_1 and x_2 on the left are at least as big as the coefficients of x_1 and x_2 in the expression for z. As we want as small an upper bound as we can get, we are led to the LP

$$\min\ 11y_1 + 18y_2 + 4y_3$$

subject to

$$\begin{array}{rcrcrcr} 2y_1 & + & y_2 & + & y_3 & \geq & 1 \\ y_1 & + & 3y_2 & & & \geq & 1 \\ & y_1, & y_2, & & y_3 & \geq & 0 \end{array}$$

Now consider a general standard maximum LP

(P) $\max\ \mathbf{c}'\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}\ ,\ \mathbf{x} \geq \mathbf{0}$,

where $\mathbf{c}$ is a given n-vector, $\mathbf{b}$ is a given m-vector, A is a given $(m \times n)$ matrix, and $\mathbf{x}$ is a variable n-vector. The dual program is defined to be the LP

(D) $\min\ \mathbf{b}'\mathbf{y}$ subject to $A'\mathbf{y} \geq \mathbf{c}$, $\mathbf{y} \geq 0$,

where $\mathbf{y}$ is a variable m-vector. Recall that A' denotes the transpose of the matrix A, so that

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}' = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$$

In a different notation the program (P) is

$$\max \sum_{j=1}^{n} c_j x_j$$

subject to

$$\begin{array}{ll} \sum_{j=1}^{n} a_{ij} x_j \le b_i & (\text{for } i = 1, \ldots, m) \\ x_j \ge 0 & (\text{for } j = 1, \ldots, n), \end{array}$$

and its dual (D) is

$$\min \sum_{i=1}^{m} b_i y_i$$

subject to

$$\begin{array}{ll} \sum_{i=1}^{m} a_{ij} y_i \ge c_j & (\text{for } j = 1, \ldots, n) \\ y_i \ge 0 & (\text{for } i = 1, \ldots, m). \end{array}$$

Note that in forming the dual program

- we have one variable for each constraint of the primal, and one constraint for each variable of the primal
- max goes to min (and conversely)
- the rôles of **b** and **c** are interchanged
- the inequalities are reversed
- all variables are non-negative.

In the diagram below columns correspond to variables of the primal and constraints of the dual, and rows to constraints of the primal and variables of the dual.

$$\begin{array}{cccc|cc} x_1 & x_2 & \cdots & x_n & & \\ c_1 & c_2 & \cdots & c_n & & \\ \hline a_{11} & a_{22} & \cdots & a_{1n} & b_1 & y_1 \\ a_{21} & & & & b_2 & y_2 \\ & & & & & \\ & & & & & \\ a_{m1} & & \cdots\cdots & a_{mn} & b_m & y_m \end{array}$$

The relationship between the primal (P) and dual (D) is symmetric, so that the dual of the dual is back to the primal.

Dual of activity analysis

Suppose that you wish to take over the firm as cheaply as possible. You want to attach 'reasonable' prices $y_1, \ldots, y_m$ to the resources owned by the firm so as to minimise the total cost $b_1 y_1 + \ldots + b_m y_m$. To be reasonable, certainly the prices y_i must be ≥ 0. Also the cost of the resources used in producing one unit of good j should be at least the resulting return c_j, that is

$$a_{1j}y_1 + a_{2j}y_2 + \ldots + a_{mj}y_m \geq c_j \quad (\text{for} \;\; j = 1, \ldots, n);$$

for otherwise the firm would wish to continue making that good.
Thus you should solve the LP

$$\min \quad \mathbf{b}'\mathbf{y} \quad \text{subject to} \quad A'\mathbf{y} \geq \mathbf{c}, \, \mathbf{y} \geq \mathbf{0}$$

which is the dual of the firm's problem. The optimal values of the dual variables y_i are the *shadow prices* of the resources. Note that to make the dimensions fit in the dual LP the y_i must be some sort of prices.

