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Critical branching Brownian motion with absorption: survival probability

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Abstract We consider branching Brownian motion on the real line with absorption at zero, in which particles move according to independent Brownian motions with the critical drift of $-\sqrt{2}$. Kesten (Stoch Process 7:9–47, 1978) showed that almost surely this process eventually dies out. Here we obtain upper and lower bounds on the probability that the process survives until some large time t. These bounds improve upon results of Kesten (Stoch Process 7:9–47, 1978), and partially confirm nonrigorous predictions of Derrida and Simon (EPL 78:60006, 2007).

Keywords Branching Brownian motion · Extinction time · Survival probability · Critical phenomena

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1 Introduction

1.1 Main results

We consider branching Brownian motion with absorption, which is constructed as follows. At time zero, there is a single particle at $x > 0$. Each particle moves independently according to one-dimensional Brownian motion with a drift of $-\mu$, and each particle independently splits into two at rate 1. Particles are killed when they reach the origin. This process was first studied in 1978 by Kesten [21], who showed that almost surely all particles are eventually killed if $\mu \geq \sqrt{2}$, whereas with positive probability there are particles alive at all times if $\mu < \sqrt{2}$. Thus, $\mu = \sqrt{2}$ is the critical value for the drift parameter.

Harris et al. [17] obtained an asymptotic result for the survival probability of this process when $\mu < \sqrt{2}$. Harris and Harris [16] focused on the subcritical case $\mu > \sqrt{2}$ and estimated the probability that the process survives until time t for large values of t. Results about the survival probability in the nearly critical case when μ is just slightly larger than $\sqrt{2}$ were obtained in [4,9,24]. Questions about the survival probability have likewise been studied for branching random walks in which particles are killed when they get below a barrier. See [1,3,12,13,19] for recent progress in this area. These questions are indeed of interest in a wide range of problems going well beyond branching process theory. Let us mention for instance: particle systems and in particular the so-called Fleming–Viot systems proposed by Burdzy et al. [10,11]; theoretical population genetics, where the Brownian trajectories of particles represent the fitness of individuals in this population, and the killing is a crude model for the effect of natural selection (this was suggested most notably by Brunet et al. in [7,8]); and nonlinear partial differential equations, as the equation satisfied by survival probabilities of branching Brownian motions with killing give rise to reaction-diffusion equations of the Fisher-KPP type which are the subject of much current research (see e.g. [14,15] and references therein).

In this paper, we consider the critical case in which $\mu = \sqrt{2}$. Let ζ be the time when the process becomes extinct, which we know is almost surely finite. Kesten showed (see Theorem 1.3 of [21]) that there exists $K > 0$ such that for all $x > 0$, we have

$$xe^{\sqrt{2}x - K(\log t)^2 - (3\pi^2 t)^{1/3}} \leq \mathbb{P}_x(\zeta > t) \leq (1+x)e^{\sqrt{2}x + K(\log t)^2 - (3\pi^2 t)^{1/3}}$$

for sufficiently large t, where $\mathbb{P}_x$ denotes the law of the process started from one particle at $x > 0$. Our main result, which is Theorem 1 below, improves upon this result. For this result, and throughout the rest of the paper, we let

$$\tau = \frac{2\sqrt{2}}{3\pi^2}, \quad c = \tau^{-1/3} = \left(\frac{3\pi^2}{2\sqrt{2}}\right)^{1/3}. \tag{1}$$

Theorem 1 *There exist positive constants C_1 and C_2 such that*

$$C_1 e^{\sqrt{2}x} \sin\left(\frac{\pi x}{ct^{1/3}}\right) t^{1/3} e^{-(3\pi^2 t)^{1/3}} \leq \mathbb{P}_x(\zeta > t) \leq C_2 e^{\sqrt{2}x} \sin\left(\frac{\pi x}{ct^{1/3}}\right) t^{1/3} e^{-(3\pi^2 t)^{1/3}} \tag{2}$$

for any $x > 0$ and $t > 0$ such that $x < ct^{1/3} - 1$. In particular, there exist positive constants C_3 and C_4 such that for any fixed $x > 0$, we have

$$C_3 x e^{\sqrt{2}x} e^{-(3\pi^2 t)^{1/3}} \leq \mathbb{P}_x(\zeta > t) \leq C_4 x e^{\sqrt{2}x} e^{-(3\pi^2 t)^{1/3}} \tag{3}$$

for sufficiently large t.

The main novelty in Theorem 1 is that the terms $e^{\pm K(\log t)^2}$ in Kesten's upper and lower bounds may be replaced by constants C_1 and C_2 respectively. Nonrigorous work of Derrida and Simon [9] indicates that it should be possible to obtain a result even sharper than Theorem 1. Indeed, equation (13) of [9] indicates that for each fixed x, we should have

$$\mathbb{P}_x(\zeta > t) \sim Ce^{-(3\pi^2 t)^{1/3}}$$

as $t \to \infty$, where C is a constant depending on x.

Note that the result (2) is only valid when $0 < x < ct^{1/3} - 1$. However, when $x = ct^{1/3} - 1$, Eq. (2) shows that the survival probability up to t is already of order 1. It is an open question whether there exists a function $\phi : \mathbb{R} \mapsto [0, 1]$ such that

$$\mathbb{P}_{ct^{1/3}+x}(\zeta > t) \to \phi(x)$$

as $t \to \infty$, where $\mathbb{P}_z$ denotes probabilities for branching Brownian motion started from a single particle at z.

A major step in the proof of Theorem 1 is to obtain sharp estimates on the extinction time ζ when the position x of the initial particle tends to infinity. For this question we obtain the following result of independent interest.

Theorem 2 *Let $\varepsilon > 0$. Then there exists a positive number $\beta > 0$, depending on ε, such that for sufficiently large x,*

$$\mathbb{P}_x(\tau x^3 - \beta x^2 < \zeta < \tau x^3 + \beta x^2) \geq 1 - \varepsilon.$$

Let $x > 0$ and let $t = \tau x^3$. Thus Theorem 2 says that if there is initially one particle at x, the extinction time of the process will be close to t (if x is large). Conversely, fix $t > 0$ and define a function

$$L(s) = c(t - s)^{1/3}. \tag{4}$$

From Theorem 2, we see that if a particle reaches $L(s)$ at time $s \in (0, t)$, then there is a good chance that a descendant of this particle will survive until time t. Our strategy for proving Theorem 1 will be to estimate the probability that a particle reaches $L(s)$ for some $s \in (0, t)$, and then argue that, up to a constant, this is the same as the probability that the process survives until time t. We note that the importance of the curve $(L(s), 0 \leq s \leq t)$ was already apparent in the work of Kesten [21], who already considered the strategy of counting the number of particles hitting that curve.

Theorem 1 gives an estimate of the probability that the process started with one particle at $x > 0$ survives until some large time t. An important open question is to determine, conditional on survival up to a large time t, what the configuration of particles will look like before time t. The complete description of the configuration of particles, conditionally upon survival up to a large time t, is known as the Yaglom conditional limit. This is in turn related to a main conjecture concerning the limiting behaviour of the Fleming–Viot process proposed by Burdzy et al. [10,11]. See [2] for a recent discussion and verification in a particular case of that conjecture.

This is the first in a series of two papers concerning the properties of critical branching Brownian motion with absorption. In the companion paper [6], we use ideas developed in this paper to obtain a precise description of the particle configuration at times $0 \le s \le t$, when the position x of the initial particle tends to infinity and $t = \tau x^3$. It seems likely that the results and methods of [6] will also shed some light on the behavior of the process conditioned to survive for a long time.

1.2 Organization of the paper

In Sects. 2 and 3, we collect some general results about branching Brownian motion killed at the boundaries of a strip. Theorems 1 and 2 are proved in Sect. 4. Throughout the paper, C will denote a positive constant whose value may change from line to line, and $\asymp$ will mean that the ratio of the two sides is bounded above and below by positive constants.

2 Branching Brownian motion in a strip

We collect in this section some results pertaining to branching Brownian motion in a strip. Consider branching Brownian motion in which each particle drifts to the left at rate $-\sqrt{2}$, and each particle independently splits into two at rate 1. Particles are killed if either they reach 0 or if they reach $L(s)$ at time s, where $L(s) \ge 0$ for all s. We assume that the initial configuration of particles is deterministic, with all particles located between 0 and $L(0)$.

Let $N(s)$ be the number of particles at time s, and denote the positions of the particles at time s by $X_1(s) \ge X_2(s) \ge \cdots \ge X_{N(s)}(s)$. Let

$$Z(s) = \sum_{i=1}^{N(s)} e^{\sqrt{2}X_i(s)} \sin\left(\frac{\pi X_i(s)}{L(s)}\right).$$

Let $(\mathcal{F}_s, s \ge 0)$ denote the natural filtration associated with the branching Brownian motion.

Let $q_s(x, y)$ denote the density of the branching Brownian motion, meaning that if initially there is a single particle at x and A is a Borel subset of $(0, L(s))$, then the expected number of particles in A at time s is

$$\int_A q_s(x, y)\, dy.$$

2.1 A constant right boundary

We first consider briefly the case in which $L(s) = L$ for all s, which was studied in [5]. The following result is Lemma 5 of [5], which relies only on standard and elementary estimates about Brownian motion.

Lemma 3 *For $s > 0$ and $x, y \in (0, L)$, let*

$$p_s(x, y) = \frac{2}{L} e^{-\pi^2 s/2L^2} e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L}\right) e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L}\right),$$

and define $D_s(x, y)$ so that

$$q_s(x, y) = p_s(x, y)(1 + D_s(x, y)).$$

Then for all $x, y \in (0, L)$, we have

$$|D_s(x, y)| \leq \sum_{n=2}^{\infty} n^2 e^{-\pi^2(n^2-1)s/2L^2}.$$

Lemma 3 allows us to approximate $q_s(x, y)$ by $p_s(x, y)$ when s is sufficiently large. We will also use the following result, which follows from (28) and (51) of [5] and is proved using Green's function estimates for Brownian motion in a strip.

Lemma 4 *For all $s \geq 0$ and all $x, y \in (0, L)$, we have*

$$\int_0^{\infty} q_s(x, y)\, ds \leq \frac{2e^{\sqrt{2}(x-y)} x(L - y)}{L}.$$

2.2 A piecewise linear right boundary

Fix $m > 0$, and fix $0 < K < L$. Also, let $t > 0$. We consider here the case in which

$$L(s) = \begin{cases} L & \text{if } \ 0 \leq s \leq t - m^{-1}(L - K) \\ K + m(t - s) & \text{if } \ t - m^{-1}(L - K) \leq s \leq t. \end{cases}$$

We will assume that $m^{-1}(L - K) \leq t/2$. Thus, the right boundary stays at L from time 0 until at least time $t/2$, but eventually moves to the left at a linear rate, reaching K at time t.

To obtain an estimate of $q_s(x, y)$, we will need the following result for the probability that a Brownian bridge crosses a line. This result is well-known and follows immediately, for example, from Proposition 3 of [23]. We let $B^{br}_{x,y,t} = (B^{br}_{x,y,t}(s), 0 \leq s \leq t)$ denote the Brownian bridge from x to y of length t.

Lemma 5 *If $x < a$ and $y < a + bt$, then*

$$\mathbb{P}(B^{br}_{x,y,t}(s) \geq a + bs \text{ for some } s \in [0,t]) = \exp\left(-\frac{2(a-x)(a+bt-y)}{t}\right). \tag{5}$$

If $x > a$ and $y > a + bt$, then

$$\mathbb{P}(B^{br}_{x,y,t}(s) \leq a + bs \text{ for some } s \in [0,t]) = \exp\left(-\frac{2(x-a)(y-a-bt)}{t}\right). \tag{6}$$

Proof Proposition 3 of [23] states that if $a > 0$ and $y < a + bt$, then

$$\mathbb{P}(B^{br}_{0,y,t}(s) \geq a + bs \text{ for some } s \in [0,t]) = \exp\left(-\frac{2a(a+bt-y)}{t}\right).$$

The result (5) follows because $(B^{br}_{0,y,t}(s) + x(t-s)/t, 0 \leq s \leq t)$ is a Brownian bridge of length t from x to y. Then (6) follows because $(-B^{br}_{x,y,t}(s), 0 \leq s \leq t)$ is a Brownian bridge of length t from $-x$ to $-y$. □

Lemma 6 *There exists a positive constant C such that if $t > 0$ and $K + mt/2 \leq 2L$, then for all $x \in [0, L]$ and all $y \in [0, K]$, we have*

$$q_t(x,y) \leq \frac{CL^4}{t^{5/2}} e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L}\right) e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{K}\right).$$

Proof First, we claim that

$$q_t(x,y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t} \cdot e^{\sqrt{2}(x-y)-t} \cdot e^t \cdot \mathbb{P}(0 \leq B^{br}_{x,y,t}(s) \leq L(s) \text{ for all } s \in [0,t]).$$

To see this, observe that the first factor is the density for standard Brownian motion, the second factor is a Girsanov term that relates Brownian motion with drift to standard Brownian motion, the third factor of e^t accounts for the branching at rate 1 (this corresponds to the so-called Many-to-One Lemma), and the fourth factor is the probability that a Brownian particle that starts at x and ends at y avoids being killed at one of the boundaries. Therefore,

$$q_t(x,y) \leq \frac{Ce^{\sqrt{2}(x-y)}}{\sqrt{t}} \mathbb{P}(0 \leq B^{br}_{x,y,t}(s) \leq L(s) \text{ for all } s \in [0,t]). \tag{7}$$

Let g denote the density of $B^{br}_{x,y,t}(t/2)$. Then

$$\begin{aligned}
&\mathbb{P}(0 \leq B^{br}_{x,y,t}(s) \leq L(s) \text{ for all } s \in [0,t]) \\
&\quad = \int_0^{L(t/2)} \mathbb{P}(0 \leq B^{br}_{x,z,t/2}(s) \leq L(s) \text{ for all } s \in [0,t/2]) \\
&\qquad \times \mathbb{P}(0 \leq B^{br}_{z,y,t/2} \leq L(t/2+s) \text{ for all } s \in [0,t/2]) g(z)\, dz.
\end{aligned} \tag{8}$$

Recall that $L(s) = L$ for all $s \in [0, t/2]$. Therefore, if $0 \leq x \leq L/2$ and $0 \leq z \leq L$, then by (6) with $a = b = 0$,

$$\begin{aligned}
&\mathbb{P}(0 \leq B^{br}_{x,z,t/2}(s) \leq L(s) \text{ for all } s \in [0, t/2]) \\
&\quad \leq \mathbb{P}(B^{br}_{x,z,t/2}(s) \geq 0 \text{ for all } s \in [0, t/2]) \\
&\quad = 1 - \mathbb{P}(B^{br}_{x,z,t/2}(s) \leq 0 \text{ for some } s \in [0, t/2]) \\
&\quad = 1 - \exp\left(-\frac{4xz}{t}\right) \\
&\quad \leq \frac{4xL}{t}.
\end{aligned} \tag{9}$$

If $L/2 \leq x \leq L$ and $0 \leq z \leq L$, then by (5) with $a = L$ and $b = 0$,

$$\begin{aligned}
&\mathbb{P}(0 \leq B^{br}_{x,z,t/2}(s) \leq L(s) \text{ for all } s \in [0, t/2]) \\
&\quad \leq \mathbb{P}(B^{br}_{x,z,t/2}(s) \leq L \text{ for all } s \in [0, t/2]) \\
&\quad = 1 - \mathbb{P}(B^{br}_{x,z,t/2}(s) \geq L \text{ for some } s \in [0, t/2]) \\
&\quad = 1 - \exp\left(-\frac{4(L-x)(L-z)}{t}\right) \\
&\quad \leq \frac{4(L-x)L}{t}.
\end{aligned} \tag{10}$$

Combining (9) and (10), we get

$$\begin{aligned}
&\mathbb{P}(0 \leq B^{br}_{x,z,t/2}(s) \leq L(s) \text{ for all } s \in [0, t/2]) \\
&\quad \leq \frac{4L}{t} \min\{x, L-x\} \leq \frac{CL^2}{t} \sin\left(\frac{\pi x}{L}\right).
\end{aligned} \tag{11}$$

If $0 \leq y \leq K/2$ and $0 \leq z \leq L$, then using the same reasoning as in (9),

$$\begin{aligned}
&\mathbb{P}(0 \leq B^{br}_{z,y,t/2}(s) \leq L(t/2+s) \text{ for all } s \in [0, t/2]) \\
&\quad \leq \mathbb{P}(B^{br}_{z,y,t/2}(s) \geq 0 \text{ for all } s \in [0, t/2]) \leq \frac{4yL}{t}.
\end{aligned} \tag{12}$$

If $K/2 \leq y \leq K$, then by (5) with $a = K + mt/2$ and $b = -m$,

$$\begin{aligned}
&\mathbb{P}(0 \leq B^{br}_{z,y,t/2}(s) \leq L(t/2+s) \text{ for all } s \in [0, t/2]) \\
&\quad \leq \mathbb{P}(B^{br}_{z,y,t/2}(s) \leq K + m(t/2 - s) \text{ for all } s \in [0, t/2]) \\
&\quad = 1 - \mathbb{P}(B^{br}_{z,y,t/2}(s) \geq K + m(t/2 - s) \text{ for some } s \in [0, t/2])
\end{aligned}$$

$$= 1 - \exp\left(\frac{4(K + mt/2 - z)(K - y)}{t}\right).$$
$$\leq \frac{4(K + mt/2)(K - y)}{t}. \tag{13}$$

From (12) and (13) and the assumption that $K + mt/2 \leq 2L$, we get

$$\begin{aligned} &\mathbb{P}(0 \leq B^{br}_{z,y,t/2}(s) \leq L(t/2 + s) \text{ for all } s \in [0, t/2]) \\ &\leq \frac{8L}{t} \min\{y, K - y\} \leq \frac{CL^2}{t} \sin\left(\frac{\pi y}{K}\right). \end{aligned} \tag{14}$$

By (8), (11), and (14),

$$\begin{aligned} \mathbb{P}(0 \leq B^{br}_{x,y,t}(s) \leq L(s) \text{ for all } s \in [0, t]) &\leq \frac{CL^4}{t^2} \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi y}{K}\right) \int_0^{L(t/2)} g(z)\, dz \\ &\leq \frac{CL^4}{t^2} \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi y}{K}\right). \end{aligned}$$

The lemma follows by combining this result with (7). □

2.3 A curved right boundary

We now consider the more general case in which the right boundary may change over time, which was studied in detail in [18]. Harris and Roberts [18] considered branching Brownian motion restricted to stay between $f(s) - L(s)$ and $f(s) + L(s)$, which is equivalent to our setting when both $f(s)$ and $L(s)$ are set equal to what we have denoted by $L(s)/2$. Assume that $s \mapsto L(s)$ is twice continuously differentiable.