Dual of diet problem
A manufacturer of nutrient pills considers that the dietician will be prepared to substitute pills for food in the diet if the pills are no more expensive than the corresponding food. His problem is to choose the price y_i for one unit of the N_i pill so that his receipts, based on the minimal nutritional requirements of the diet, are maximised. Thus he wishes to maximise $b_1y_1 + \ldots + b_my_m$ subject to $y_i \geq 0$ (of course) and

$$a_{1j}y_1 + a_{2j}y_2 + \ldots + a_{mj}y_m \leq c_j \quad (\text{for} \;\; j = 1, ..n).$$

Thus he should solve the LP

$$\max \quad \mathbf{b}'\mathbf{y} \quad \text{subject to} \quad A'\mathbf{y} \leq \mathbf{c}, \quad \mathbf{y} \geq \mathbf{0}$$

which is the dual of the dietician's problem. The optimal values of the dual variables y_i are the shadow prices for the nutrients.

3.2 Duality theorem for standard LP's

We now build up to the fundamental duality theorem. The first result should come as no surprise, given that the dual LP was introduced as a way of upper bounding the value in the primal.

Weak duality theorem If $\mathbf{x}$ and $\mathbf{y}$ are feasible for the dual programs (P) and (D) respectively, then $\mathbf{c}'\mathbf{x} \leq \mathbf{b}'\mathbf{y}$.

Let us prove this result carefully, since the proof method will also be useful later. From the inequalities $y_i \geq 0$ and $(A\mathbf{x})_i \leq b_i$ we obtain

$$\mathbf{y}'\mathbf{b} - \mathbf{y}'(A\mathbf{x}) = \sum_i y_i(b_i - (A\mathbf{x})_i) \geq 0.$$

Similarly, since $(\mathbf{y}'A)_j = (A'\mathbf{y})_j \geq c_j$ and $x_j \geq 0$, we have

$$(\mathbf{y}'A)\mathbf{x} - \mathbf{c}'\mathbf{x} = \sum_j ((\mathbf{y}'A)_j - c_j)x_j \geq 0.$$

Hence

$$\mathbf{y}'\mathbf{b} \geq \mathbf{y}'(A\mathbf{x}) = (\mathbf{y}'A)\mathbf{x} \geq \mathbf{c}'\mathbf{x},$$

as required. □

For an activity analysis, the weak duality theorem says that the profit from producing goods cannot exceed the shadow cost of resources consumed. For a diet problem it says that the total food cost must be at least the total shadow cost of the nutrients.

Optimality test Let $\mathbf{x}^*$ and $\mathbf{y}^*$ be feasible for the dual programs (P) and (D) respectively, and such that the corresponding objective function values are equal, that is $\mathbf{c}'\mathbf{x}^* = \mathbf{b}'\mathbf{y}^*$. Then $\mathbf{x}^*$ and $\mathbf{y}^*$ are both optimal.

To see why this is true, consider any feasible vector $\mathbf{x}$ for (P). By the weak duality theorem

$$\mathbf{c}'\mathbf{x} \leq \mathbf{b}'\mathbf{y}^* = \mathbf{c}'\mathbf{x}^*.$$

Thus $\mathbf{x}^*$ maximises $\mathbf{c}'\mathbf{x}$ over all vectors $\mathbf{x}$ feasible for (P). Similarly $\mathbf{y}^*$ is optimal for (D). $\square$

Example Consider again the chemist's problem. Let us use the optimality test to see that $\mathbf{x}^* = (3, 5)'$ is an optimal solution.
Note first that $\mathbf{x}^*$ is feasible for (P) and has objective function value 8. But we may also check easily that $\mathbf{y}^* = (2/5, 1/5, 0)'$ is feasible for the dual program (D), and also has objective function value 8. Hence by the optimality test $\mathbf{x}^*$ is optimal for (P) (and $\mathbf{y}^*$ is optimal for (D)). How did we know to look at the vector $\mathbf{y}^*$? The plot thickens!

Duality theorem If both a linear program and its dual are feasible, then both have optimal solutions and the values of the two programs are equal. If exactly one is feasible then it has an unbounded optimum.

Here, by the *value* of a program we mean the optimal value of the objective function. If a program has an unbounded optimum then of course it has no optimal solution. Note that it is possible for both primal and dual programs to be infeasible.