Fix a point x such that $0 < x < L(0)$. Following the analysis in [18], let $(\xi_t)_{t \geq 0}$ be a standard Brownian motion started at x, and define

$$\begin{aligned} G(s) = &\exp\left(\frac{1}{2}\int_0^s L'(u)\, d\xi_u - \frac{1}{8}\int_0^s L'(u)^2\, du + \int_0^s \frac{\pi^2}{2L(u)^2}\, du\right) \\ &\times \exp\left(\frac{L'(s)}{2L(s)}(\xi_s - L(s)/2)^2 - \int_0^s \left(\frac{L''(u)}{2L(u)}(\xi_u - L(u)/2)^2 + \frac{L'(u)}{2L(u)}\right) du\right). \end{aligned}$$

Also, define

$$V(s) = G(s) \sin\left(\frac{\pi \xi_s}{L(s)}\right) \mathbf{1}_{\{0 < \xi_u < L(u)\, \forall u \leq s\}}. \tag{15}$$

It is shown in [18] using Itô's formula (see Lemma 4.2 of [18] and the discussion immediately following that result) that the process $(V(s), s \geq 0)$ is a martingale.

We now write $G(s)$ as a product of three terms $G(s) = A(s)B(s)C(s)$ as follows:

$$A(s) = \exp\left(\frac{1}{2}\int_0^s L'(u)\, d\xi_u - \frac{1}{8}\int_0^s L'(u)^2\, du\right)$$

$$B(s) = \exp\left(\int_0^s \frac{\pi^2}{2L(u)^2}\, du - \int_0^s \frac{L'(u)}{2L(u)}\, du\right)$$

$$C(s) = \exp\left(\frac{L'(s)}{2L(s)}(\xi_s - L(s)/2)^2 - \int_0^s \frac{L''(u)}{2L(u)}(\xi_u - L(u)/2)^2\, du\right).$$

This leads to the following result about the expectation of $Z(s)$.

Lemma 7 *Suppose initially there is a single particle at x. Then*

$$\mathbb{E}_x[Z(s)] = e^{\sqrt{2}x} B(s)^{-1}\, \mathbb{E}_x[V(s)A(s)^{-1}C(s)^{-1}].$$

Proof Recall that $(\xi_t)_{t\geq 0}$ is standard Brownian motion with $\xi_0 = x$. By the well-known Many-to-One Lemma for branching Brownian motion (see, for example, equation (3) of [16]),

$$\mathbb{E}_x[Z(s)] = e^s\, \mathbb{E}_x\left[e^{\sqrt{2}(\xi_s - \sqrt{2}s)} \sin\left(\frac{\pi(\xi_s - \sqrt{2}s)}{L(s)}\right)\mathbf{1}_{\{0<\xi_u - \sqrt{2}u < L(u)\ \forall u \leq s\}}\right].$$

Using Girsanov's Theorem to relate Brownian motion with drift to standard Brownian motion,

$$\begin{aligned}
\mathbb{E}_x[Z(s)] &= e^s\, \mathbb{E}_x\left[e^{-s-\sqrt{2}(\xi_s - x)} \cdot e^{\sqrt{2}\xi_s} \sin\left(\frac{\pi\xi_s}{L(s)}\right)\mathbf{1}_{\{0<\xi_u<L(u)\ \forall u\leq s\}}\right] \\
&= e^{\sqrt{2}x}\, \mathbb{E}_x\left[\sin\left(\frac{\pi\xi_s}{L(s)}\right)\mathbf{1}_{\{0<\xi_u<L(u)\ \forall u\leq s\}}\right] \\
&= e^{\sqrt{2}x}\, \mathbb{E}_x\left[\frac{V(s)}{G(s)}\right] \\
&= e^{\sqrt{2}x} B(s)^{-1}\, \mathbb{E}_x[V(s)A(s)^{-1}C(s)^{-1}],
\end{aligned}$$

as claimed. □

3 The case $L(s) = c(t-s)^{1/3}$

Fix any time $t > 0$, and for $0 \leq s \leq t$, define

$$L(s) = c(t-s)^{1/3},$$

where c was defined in (1). This right boundary was previously considered by Kesten [21]. Note that for $0 < s < t$,

$$L'(s) = -\frac{c}{3}(t-s)^{-2/3}$$

and

$$L''(s) = -\frac{2c}{9}(t-s)^{-5/3}.$$

Also, a straightforward calculation gives

$$B(s)^{-1} = \exp(-(3\pi^2)^{1/3}(t^{1/3} - (t-s)^{1/3})) \left(\frac{t-s}{t}\right)^{1/6}.$$

We consider in this section branching Brownian motion with drift $-\sqrt{2}$ in which particles are killed if they reach 0 or $L(s)$ at time s. All particles will be killed by time t because $L(t) = 0$. We define $X_i(s)$, $N(s)$, and $Z(s)$ as in Sect. 2.

3.1 Estimating $\mathbb{E}_x[Z(s)]$

In this section, we will estimate $\mathbb{E}_x[Z(s)]$ when $0 < s < t$. In view of Lemma 7, this will require bounds on $A(s)$ and $C(s)$, which we present in Lemmas 8 and 9 below. Note that the constants $c_1, \ldots, c_6$ in these lemmas and in Proposition 10 do not depend on the initial position x of the Brownian motion $(\xi_t)_{t\geq 0}$.

Lemma 8 *There exist positive constants c_1 and c_2 such that for all $s \in (0, t)$, almost surely on the event $\{0 < \xi_u < L(u)\ \forall u \leq s\}$ we have*

$$\exp(-c_1(t-s)^{-1/3}) \leq C(s) \leq \exp(c_2(t-s)^{-1/3}).$$

Proof On the event $\{0 < \xi_u < L(u)\ \forall u \leq s\}$, we have

$$\begin{aligned} C(s) &\leq \exp\left(\int_0^s \left|\frac{L''(u)}{2L(u)}(\xi_u - L(u)/2)^2\right| du\right) \\ &\leq \exp\left(\int_0^s \left|\frac{L''(u)L(u)}{8}\right| du\right) \\ &= \exp\left(\frac{c^2}{36}\int_0^s (t-u)^{-4/3}\, du\right) \\ &\leq \exp\left(\frac{c^2}{12}(t-s)^{-1/3}\right). \end{aligned} \tag{16}$$

On the other hand, on the event $\{0 < \xi_u < L(u)\ \forall u \leq s\}$,

$$C(s) \geq \exp\left(\frac{L'(s)}{2L(s)}(\xi_s - L(s)/2)^2\right)$$
$$\geq \exp\left(-\frac{c^2}{24}(t-s)^{-1/3}\right). \tag{17}$$

The result follows from (16) and (17). □

Lemma 9 *There exist positive constants c_3 and c_4 such that for all $s \in (0,t)$, almost surely on the event $\{0 < \xi_u < L(u)\ \forall u \leq s\}$ we have*

$$\exp(-c_3(t-s)^{-1/3}) \leq A(s) \leq \exp(c_4(t-s)^{-1/3}).$$

Proof Observe that

$$\int_0^s L'(u)^2\,du = \frac{c^2}{3}\left((t-s)^{-1/3} - t^{-1/3}\right) \leq \frac{c^2}{3}(t-s)^{-1/3}.$$

Therefore,

$$\exp\left(\frac{1}{2}\int_0^s L'(u)\,d\xi_u\right)\exp\left(-\frac{c^2}{24}(t-s)^{-1/3}\right) \leq A(s) \leq \exp\left(\frac{1}{2}\int_0^s L'(u)\,d\xi_u\right),$$

so it suffices to prove the result with $\exp(\frac{1}{2}\int_0^s L'(u)\,d\xi_u)$ in place of $A(s)$.

Using the (stochastic) Integration by Parts Formula and the fact that L' has finite variation,

$$\int_0^s L'(u)\,d\xi_u = L'(s)\xi_s - L'(0)\xi_0 - \int_0^s L''(u)\xi_u\,du.$$

On the event $\{0 < \xi_u < L(u)\,\forall u \leq s\}$, we have $0 \leq -L'(s)\xi_s \leq \frac{c^2}{3}(t-s)^{-1/3}$, which is also valid for $s = 0$, and

$$0 \leq -\int_0^s L''(u)\xi_u\,du \leq \frac{2c^2}{9}\int_0^s (t-u)^{-4/3}\,du \leq \frac{2c^2}{3}(t-s)^{-1/3}.$$

These inequalities yield the conclusion. □

Proposition 10 *There exist positive constants c_5 and c_6 such that for all $s \in (0,t)$,*

$$Z(0)B(s)^{-1}\exp(-c_5(t-s)^{-1/3}) \leq \mathbb{E}_x[Z(s)] \leq Z(0)B(s)^{-1}\exp(c_6(t-s)^{-1/3}).$$

Proof First, suppose that initially there is a single particle at x with $0 < x < L(0)$. Recall the definition of $V(s)$ from (15). Because $V(s) = 0$ outside of the event $\{0 < \xi_u < L(u)\ \forall u \leq s\}$, it follows from Lemmas 7, 8, and 9 that there are constants c_7 and c_8 such that

$$e^{\sqrt{2}x} B(s)^{-1}\, \mathbb{E}_x[V(s)] \exp\left(-c_7(t-s)^{-1/3}\right) \\ \leq \mathbb{E}_x[Z(s)] \leq e^{\sqrt{2}x} B(s)^{-1}\, \mathbb{E}_x[V(s)] \exp(c_8(t-s)^{-1/3}).$$

Because $(V(s), s \geq 0)$ is a martingale,

$$e^{\sqrt{2}x}\, \mathbb{E}_x[V(s)] = e^{\sqrt{2}x} V(0) = e^{\sqrt{2}x} G(0) \sin\left(\frac{\pi x}{L(0)}\right) = Z(0)G(0).$$

The result follows because

$$1 \geq G(0) = \exp\left(\frac{L'(0)}{2L(0)}(\xi_0 - L(0)/2)^2\right) \geq \exp\left(\frac{L'(0)L(0)}{8}\right) = \exp\left(-\frac{c^2}{24} t^{-1/3}\right).$$

□

Note that because $B(s)$ and the constants c_5 and c_6 do not depend on the position x of the initial particle, Proposition 10 and the corollary below hold for general initial configurations by summing over the particles.

Corollary 11 *Let $(\mathcal{F}_u, u \geq 0)$ be the natural filtration associated with the branching Brownian motion. Let $0 < r < s < t$. Let*

$$B_r(s) = \exp\left(\int_r^s \frac{\pi^2}{2L(u)^2}\, du - \int_r^s \frac{L'(u)}{2L(u)}\, du\right) \\ = \exp\left((3\pi^2)^{1/3}((t-r)^{1/3} - (t-s)^{1/3})\right)\left(\frac{t-r}{t-s}\right)^{1/6}.$$

Then

$$Z(r)B_r(s)^{-1} \exp(-c_5(t-s)^{-1/3}) \leq \mathbb{E}[Z(s)|\mathcal{F}_r] \leq Z(r)B_r(s)^{-1} \exp(c_6(t-s)^{-1/3}),$$

where c_5 and c_6 are the constants from Proposition 10.

Proof Apply the Markov Property at time r, and then apply Proposition 10 with $t^* = t - r$ and $L^*(u) = c(t^* - u)^{1/3} = c(t - r - u)^{1/3} = L(u + r)$. □

3.2 Bounding the density

We now use the estimate of $\mathbb{E}[Z(s)]$ from Proposition 10 to obtain bounds on the density. For $0 \leq r < s < t$, let $q_{r,s}(x, y)$ represent the density of particles at time s

that are descended from a particle at the location x at time r. That is, if A is a Borel subset of $(0, L(s))$, then the expected number of particles in A at time s descended from the particle which is at x at time r is

$$\int_A q_{r,s}(x, y)\, dy.$$

Note that $q_s(x, y) = q_{0,s}(x, y)$. For $x, y > 0$ and $0 \le r \le s \le t$, let

$$\psi_{r,s}(x, y) = \frac{1}{L(s)} e^{-(3\pi^2)^{1/3}((t-r)^{1/3}-(t-s)^{1/3})} \left(\frac{t-s}{t-r}\right)^{1/6} \times e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L(r)}\right) e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L(s)}\right).$$

This expression becomes simpler if we view the process from time t, as we get

$$\psi_{t-u,t-v}(x, y) = \frac{1}{c} e^{-(3\pi^2)^{1/3}(u^{1/3}-v^{1/3})} \left(\frac{1}{uv}\right)^{1/6} e^{\sqrt{2}x} \sin\left(\frac{\pi x}{cu^{1/3}}\right) e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{cv^{1/3}}\right).$$

Proposition 12 *Fix a positive constant b. There exists a constant $A > 0$ and positive constants C' and C'', with C'' depending on b, such that if $r + L(r)^2 \le s \le t - A$, then*

$$q_{r,s}(x, y) \ge C' \psi_{r,s}(x, y), \tag{18}$$

and if $r + bL(r)^2 \le s \le t - A$, then

$$q_{r,s}(x, y) \le C'' \psi_{r,s}(x, y). \tag{19}$$

Proof Let $\mathbb{E}_{r,x}$ denote expectation for the process starting from a single particle at x at time r. Note that if $r < u < s$, then

$$q_{r,s}(x, y) = \int_0^{L(u)} q_{r,u}(x, z) q_{u,s}(z, y)\, dz. \tag{20}$$

We first prove the upper bound. We may assume $b \le 1$. Assume $r + bL(r)^2 \le s \le t - A$. Let $u = s - bL(s)^2$. Note that $u > r$ because $L(s) < L(r)$. Let $m = -2L'(s) = (2c/3)(t-s)^{-2/3}$. For $u \le v \le s$, let

$$\hat{L}(v) = \begin{cases} L(u) & \text{if } u \le v \le s - m^{-1}(L(u) - L(s)) \\ L(s) + m(s - v) & \text{if } s - m^{-1}(L(u) - L(s)) \le v \le s. \end{cases}$$

Note that $\hat{L}(v) \ge L(v)$ for all $v \in [u, s]$ (see Fig. 1). Therefore, if we define $\hat{q}_{u,s}(z, y)$ in the same way as $q_{u,s}(z, y)$, except that for $v \in [u, s]$, particles are killed when they reach $\hat{L}(v)$ instead of when they reach $L(v)$, then

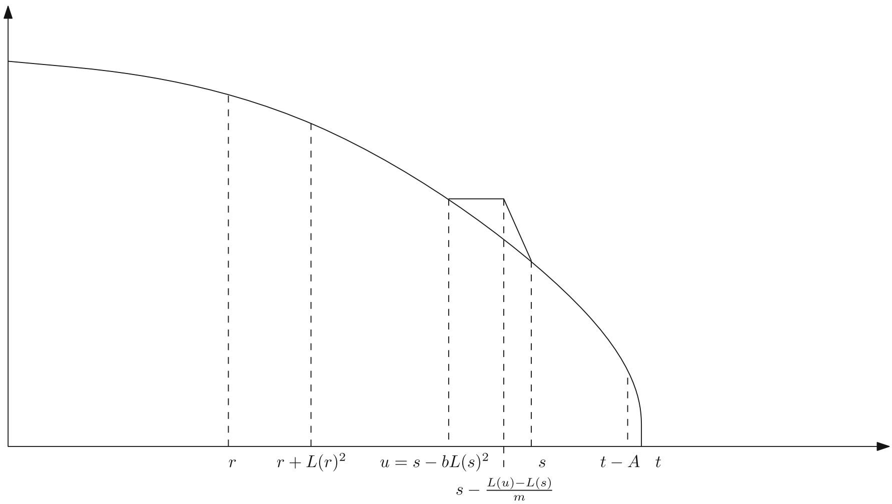

Fig. 1 The function $\hat{L}$

$$q_{u,s}(z,y) \le \hat{q}_{u,s}(z,y). \tag{21}$$

We now wish to apply Lemma 6 with $K = L(s)$, $L = L(u)$ and $t = s - u$. We need to check first that $L(s) + m(s-u)/2 \le 2L(u)$ and second that $m^{-1}(L(u) - L(s)) \le (s-u)/2$. For the first condition, as long as A is chosen to be large enough that $L(t-A) \ge c^3/3$, we have

$$L(s) + \frac{m(s-u)}{2} = L(s) + \frac{mbL(s)^2}{2} = L(s) + \frac{bc^3}{3} \le 2L(s) \le 2L(u).$$

The second condition also holds because

$$m^{-1}(L(u) - L(s)) \le m^{-1}|L'(s)|(s-u) = \frac{s-u}{2}.$$

Therefore, by Lemma 6,

$$\hat{q}_{u,s}(z,y) \le \frac{CL(u)^4}{(bL(s)^2)^{5/2}} e^{\sqrt{2}z} \sin\left(\frac{\pi z}{L(u)}\right) e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L(s)}\right). \tag{22}$$

Note that

$$L(u) - L(s) \le -L'(s)(s-u) = \frac{bc^3}{3}. \tag{23}$$

Therefore, if A is large enough that $L(t-A) \ge c^3/3$, then $L(u) \le 2L(s)$, so combining (20), (21), (22), we get

$$\begin{aligned} q_{r,s}(x,y) &\le \frac{C}{L(s)} e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L(s)}\right) \int_0^{L(u)} e^{\sqrt{2}z} \sin\left(\frac{\pi z}{L(u)}\right) q_{r,u}(x,z)\,dz \\ &= \frac{C}{L(s)} e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L(s)}\right) \mathbb{E}_{r,x}[Z(u)], \end{aligned}$$

where above, we recall that the subscript indicates that we start with one particle at time r and at position x. Therefore, using Corollary 11 to bound $\mathbb{E}_{r,x}[Z(u)]$,

$$\begin{aligned} q_{r,s}(x,y) \le \frac{C}{L(s)} e^{-(3\pi^2)^{1/3}((t-r)^{1/3}-(t-u)^{1/3})} \left(\frac{t-u}{t-r}\right)^{1/6} \\ \times e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L(r)}\right) e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L(s)}\right). \end{aligned}$$

The upper bound (19) now follows because $(t-u)^{1/3} \le (t-s)^{1/3} + bc^2/3$ by (23) and $t-u = (t-s)+(s-u) \le C(t-s)$.