We shall see why the theorem is true by looking carefully at the simplex method. Recall that when we solve the chemist problem by the simplex method, the z-equation corresponding to the optimal tableau is $z + \frac{2}{5}x_3 + \frac{1}{5}x_4 = 8$, which suggests that $\frac{2}{5}$ is the marginal value of the resource R (since $-\frac{2}{5}$ is the reduced profit of the slack variable x_3 corresponding to R). But the optimal value y_1^* of the corresponding dual variable also attaches a value to the resource R (we shall discuss this more later). Perhaps then we can read off an optimal dual solution from the coefficients of the slack variables in the optimal z-equation?

Proof of the duality theorem

(i) Suppose first that both the programs (P) and (D) are feasible. We must show that they each have optimal solutions, which achieve the same objective function value. For simplicity let us assume that $\mathbf{b} \geq \mathbf{0}$. Add slack variables $s_1, \ldots, s_m$, and use them to start the simplex method. Since the program (D) is feasible, the program (P) does not have an unbounded optimum (by the weak duality theorem), and so the simplex method must terminate, with a final optimal tableau.

z	1	. . . $-c_j$ . . .	. . 0 . .	0
s_1	0	.		
.	.	.		
		$\mathbf{a}_j$	I_m	$\mathbf{b}$
.	.	.		
s_m	0	.		
z	1	. . . $-c_j + \mathbf{y}'\mathbf{a}_j$ . . .	y_1 . . . y_m	$\mathbf{y}'\mathbf{b}$
	0	.		
	.	.		
	.	.		
	0	.		

The z-row in the final tableau was obtained by adding on to the initial z-row various multiples of the constraint rows of the intervening tableaux; and thus it could be obtained from the initial z-row by adding some multiples of the constraint rows of the *initial* tableau, say $y_1 \times R_1, y_2 \times R_2, \ldots, y_m \times R_m$. Observe that the 'simplex multiplier' y_i must in fact be the coefficient of the ith slack variable s_i in row R_0 of the final tableau, since the corresponding column of the original tableau has a 1 in row R_i and 0's elsewhere. Similarly, if we denote by $\mathbf{a}_j$ the column of the matrix A corresponding to the variable x_j, then in the final tableau the coefficient of x_j in row R_0 (which is minus the reduced profit) equals $-c_j + \mathbf{y}'\mathbf{a}_j$, which may be written as $-c_j + (\mathbf{y}'A)_j$. Here we have written $\mathbf{y} = (y_1, \ldots, y_m)'$. Further the (primal) basic feasible solution corresponding to the tableau has objective function value $\mathbf{y}'\mathbf{b}$.

But since the tableau is optimal, all the reduced profits are ≤ 0, and so $-\mathbf{c}' + \mathbf{y}'A \geq \mathbf{0}'$, and $\mathbf{y}' \geq \mathbf{0}'$; and thus the vector $\mathbf{y}$ is feasible for the dual program (D). The optimality test now shows that both $\mathbf{x}$ and $\mathbf{y}$ are optimal, as required.
(ii) Now suppose that exactly one of the programmes, say (P), is feasible. Then we can start the simplex method but we cannot reach an optimal tableau, or else (D) would have a feasible solution as we have just seen. Hence (P) must have an unbounded optimum. □

Next we consider the important idea of complementary slackness.
Complementary slackness (or equilibrium) theorem
Let $\mathbf{x}$ and $\mathbf{y}$ be feasible solutions for the standard maximum problem (P) and its dual (D). Then $\mathbf{x}$ and $\mathbf{y}$ are optimal if and only if

$$\text{for each } i = 1, \ldots, m, \text{ either } (A\mathbf{x})_i = b_i \text{ or } y_i = 0 \quad \text{(or both);}$$

and

$$\text{for each } j = 1, \ldots, n, \text{ either } x_j = 0 \text{ or } (A'\mathbf{y})_j = c_j \quad \text{(or both).}$$

Proof Suppose that the feasible vectors $\mathbf{x}$ and $\mathbf{y}$ satisfy the above complementary slackness (CS) conditions. Then, from the proof of the weak duality theorem, we see that

$$\mathbf{y}'\mathbf{b} = \mathbf{y}'A\mathbf{x} = \mathbf{c}'\mathbf{x},$$

since all the non-negative summands are in fact zero. Hence by the optimality test, both $\mathbf{x}$ and $\mathbf{y}$ are optimal.