We next prove the lower bound. Assume that $r + L(r)^2 \le s \le t - A$. Let $u = s - L(s)^2/2$. Note that $u > r$ because $L(s) < L(r)$. For $0 \le z \le L(s)$, define $\tilde{q}_{u,s}(z,y)$ in the same way as $q_{u,s}(z,y)$ except that for $v \in [u,s]$, particles are killed when they reach $L(s)$ instead of when they reach $L(v)$. Then

$$q_{u,s}(z,y) \ge \tilde{q}_{u,s}(z,y). \tag{24}$$

By Lemma 3, if $0 \le z \le L(s)$, then because

$$\sum_{n=2}^{\infty} n^2 e^{-\pi^2(n^2-1)(s-u)/2L(s)^2} = \sum_{n=2}^{\infty} n^2 e^{-\pi^2(n^2-1)/4} < 1,$$

we have

$$\tilde{q}_{u,s}(z,y) \ge \frac{C}{L(s)} e^{-\pi^2(s-u)/2L(s)^2} e^{\sqrt{2}z} \sin\left(\frac{\pi z}{L(s)}\right) e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L(s)}\right). \tag{25}$$

By (20), (24), and (25),

$$q_{r,s}(x,y) \ge \frac{C}{L(s)} e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L(s)}\right) \int_0^{L(s)} e^{\sqrt{2}z} \sin\left(\frac{\pi z}{L(s)}\right) q_{r,u}(x,z)\,dz.$$

Using (23) with $b = 1/2$, we get $L(u) - L(s) \le c^3/6$. Therefore, there is a positive constant C such that $\sin(\pi z/L(s)) \ge C \sin(\pi z/L(u))$ for all $z \le L(u) - c^3$. It follows that

$$q_{r,s}(x, y) \geq \frac{C}{L(s)} e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L(s)}\right) \left(\mathbb{E}_{r,x}[Z(u)] - \int_{L(u)-c^3}^{L(u)} e^{\sqrt{2}z} \sin\left(\frac{\pi z}{L(u)}\right) q_{r,u}(x, z)\, dz\right). \tag{26}$$

By Corollary 11,

$$\mathbb{E}_{r,x}[Z(u)] \geq C e^{-(3\pi^2)^{1/3}((t-r)^{1/3}-(t-u)^{1/3})} \left(\frac{t-u}{t-r}\right)^{1/6} e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L(r)}\right). \tag{27}$$

Also, because $u - r = s - L(s)^2/2 - r \geq L(r)^2 - L(s)^2/2 \geq L(r)^2/2$, we can apply the upper bound (19) to get

$$\begin{aligned} &\int_{L(u)-c^3}^{L(u)} e^{\sqrt{2}z} \sin\left(\frac{\pi z}{L(u)}\right) q_{r,u}(x, z)\, dz \\ &\leq \frac{C}{L(u)} e^{-(3\pi^2)^{1/3}((t-r)^{1/3}-(t-u)^{1/3})} \left(\frac{t-u}{t-r}\right)^{1/6} \\ &\quad \times e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L(r)}\right) \int_{L(u)-c^3}^{L(u)} \sin\left(\frac{\pi z}{L(u)}\right)^2 dz \\ &\leq \frac{C}{L(u)^3} e^{-(3\pi^2)^{1/3}((t-r)^{1/3}-(t-u)^{1/3})} \left(\frac{t-u}{t-r}\right)^{1/6} e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L(r)}\right). \end{aligned} \tag{28}$$

Choosing A sufficiently large, the lower bound (18) now follows from (26), (27), (28), and the fact that $t - u \geq t - s$. □

3.3 Particles hitting the right boundary

For $0 \leq s < u \leq t$, let $R_{s,u}$ denote the number of particles that are killed at $L(r)$ for some $r \in [s, u]$. Let $\mathbb{E}_{s,x}$ denote expectation for the process started from a single particle at x at time s.

Lemma 13 *If* $0 \leq s < u < t$, *then*

$$\mathbb{E}_{s,x}[R_{s,u}] \leq \frac{x e^{\sqrt{2}x} e^{-\sqrt{2}L(u)}}{L(u)}.$$

Proof For branching Brownian motion with absorption only at the origin, if we define

$$M(s) = \sum_{i=1}^{N(s)} X_i(s) e^{\sqrt{2}X_i(s)},$$

then it is well-known (see, for example, Lemma 2 of [16]) that the process $(M(s), s \geq 0)$ is a martingale. Now, for $u \in [s, t]$, let

$$M_s(u) = \sum_{i=1}^{N(u)} X_i(u) e^{\sqrt{2} X_i(u)} + L(u) e^{\sqrt{2} L(u)} R_{s,u}. \tag{29}$$

We claim that the process $(M_s(u), s \leq u \leq t)$ is a supermartingale for branching Brownian motion with killing both at the origin and at the right boundary $L(\cdot)$. To see this, observe that because the process $(M(s), s \geq 0)$ is a martingale when there is no killing at the right boundary, this process would still be a martingale if particles were stopped, but not killed, upon reaching the right boundary. (Essentially, this follows from the simple fact that a martingale, stopped at a stopping time, is still a martingale. See for instance Theorem 5.1 and Corollary 5.4 in [20] for a related statement and proof.) Because the function $u \mapsto L(u)$ is decreasing and because $x \mapsto x e^{\sqrt{2}x}$ is increasing, the process becomes a supermartingale if particles, after hitting the right boundary, follow the right boundary until time t. This is the process defined in (29) because there will be $R_{s,u}$ particles at $L(u)$ at time u.

Because the process defined in (29) is a supermartingale, we have

$$\begin{aligned} x e^{\sqrt{2}x} &= \mathbb{E}_{s,x}[M_s(s)] \geq \mathbb{E}_{s,x}[M_s(u)] \geq \mathbb{E}_{s,x}[L(u) e^{\sqrt{2} L(u)} R_{s,u}] \\ &= L(u) e^{\sqrt{2} L(u)}\, \mathbb{E}_{s,x}[R_{s,u}]. \end{aligned}$$

The result follows. □

Lemma 14 *There is a constant $A > 0$ such that for all s, u, and x such that $s \geq 0$, $0 < x < L(s)$, and $s + L(s)^2 \leq u \leq t - A$, we have*

$$\mathbb{E}_{s,x}[R_{u,u+1}] \asymp \frac{1}{L(u)^2} e^{-(3\pi^2)^{1/3}(t-s)^{1/3}} \left(\frac{t-u}{t-s}\right)^{1/6} e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L(s)}\right).$$

Proof We adapt ideas from the proofs of Lemma 15 and Proposition 16 in [5]. By applying the Markov property at time u, we get

$$\mathbb{E}_{s,x}[R_{u,u+1}] = \int_0^{L(u)} q_{s,u}(x, y)\, \mathbb{E}_{u,y}[R_{u,u+1}]\, dy. \tag{30}$$

Let $(\xi_r)_{r \geq 0}$ be standard Brownian motion with $\xi_0 = 0$. Because a particle at time u will have on average e descendants at time $u + 1$ if no particles are killed, the expectation $\mathbb{E}_{u,y}[R_{u,u+1}]$ is bounded above by e times the probability that a particle started from y at time u is to the right of $L(u + 1)$ at some time before time $u + 1$. Therefore, it follows from the Reflection Principle and the inequality

$$\int_z^{\infty} e^{-x^2/2}\, dx \leq z^{-1} e^{-z^2/2},$$

valid for $z > 0$, that if $y < L(u+1)$, then

$$\begin{aligned}\mathbb{E}_{u,y}[R_{u,u+1}] &\leq e\ \mathbb{P}\left(\max_{0\leq r\leq 1}(\xi_r - \sqrt{2}r) \geq L(u+1) - y\right)\\ &\leq e\ \mathbb{P}\left(\max_{0\leq r\leq 1}\xi_r \geq L(u+1) - y\right)\\ &\leq 2e\ \mathbb{P}(\xi_1 \geq L(u+1) - y)\\ &\leq \frac{C}{L(u+1) - y}e^{-(L(u+1)-y)^2/2}.\end{aligned}$$

Therefore, letting $\alpha = L(u) - L(u+1)$ and requiring A to be large enough that $L(t - A + 1) > 1$, we have that $0 \leq \alpha \leq C$ and (using the change of variable $z = L(u) - y$)

$$\begin{aligned}&\int_0^{L(u)-\alpha-1} e^{-\sqrt{2}y}\sin\left(\frac{\pi y}{L(u)}\right)\mathbb{E}_{u,y}[R_{u,u+1}]\,dy\\ &\leq C\int_0^{L(u)-\alpha-1} e^{-\sqrt{2}y}\sin\left(\frac{\pi y}{L(u)}\right)\frac{1}{L(u+1)-y}e^{-(L(u+1)-y)^2/2}\,dy\\ &\leq Ce^{-\sqrt{2}L(u)}\int_{\alpha+1}^{L(u)} e^{\sqrt{2}z}\cdot\frac{\pi z}{L(u)}\cdot\frac{1}{z-\alpha}e^{-(z-\alpha)^2/2}\,dz\\ &\leq \frac{Ce^{-\sqrt{2}L(u)}}{L(u)}.\end{aligned} \tag{31}$$

Using the bound $\mathbb{E}_{u,y}[R_{u,u+1}] \leq e$, we get

$$\int_{L(u)-\alpha-1}^{L(u)} e^{-\sqrt{2}y}\sin\left(\frac{\pi y}{L(u)}\right)\mathbb{E}_{u,y}[R_{u,u+1}]\,dy \leq \frac{Ce^{-\sqrt{2}L(u)}}{L(u)}. \tag{32}$$

Combining (31) and (32) with (30) and Proposition 12, and using the fact that $e^{-\sqrt{2}L(u)} = e^{-(3\pi^2)^{1/3}(t-u)^{1/3}}$, we get, for A large enough,

$$\mathbb{E}_{s,x}[R_{u,u+1}] \leq \frac{C}{L(u)^2}e^{-(3\pi^2)^{1/3}(t-s)^{1/3}}\left(\frac{t-u}{t-s}\right)^{1/6}e^{\sqrt{2}x}\sin\left(\frac{\pi x}{L(s)}\right),$$

which is the upper bound in the statement of the lemma.

Next, observe that for $y \in [L(u) - 1, L(u)]$, we have

$$\mathbb{E}_{u,y}[R_{u,u+1}] \geq \mathbb{P}(\xi_1 - \sqrt{2} \geq L(u+1) - y) \geq \mathbb{P}(\xi_1 \geq 1 + \sqrt{2}) \geq C.$$

Thus, by (30) and Proposition 12,

$$\mathbb{E}_{s,x}[R_{u,u+1}] \geq \frac{C}{L(u)} e^{-(3\pi^2)^{1/3}((t-s)^{1/3}-(t-u)^{1/3})} \left(\frac{t-u}{t-s}\right)^{1/6}$$
$$\times e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L(s)}\right) \int_{L(u)-1}^{L(u)} e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L(u)}\right) dy$$
$$\geq \frac{C}{L(u)^2} e^{-(3\pi^2)^{1/3}(t-s)^{1/3}} \left(\frac{t-u}{t-s}\right)^{1/6} e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L(s)}\right),$$

which gives the required lower bound. □

Lemma 15 *There is a constant $A_0 > 0$ and positive constants C' and C'' such that if $0 \leq s \leq t - A_0$ and $0 < x < L(s)$, then*

$$C'h(s,x) \leq \mathbb{E}_{s,x}[R_{s,t}] \leq C''(h(s,x) + j(s,x)), \tag{33}$$

where

$$h(s,x) = e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L(s)}\right)(t-s)^{1/3} \exp(-(3\pi^2(t-s))^{1/3}) \tag{34}$$

and

$$j(s,x) = xe^{\sqrt{2}x}(t-s)^{-1/3} \exp(-(3\pi^2(t-s))^{1/3}).$$

Also, if $0 < \alpha < \beta < 1$, then

$$C'h(s,x) \leq \mathbb{E}_{s,x}[R_{s+\alpha(t-s),s+\beta(t-s)}] \leq C''h(s,x), \tag{35}$$

where the constants C' and C'' depend on α and β.

Proof If $u = s + L(s)^2$, then for sufficiently large A_0,

$$L(s) - L(u) \leq -L'(u)(u-s) = \frac{c^3}{3}\left(\frac{t-s}{t-u}\right)^{2/3} \leq C.$$

Therefore, by Lemma 13, using that $\sqrt{2}L(s) = (3\pi^2(t-s))^{1/3}$,

$$0 \leq \mathbb{E}_{s,x}[R_{s,s+L(s)^2}] \leq \frac{Cxe^{\sqrt{2}x}e^{-\sqrt{2}L(s)}}{L(s)} \leq Cj(s,x). \tag{36}$$

We may choose A_0 to be large enough that $s + L(s)^2 \leq t - A - 1$ whenever $0 \leq s \leq t - A_0$, where A is the constant from Lemma 14. By Lemma 14,

$$\begin{aligned}\mathbb{E}_{s,x}[R_{s+L(s)^2,t-A}] &\asymp \frac{e^{-(3\pi^2)^{1/3}(t-s)^{1/3}}}{(t-s)^{1/6}} e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L(s)}\right) \int_{s+L(s)^2}^{t-A} \frac{(t-u)^{1/6}}{L(u)^2}\, du \\ &\asymp \frac{e^{-(3\pi^2)^{1/3}(t-s)^{1/3}}}{(t-s)^{1/6}} e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L(s)}\right) \int_{s+L(s)^2}^{t-A} \frac{1}{(t-u)^{1/2}}\, du \\ &\asymp h(s,x). \end{aligned} \tag{37}$$

Because particles branch at rate one, $\mathbb{E}_{s,x}[R_{t-A,t}]$ is at most e^A times the expected number of particles between 0 and $L(t-A)$ at time $t-A$. Therefore, by Proposition 12,

$$\begin{aligned}\mathbb{E}_{s,x}[R_{t-A,t}] &\le e^A \int_0^{L(t-A)} q_{s,t-A}(x,y)\, dy \\ &\le \frac{Ce^{-(3\pi^2)^{1/3}(t-s)^{1/3}}}{(t-s)^{1/6}} e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L(s)}\right) \int_0^{L(t-A)} e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L(s)}\right) dy \\ &\le \frac{Ch(s,x)}{(t-s)^{5/6}}. \end{aligned} \tag{38}$$

The result (33) follows from (36), (37), and (38). The result (35) follows from the reasoning in (37), using $s+\alpha(t-s)$ and $s+\beta(t-s)$ as the limits of integration. □

Lemma 16 *Let* $0 < \alpha < \beta < 1$. *Let* A_0 *be the constant defined in Lemma* 15. *Then there exist positive constants* C' *and* C'' *depending on* α *and* β *such that if* $t \ge A_0$ *and* $0 < x < L(0) - 1$, *then*

$$\mathbb{E}_{0,x}[R^2_{\alpha t,\beta t}] \le Ch(0,x).$$

Proof The proof is similar to the proof of Proposition 18 in [5]. Throughout this proof, we write $R = R_{\alpha t,\beta t}$. Note that $R^2 = R + 2Y$, where Y is the number of distinct pairs of particles that reach $L(s)$ for some $s \in [\alpha t, \beta t]$. A branching event at the location y at time s produces, on average, $(\mathbb{E}_{s,y}[R])^2$ pairs of particles that reach the right boundary and have their most recent common ancestor at time s. Therefore, by Lemma 15, we may write

$$\begin{aligned}\mathbb{E}_{0,x}[R^2] &= \mathbb{E}_{0,x}[R] + 2\int_0^{\beta t}\int_0^{L(s)} q_{0,s}(x,y)(\mathbb{E}_{s,y}[R])^2\, dy\, ds \\ &\le \mathbb{E}_{0,x}[R] + C\int_0^{\beta t}\int_0^{L(s)} q_{0,s}(x,y)(h(s,y)^2 + j(s,y)^2)\, dy\, ds. \end{aligned} \tag{39}$$

We bound separately the term involving $h(s, y)^2$ and the term involving $j(s, y)^2$. We also treat separately the cases $s \le L(0)^2$ and $s \ge L(0)^2$.

By Proposition 12 and (34),

$$\begin{aligned}
&\int_{L(0)^2}^{\beta t} \int_0^{L(s)} q_{0,s}(x, y) h(s, y)^2 \, dy \, ds \\
&\le C e^{-(3\pi^2)^{1/3} t^{1/3}} e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L(0)}\right) \int_{L(0)^2}^{\beta t} \int_0^{L(s)} \frac{1}{L(s)} \left(\frac{t-s}{t}\right)^{1/6} e^{(3\pi^2)^{1/3}(t-s)^{1/3}} \\
&\quad \times e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L(s)}\right) \left\{(t-s)^{1/3} e^{-(3\pi^2)^{1/3}(t-s)^{1/3}} e^{\sqrt{2}y} \sin\left(\frac{\pi y}{L(s)}\right)\right\}^2 \, dy \, ds \\
&\le \frac{C e^{-(3\pi^2)^{1/3} t^{1/3}}}{t^{1/6}} e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L(0)}\right) \int_{L(0)^2}^{\beta t} e^{-(3\pi^2)^{1/3}(t-s)^{1/3}} (t-s)^{1/2} \\
&\quad \times \int_0^{L(s)} e^{\sqrt{2}y} \sin^3\left(\frac{\pi y}{L(s)}\right) \, dy \, ds \\
&\le \frac{C e^{-(3\pi^2)^{1/3} t^{1/3}}}{t^{1/6}} e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L(0)}\right) \int_{L(0)^2}^{\beta t} e^{-(3\pi^2)^{1/3}(t-s)^{1/3}} (t-s)^{1/2} \frac{e^{\sqrt{2}L(s)}}{L(s)^3} \, ds \\
&\le \frac{C e^{-(3\pi^2)^{1/3} t^{1/3}}}{t^{1/6}} e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L(0)}\right) \int_{L(0)^2}^{\beta t} \frac{1}{(t-s)^{1/2}} \, ds \\
&\le C h(0, x).
\end{aligned} \tag{40}$$