Conversely, suppose that $\mathbf{x}$ and $\mathbf{y}$ are optimal. Then by the duality theorem we have $\mathbf{y}'\mathbf{b} = \mathbf{c}'\mathbf{x}$. But by the proof of the weak duality theorem, if any of the CS conditions failed, then we would have $\mathbf{y}'\mathbf{b} > \mathbf{c}'\mathbf{x}$. Hence the CS conditions must hold. $\square$

Let us try again to show in the chemist's example that $\mathbf{x}^* = (3,5)'$ is optimal for (P). *If* $\mathbf{x}^*$ is optimal for (P) and a vector $\mathbf{y}^* = (y_1, y_2, y_3)'$ is optimal for (D), then by complementary slackness, $\mathbf{y}^*$ satisfies both constraints of (D) at equality since both coordinates of $\mathbf{x}^*$ are strictly positive; and $y_3 = 0$ since $\mathbf{x}^*$ satisfies the third inequality not at equality. Thus

$$2y_1 + y_2 = 1 \text{ and } y_1 + 3y_2 = 1,$$

and so $\mathbf{y}^* = (\frac{2}{5}, \frac{1}{5}, 0)'$. But this vector $\mathbf{y}^*$ is indeed feasible for (D), and so by complementary slackness again, $\mathbf{x}^*$ is optimal for (P) (and $\mathbf{y}^*$ for (D)).

For an activity analysis, the first set of CS conditions say that if an optimum production program leaves some of the ith resource unused then the shadow price of that resource must be zero; and the second set of CS conditions say that the jth good will not be produced if the shadow cost of the resources it uses is strictly greater than the resulting return. In other words, resources in excess supply are 'free', and processes that make a loss (costed at shadow prices) will not be used.

For a diet problem, we find that if a nutrient is oversupplied in an optimum diet then the shadow price of the corresponding pill is zero; and a given food will not be used if it is 'overpriced', that is, if it costs more than the shadow price of the equivalent pills.

3.3 Duality in general

Let us now extend the definition given above for the dual of an LP. Suppose that in the primal (P) certain of the inequalities $\leq$ are replaced by equations and certain of the variables x_j may now be unrestricted in sign. Then the dual program (D) is as before except that the dual variables corresponding to equality constraints in (P) are now unrestricted in sign, and the dual inequalities $\geq$ which correspond to unrestricted primal variables x_j are replaced by equations. Thus we have a more general recipe for forming dual programs (though we shall not consider maximising problems with any $\geq$ constraints, or minimising problems with any $\leq$ constraints).

In particular, the dual of the 'canonical maximum problem'

$$\max\ \mathbf{c}'\mathbf{x} \text{ subject to } A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq 0$$

is

$$\min\ \mathbf{b}'\mathbf{y} \text{ subject to } A'\mathbf{y} \geq \mathbf{c},\ \mathbf{y} \text{ unrestricted.}$$

Similarly, the dual of the 'canonical minimum problem'

$$\min\ \mathbf{c}'\mathbf{x} \text{ subject to } A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq 0$$

is

$$\max\ \mathbf{b}'\mathbf{y} \text{ subject to } A'\mathbf{y} \leq \mathbf{c},\ \mathbf{y} \text{ unrestricted.}$$

The Duality Theorem stated earlier holds for **any** dual pair of linear programs. Also, the complementary slackness results extend to the present more general case. For both the canonical maximum and minimum problems mentioned above, feasible solutions $\mathbf{x}$ for the primal and $\mathbf{y}$ for the dual are optimal if and only if

$$\text{for each } j = 1, \ldots, n\ ,\ \text{ if } \ x_j \neq 0 \ \text{ then } \ (A'\mathbf{y})_j = c_j\ .$$

3.4 Further thoughts on the simplex method

Refer back to the proof of the duality theorem, but now consider any simplex tableau we reach, not just the optimal tableau. As we saw earlier, the reduced profit of x_j (which is minus the coefficient of x_j in row R_0) equals $c_j - \mathbf{y}'\mathbf{a}_j$, where $\mathbf{y} = (y_1, \ldots, y_m)'$ is the vector of simplex multipliers. Also the reduced profit of the slack variable s_i is just y_i.