A similar computation gives

$$\begin{aligned}
&\int_{L(0)^2}^{\beta t} \int_0^{L(s)} q_{0,s}(x, y) j(s, y)^2 \, dy \, ds \\
&\le C e^{-(3\pi^2)^{1/3} t^{1/3}} e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L(0)}\right) \int_{L(0)^2}^{\beta t} \int_0^{L(s)} \frac{1}{L(s)} \left(\frac{t-s}{t}\right)^{1/6} e^{(3\pi^2)^{1/3}(t-s)^{1/3}} \\
&\quad \times e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L(s)}\right) \left\{(t-s)^{-1/3} e^{-(3\pi^2)^{1/3}(t-s)^{1/3}} y e^{\sqrt{2}y}\right\}^2 \, dy \, ds
\end{aligned}$$

$$\leq \frac{Ce^{-(3\pi^2)^{1/3}t^{1/3}}}{t^{1/6}} e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L(0)}\right) \int_{L(0)^2}^{\beta t} e^{-(3\pi^2)^{1/3}(t-s)^{1/3}} \frac{1}{(t-s)^{5/6}}$$
$$\times \int_0^{L(s)} e^{\sqrt{2}y} y^2 \sin\left(\frac{\pi y}{L(s)}\right) ds$$
$$\leq \frac{Ce^{-(3\pi^2)^{1/3}t^{1/3}}}{t^{1/6}} e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L(0)}\right) \int_{L(0)^2}^{\beta t} \frac{1}{(t-s)^{1/2}} ds$$
$$\leq Ch(0,x). \tag{41}$$

It remains to bound from above the two integrals between 0 and $L(0)^2$. If $0 \leq s \leq L(0)^2$, then $t^{1/3} - (t-s)^{1/3} \leq C$, and $\sin(\pi y/L(s)) \leq C \sin(\pi y/L(0))$ for all $y \in [0, L(s)]$. Also, because $q_{0,s}(x, y)$ is bounded above by the density that would be obtained if particles were killed at $L(0)$, rather than $L(r)$, for $r \in [0, s]$, Lemma 4 implies that

$$\int_0^{L(0)^2} q_{0,s}(x,y)\, ds \leq \frac{2e^{\sqrt{2}(x-y)} x(L(0)-y)}{L(0)}.$$

Thus

$$\int_0^{L(0)^2} \int_0^{L(s)} q_{0,s}(x,y) h(s,y)^2 \, dy\, ds$$
$$\leq C \int_0^{L(0)^2} \int_0^{L(s)} q_{0,s}(x,y) \left\{ e^{\sqrt{2}y} \sin\left(\frac{\pi y}{L(s)}\right) (t-s)^{1/3} e^{-(3\pi^2)^{1/3}(t-s)^{1/3}} \right\}^2 dy\, ds$$
$$\leq Ce^{-2(3\pi^2)^{1/3}t^{1/3}} t^{2/3} \int_0^{L(0)} e^{2\sqrt{2}y} \sin^2\left(\frac{\pi y}{L(0)}\right) \left(\int_0^{L(0)^2} q_{0,s}(x,y)\, ds \right) dy$$
$$\leq Cxe^{\sqrt{2}x} e^{-2(3\pi^2)^{1/3}t^{1/3}} t^{2/3} \int_0^{L(0)} e^{\sqrt{2}y} \sin^2\left(\frac{\pi y}{L(0)}\right) \frac{L(0)-y}{L(0)} dy$$
$$\leq Cxe^{\sqrt{2}x} e^{-2(3\pi^2)^{1/3}t^{1/3}} t^{2/3} \cdot \frac{e^{\sqrt{2}L(0)}}{L(0)^3}$$
$$\leq Cxe^{\sqrt{2}x} e^{-(3\pi^2)^{1/3}t^{1/3}} t^{-1/3}.$$

Because

$$xt^{-1/3} \leq Ct^{1/3} \sin(\pi x/L(0)) \tag{42}$$

when $0 < x < L(0) - 1$, it follows that

$$\int_0^{L(0)^2} \int_0^{L(s)} q_{0,s}(x, y) h(s, y)^2 \, dy \, ds \leq Ch(0, x). \tag{43}$$

Likewise, using that $y(t-s)^{-1/3} \leq C$ for $y \leq L(s)$, we get

$$\begin{aligned}
&\int_0^{L(0)^2} \int_0^{L(s)} q_{0,s}(x, y) j(s, y)^2 \, dy \, ds \\
&\leq C \int_0^{L(0)^2} \int_0^{L(s)} q_{0,s}(x, y) \left\{ e^{\sqrt{2}y} e^{-(3\pi^2)^{1/3}(t-s)^{1/3}} \right\}^2 dy \, ds \\
&\leq C e^{-2(3\pi^2)^{1/3} t^{1/3}} \int_0^{L(0)} e^{2\sqrt{2}y} \left(\int_0^{L(0)^2} q_{0,s}(x, y) \, ds \right) dy \\
&\leq C x e^{\sqrt{2}x} e^{-2(3\pi^2)^{1/3} t^{1/3}} \int_0^{L(0)} e^{\sqrt{2}y} \cdot \frac{L(0) - y}{L(0)} \, dy \\
&\leq C x e^{\sqrt{2}x} e^{-(3\pi^2)^{1/3} t^{1/3}} t^{-1/3}. \\
&\leq Ch(0, x).
\end{aligned} \tag{44}$$

The result follows from (39), (40), (41), (43), (44), and Lemma 15. □

Corollary 17 *Let A_0 be the constant defined in Lemma 15. If there is a single particle at x at time zero, where $0 < x < L(0) - 1$, then for $t \geq A_0$,*

$$\mathbb{P}_x(R_{0,t} > 0) \asymp e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L(0)}\right) t^{1/3} \exp(-(3\pi^2 t)^{1/3}).$$

Likewise, if $0 < \alpha < \beta < 1$, then there are positive constants $C'_{\alpha,\beta}$ and $C''_{\alpha,\beta}$, depending on α and β such that for all $t \geq A_0$,

$$\begin{aligned}
&C'_{\alpha,\beta} e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L(0)}\right) t^{1/3} \exp(-(3\pi^2 t)^{1/3}) \\
&\quad \leq \mathbb{P}_x(R_{\alpha t,\beta t} > 0) \leq C''_{\alpha,\beta} e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L(0)}\right) t^{1/3} \exp(-(3\pi^2 t)^{1/3}).
\end{aligned}$$

Proof Note that $j(0, x) \leq Ch(0, x)$ when $x < L(0) - 1$ by (42). Therefore, by Lemma 15 with $s = 0$ and Markov's Inequality,

$$\mathbb{P}_x(R_{\alpha t,\beta t} > 0) \leq \mathbb{P}_x(R_{0,t} > 0) \leq \mathbb{E}_x[R_{0,t}] \leq C(h(0, x) + j(0, x)) \leq Ch(0, x).$$

For the lower bound, we use a standard second moment argument and apply Lemmas 15 and 16 to get

$$\mathbb{P}_x(R_{0,t} > 0) \geq \mathbb{P}_x(R_{\alpha t,\beta t} > 0) \geq \frac{(\mathbb{E}_{0,x}[R_{\alpha t,\beta t}])^2}{\mathbb{E}_{0,x}[R^2_{\alpha t,\beta t}]} \geq \frac{Ch(0, x)^2}{h(0, x)} = Ch(0, x).$$

The result follows. □

4 Proofs of main results

In this section, we prove Theorems 1 and 2. The key to these proofs is Proposition 20 below. We first recall the following result due to Neveu [22].

Lemma 18 *Consider branching Brownian motion with drift $-\sqrt{2}$ and no absorption, started with a single particle at the origin. For each $y \geq 0$, let $K(y)$ be the number of particles that reach $-y$ in a modified process in which particles are killed upon reaching $-y$. Then there exists a random variable W, with $\mathbb{P}(0 < W < \infty) = 1$ and $\mathbb{E}[W] = \infty$, such that*

$$\lim_{y \to \infty} ye^{-\sqrt{2}y} K(y) = W \ a.s.$$

To prove Proposition 20, we will use the following result about the survival probability of a Galton–Watson process, which is Lemma 13 of [4].

Lemma 19 *Let $(p_k)_{k=0}^{\infty}$ be a sequence of nonnegative numbers that sum to 1, and let X be a random variable such that $\mathbb{P}(X = k) = p_k$ for all nonnegative integers k. Let q be the extinction probability of a Galton–Watson process with offspring distribution $(p_k)_{k=1}^{\infty}$ started with a single individual. Then*

$$1 - q \geq \frac{2(\mathbb{E}[X] - 1)}{\mathbb{E}[X(X - 1)]}.$$

Proposition 20 *Fix $t > 0$, and suppose that initially there is a single particle at $x = ct^{1/3}$. Then there are constants $A > 0$ and $C > 0$ such that if $t \geq A$, the probability that there is at least one particle remaining at time t is at least C.*

Proof We prove this result by constructing a branching process that resembles a discrete-time Galton–Watson process but allows individuals to have different offspring distributions. We will show that the probability that this branching process survives is bounded below by a positive constant, and that survival of this branching process implies that the branching Brownian motion survives until at least time $t - A$. This will in turn give the branching Brownian motion a positive probability of surviving until time t, which will imply the result.