For each basic variable the reduced profit is zero, so if x_j is basic then $c_j - \mathbf{y}'\mathbf{a}_j = 0$, that is, $(\mathbf{y}'A)_j = c_j$; and if s_i is basic then $y_i = 0$. Another way of saying the above is that $\mathbf{y}$ satisfies $\mathbf{y}'B = \tilde{\mathbf{c}}'_B$, where B is the $m \times m$ basis matrix with columns the column vectors of the original extended matrix $(A : I)$ corresponding to the m basic variables, $\tilde{\mathbf{c}}'$ is the 'stretched' row $(n+m)$-vector $(\mathbf{c}', \mathbf{0}')$ and $\tilde{\mathbf{c}}'_B$ contains the values from $\tilde{\mathbf{c}}'$ corresponding to the basic variables. (See section 1.6.)

In economic terms, we may think of the simplex multipliers $y_1, \ldots, y_m$ as temporary shadow prices. We think of solving the equation $\mathbf{y}'B = \tilde{\mathbf{c}}'_B$ as assigning temporary shadow prices to the resources in such a way that the total shadow cost of the resources consumed by each basic activity just matches the return from that activity. The quantity $z_j = \mathbf{y}'\mathbf{a}_j$ is the current shadow cost of the resources consumed by operating activity j at unit level. If none of the non-basic activities yields more than it consumes (that is, if each reduced profit $c_j - z_j \leq 0$) then the current solution is optimal. Computing the quantities $z_j = \mathbf{y}'\mathbf{a}_j$ is referred to as 'pricing out the columns'. (This idea is taken a step further in the 'revised simplex method' and in the Dantzig-Wolfe decomposition procedure.)

It is also helpful to refer more to the current tableau. A little linear algebra will show that the tableau column corresponding to the original column $\mathbf{a}_j$ is $B^{-1}\mathbf{a}_j$. Indeed, the row operations that take the initial matrix B to an $m \times m$ identity matrix in the present tableau, take each initial column vector $\mathbf{a}_j$ to $B^{-1}\mathbf{a}_j$, and in particular take the initial $m \times m$ identity matrix to B^{-1}.

Now $\mathbf{y}'B = \mathbf{c}'_B$, and so $\mathbf{y}' = \mathbf{c}'_B B^{-1}$. Thus $z_j = \mathbf{y}'\mathbf{a}_j = \mathbf{c}'_B B^{-1}\mathbf{a}_j = \mathbf{c}'_B\mathbf{d}$, where $\mathbf{d} = B^{-1}\mathbf{a}_j$ is the corresponding column in the present simplex tableau. The vector $\mathbf{d}$ is obtained by solving $B\mathbf{d} = \mathbf{a}_j$, and $\mathbf{d}$ thus gives the mix of basic activities that uses exactly the same amounts of resources as the activity j (operating at unit level). Thus we see that pushing a non-basic variable x_j up from 0 to t yields a return of $c_j t$ but forces the profit from the current basic activities down by $\mathbf{c}'_B(t\mathbf{d}) = z_j t$.

3.5 Sensitivity analysis

A great attraction of linear economic models is that as well as being able to find optimal solutions easily it is also easy to perform certain sensitivity or 'post-optimality' analyses. We shall investigate what we can say when certain parts of the input data may change slightly.

(a) **Objective function coefficients** c_j

Suppose that the profit vector $\mathbf{c}$ changes to $\mathbf{c} + t\,\mathbf{d}$ where the parameter t is ≥ 0. The current optimal solution $\mathbf{x}^*$ of course stays feasible, and the optimal profit will increase by $t\,\mathbf{d}'\mathbf{x}^*$ as long as t is small enough so that $\mathbf{x}^*$ remains optimal. (It must increase by at least this quantity.)

Back to the chemist example. Consider first the case $\mathbf{d} = (0, 0, 1, 0, 0)'$. We are increasing by t the profit contribution of activity 3 which is at present non-basic. (Perhaps we can sell off ingredient R at $t\times$ £1000 per ton.) None of the quantities z_j change. The reduced profit $c_3 - z_3$ of activity 3 is now $t - \frac{2}{5}$, and no other reduced profit is changed. Hence $\mathbf{x}^*$ remains optimal as long as $0 \leq t \leq \frac{2}{5}$, and if $t > \frac{2}{5}$ we would need to introduce x_3 into the basis and pivot.

Now consider the case $\mathbf{d} = (1, 0, 0, 0, 0)'$ say, so that we are increasing the profit contribution of activity 1, which is currently basic. We must obtain the new reduced profits, for example from computing $c_j - z_j$, where

$$z_j = (c_1 + t, c_2, c_5).(t_{1j}, t_{2j}, t_{3j}).$$

Here t_{ij} is the entry in row R_i and column x_j of the present tableau. Thus each value z_j increases by $t\, t_{1j}$, and so the reduced profit of x_j decreases by this amount for the two non-basic variables x_3 and x_4. For x_3 the reduced profit was $-\frac{2}{5}$, and $t_{13} = \frac{3}{5}$; and so we find that x_3 now has reduced profit $-\frac{2}{5} - \frac{3t}{5}$. Similarly, x_4 now has reduced profit $-\frac{1}{5} + \frac{t}{5}$. Thus the current solution remains optimal for $0 \leq t \leq 1$, and if $t > 1$ we must introduce x_4 into the basis.

For a general vector $\mathbf{d}$ we would proceed as above. There will be a t_1 such that the current solution stays optimal for $0 \leq t \leq t_1$, a $t_2 > t_1$ such that another solution is optimal for $t_1 \leq t \leq t_2$, and after a finite number of steps we reach a t_k such that some solution is optimal for all $t \geq t_k$.

(b) **right-hand side b**

Suppose that $\mathbf{x}^*$ and $\mathbf{y}^*$ are optimal solutions for an activity analysis LP and its dual. By the duality theorem, the optimal profit $\mathbf{c}'\mathbf{x}^*$ satisfies

$$\mathbf{c}'\mathbf{x}^* = \mathbf{b}'\mathbf{y}^* = b_1 y_1^* + b_2 y_2^* + \ldots + b_m y_m^*.$$

Now suppose that we find an extra unit of resource i, so that b_i should be replaced by $b_i + 1$. The constraints in the dual problem are not altered, so that $\mathbf{y}^*$ is still feasible there. If $\mathbf{y}^*$ is still optimal (as will often be the case) then the increase in return is y_i^* (and always is at most y_i^*). Thus the shadow price y_i^* is the marginal value of the ith resource, or more generally the marginal value of relaxing the ith constraint.

In the chemist example we have seen that an optimal dual solution is $\mathbf{y}^* = (\frac{2}{5}, \frac{1}{5}, 0)'$. Thus if he were given one more unit of ingredient R then his profit should increase by $\frac{2}{5}\times$ £1000. Similarly he would pay up to $\frac{1}{5}\times$ £1000 for an extra unit of ingredient S; and he would not wish to advertise Acidic.

If we add $t(1, 0, 0)'$ (where $t \geq 0$) to the right-hand side $\mathbf{b}$ we know that the profit should increase by $\frac{2}{5}t$, at least if t is small. But how big can t be? Clearly the profit does increase at this rate as long as the current optimal dual solution $\mathbf{y}^* = (\frac{2}{5}, \frac{1}{5}, 0)'$ remains optimal (since $z = \mathbf{b}'\mathbf{y}^*$). This holds as long as the current basis remains feasible, in other words as long as none of the current basic variables are forced negative, as can be seen by considering the final simplex tableau.

Suppose then that we insist on keeping the same basis, with vector $\mathbf{x_B} = (x_1, x_2, x_5)'$ of basic variables and basis matrix B. If $\mathbf{b}$ changes to $\mathbf{b} + t\mathbf{d}$ where $\mathbf{d} = (1, 0, 0)'$ say, then we want t small enough so that we have a non-negative solution to

$$B\mathbf{x_B} = \mathbf{b} + t\mathbf{d}.$$

But the unique solution is

$$\mathbf{x_B} \quad = \quad B^{-1}\mathbf{b} + tB^{-1}\mathbf{d}$$

$$
\begin{aligned}
&= \mathbf{x}_B^* + t(\frac{3}{5}, -\frac{1}{5}, -\frac{3}{5})' \\
&= (3 + \frac{3}{5}t, 5 - \frac{1}{5}t, 1 - \frac{3}{5}t)'.
\end{aligned}
$$

[From the coefficients of the slack variable x_3 in the final tableau we see that B^{-1} has first column $B^{-1}(1,0,0)' = (\frac{3}{5}, -\frac{1}{5}, -\frac{3}{5})'$.] Hence for $0 \le t \le \frac{5}{3}$ the current basis remains feasible so the optimal dual solution does not change, and so the marginal value of the first resource is indeed $\frac{2}{5}$. If we wish to make $t > \frac{5}{3}$ we may use the dual simplex method (see below) to continue the investigation. The marginal value of the first resource will then be at most $\frac{2}{5}$ (since $\mathbf{y}^*$ always stays feasible).

(c) **New activity**

Suppose that to make each unit of a possible new product j requires 2 units of each of the resources R and S, and yields unit profit. Is it worth making?

To answer this question, we 'price out the column' as earlier. From the shadow prices $\mathbf{y}^* = (\frac{2}{5}, \frac{1}{5}, 0)'$ we compute $z_j = \mathbf{y}^{*\prime}\mathbf{a_j} = (\frac{2}{5} \times 2) + (\frac{1}{5} \times 2) = \frac{6}{5}$. Thus the reduced profit is $c_j - z_j = -\frac{1}{5}$, and so we should not make the product. In economic terms, to make one unit of the new good uses up resources of shadow value z_j and yields a return of $c_j < z_j$.

(d) **New constraint**

Suppose that, having solved his original problem, the chemist finds that he should have included the constraint $x_1 + 2x_2 \le 12$. Thus his old optimal solution is no longer feasible. He may proceed as follows.

Introduce a slack variable $x_6 \ge 0$ to yield $x_1 + 2x_2 + x_6 = 12$. Now use the rows of the old optimal tableau to substitute out for the (old) basic variables x_1, x_2. We thus obtain a constraint not involving the basic variables, namely

$$
-\frac{1}{5}x_3 - \frac{3}{5}x_4 + x_6 = -1.
$$

We may now add a corresponding row and column to the old optimal tableau to obtain the following tableau, which is not a (primal) simplex tableau since it corresponds to $x_6 < 0$. Read on!

Dual simplex method [Not examinable] This is essentially a disguised way of applying the simplex method to the dual of a problem, and it is used to 'restore feasibility'.

Suppose that we have a tableau as for the usual simplex method except that

- it is 'optimal' or 'dual feasible' in that each reduced profit is ≤ 0
- it may not be primal feasible, that is some basic variables may be set < 0.

The dual simplex method chooses a pivot element so that when we pivot (exactly as before) we maintain these conditions and move towards primal feasibility. First we choose as leaving variable a basic variable which is < 0 (say the most negative). Suppose that the leaving variable corresponds to row R_r. The entering variable is then determined by computing for each non-basic variable the absolute value of the ratio of the reduced profit to the corresponding entry in row R_r whenever the latter is < 0, and finding where the minimum is attained.

Consider the chemist example as altered above. The leaving variable is x_6, corresponding to row R_4. We compute the two ratios $\frac{2}{5}/\frac{1}{5} = 2$ and $\frac{1}{5}/\frac{3}{5} = \frac{1}{3}$, and find that the entering variable is x_4. Now pivot on the entry $-\frac{3}{5}$ in row R_4, column x_4 exactly as before. We obtain a tableau which is primal feasible and hence is optimal. The new optimal solution has $x_1 = \frac{10}{3}$, $x_2 = \frac{13}{3}$ and yields profit $\frac{23}{3}$. If no entry in the pivot row R_r is < 0 then the problem is infeasible.