Let $C' = C'_{1/3,2/3}$, where $C'_{1/3,2/3}$ is the constant from Corollary 17 with $\alpha = 1/3$ and $\beta = 2/3$. Consider the setting of Lemma 18, in which we have branching Brownian motion with drift $-\sqrt{2}$ and no absorption. For $y > 0$, let $K(y)$ denote the number of particles that reach $-y$, if particles are killed upon reaching $-y$. For $\xi > 0$, let $K_\xi(y)$ be the number of these particles that reach y before time ξ. Because the random variable W in Lemma 18 has infinite expected value, it follows from Lemma 18 and Fatou's Lemma that we can choose $y > 0$ sufficiently large that

$$\mathbb{E}[ye^{-\sqrt{2}y}K(y)] \geq \frac{3 \cdot 2^{1/3}c}{C'}.$$

We can then choose a real number $\xi > 0$ and a positive integer M sufficiently large that

$$\mathbb{E}[ye^{-\sqrt{2}y}(K_\xi(y) \wedge M)] \geq \frac{2 \cdot 2^{1/3}c}{C'}. \tag{45}$$

Let A_0 be defined as in Corollary 17. Choose A to be large enough that the following hold:

$$A \geq \max\{A_0 + \xi, 2\xi\} \tag{46}$$
$$cA^{1/3} \geq 2y \tag{47}$$
$$cA^{1/3} - c(A-\xi)^{1/3} \leq \frac{y}{2}. \tag{48}$$

Let $t \geq A$, and let $L(s) = c(t-s)^{1/3}$ for $0 \leq s \leq t$.

We now construct the branching process inductively (see Fig. 2). Let $T_0 = \{0\}$. Suppose that $T_n = \{t_{n,1}, t_{n,2}, \ldots, t_{n,m_n}\}$, which will imply that at the nth stage of the process, there are particles at positions $L(t_{n,1}), \ldots, L(t_{n,m_n})$ at times $t_{n,1}, \ldots, t_{n,m_n}$. For $j = 1, 2, \ldots, m_n$, if $t_{n,j} \geq t - A$, then we put $t_{n,j}$ in the set T_{n+1}. If $t_{n,j} < t - A$, then we follow the trajectories after time $t_{n,j}$ of the descendants of the particle that reached $L(t_{n,j})$ at time $t_{n,j}$ until either time $t_{n,j} + \xi$, or until the descendant particles reach $L(t_{n,j}) - y$, which is positive by (47). Denote the times, before time $t_{n,j} + \xi$, at which descendant particles reach $L(t_{n,j}) - y$ by $u_{n,j,1} < \cdots < u_{n,j,\ell_{n,j}}$. For $\ell = 1, \ldots, \ell_{n,j} \wedge M$, if at least one descendant of the particle that reaches $L(t_{n,j}) - y$ at time $u_{n,j,\ell}$ later reaches $L(s)$ at some time $s \in [u_{n,j,\ell} + (t - u_{n,j,\ell})/3, u_{n,j,\ell} + 2(t - u_{n,j,\ell})/3]$, then we put the smallest time s at which this occurs in the set T_{n+1}. For $n \geq 0$, let Z_n be the cardinality of T_n.

The next step is to obtain bounds on the moments of Z_1 which are valid for all $t \geq A$. Write $u_i = u_{0,1,i}$. Then particles reach $L(0) - y$ at times $u_1, \ldots, u_{\ell_{0,1}}$. Observe that

$$Z_1 = \xi_1 + \cdots + \xi_{\ell_{0,1} \wedge M}, \tag{49}$$

where $\xi_i = 1$ if the particle that reaches $L(0) - y$ at time u_i has a descendant that reaches $L(s)$ at some time $s \in [u_i + (t-u_i)/3, u_i + 2(t-u_i)/3]$ and $\xi_i = 0$ otherwise.

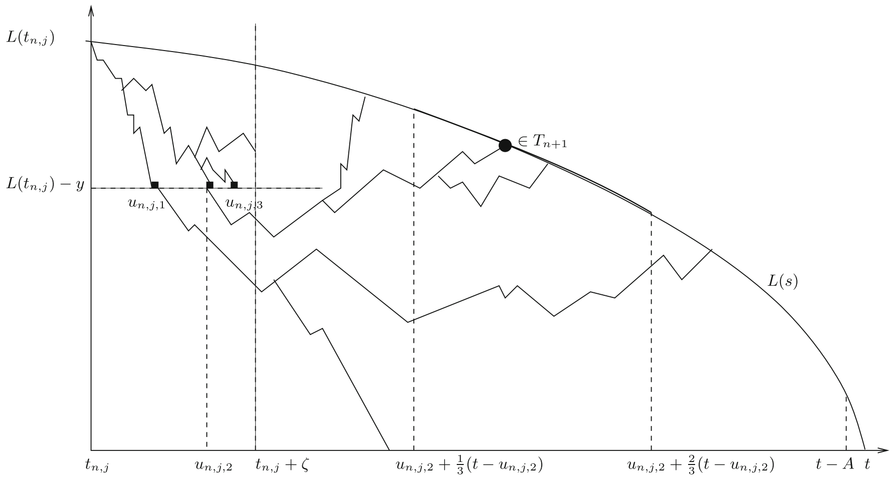

Fig. 2 Construction of the branching process T_n, $n \geq 1$. Here we look at particle j in generation n alive at time $t_{n,j}$. It has three descendants that hit level $L(t_{n,j}) - y$ figured by three squares. The second particle has a descendant that hits L between time $u_{n,j,2} + (t - u_{n,j,2})/3$ and $u_{n,j,2} + 2(t - u_{n,j,2})/3$). The first of these descendants, indicated by a *black dot*, belongs to T_{n+1}

Let $\mathcal{G}$ be the σ-field generated by $u_1, \ldots, u_{\ell_{0,1}}$. By Corollary 17, if $t \geq A$, then

$$\begin{aligned} &C' e^{\sqrt{2}(x-y)} \sin\left(\frac{\pi(x-y)}{L(u_i)}\right)(t-u_i)^{1/3} \exp(-(3\pi^2(t-u_i))^{1/3}) \\ &\quad \leq \mathbb{P}(\xi_i = 1|\mathcal{G}) \leq C e^{\sqrt{2}(x-y)} \sin\left(\frac{\pi(x-y)}{L(u_i)}\right)(t-u_i)^{1/3} \exp(-(3\pi^2(t-u_i))^{1/3}). \end{aligned} \tag{50}$$

Because $A \geq A_0 + \xi$ by (46), there is a constant C such that if $t \geq A$ then

$$\begin{aligned} 1 = e^{\sqrt{2}x} \exp(-(3\pi^2 t)^{1/3}) &\leq e^{\sqrt{2}x} \exp(-(3\pi^2(t-u_i))^{1/3}) \\ &\leq e^{\sqrt{2}x} \exp(-(3\pi^2(t-\xi))^{1/3}) \leq C. \end{aligned} \tag{51}$$

Because $A \geq 2\xi$ by (46), if $t \geq A$ then

$$(t/2)^{1/3} \leq (t-\xi)^{1/3} \leq (t-u_i)^{1/3} \leq t^{1/3}. \tag{52}$$

Therefore, using again that $A \geq 2\xi$, we get, when $t \geq A$,

$$\begin{aligned} \sin\left(\frac{\pi(x-y)}{L(u_i)}\right) = \sin\left(\frac{\pi(L(u_i)-x+y)}{L(u_i)}\right) &\leq \frac{\pi(L(u_i)-x+y)}{L(u_i)} \\ &\leq \frac{\pi y}{L(u_i)} \leq \frac{\pi y}{c(t-\xi)^{1/3}} \leq \frac{2^{1/3}\pi y}{ct^{1/3}}. \end{aligned} \tag{53}$$

By (48),

$$x - L(u_i) \leq L(0) - L(\xi) = ct^{1/3} - c(t-\xi)^{1/3} \leq y/2$$

for $t \geq A$. Using this result and the fact that $\sin(x) \geq 2x/\pi$ for $0 \leq x \leq \pi/2$, we get

$$\sin\left(\frac{\pi(x-y)}{L(u_i)}\right) = \sin\left(\frac{\pi(L(u_i)-x+y)}{L(u_i)}\right) \geq \frac{2(L(u_i)-x+y)}{L(u_i)} \geq \frac{y}{L(u_i)} \geq \frac{y}{ct^{1/3}}. \tag{54}$$

Combining (50), (51), (52), (53), and (54), we get

$$\frac{C'}{2^{1/3}c} y e^{-\sqrt{2}y} \leq \mathbb{P}(\xi_i = 1|\mathcal{G}) \leq C y e^{-\sqrt{2}y}. \tag{55}$$

Because $\ell_{0,1}$ has the same distribution as $K_\xi(y)$, it follows from (45), (49), and (55) that

$$\mathbb{E}[Z_1] \geq \frac{C'}{2^{1/3}c} y e^{-\sqrt{2}y}\, \mathbb{E}[K_\xi(y) \wedge M] \geq 2. \tag{56}$$

From (49), we see that $Z_1 \leq M$ so

$$\mathbb{E}[Z_1^2] \leq M^2 \leq C. \tag{57}$$

For $n \geq 0$, let $q_{t,n} = \mathbb{P}(T_n = \emptyset)$. Let $q_t = \lim_{n\to\infty} q_{t,n} = \mathbb{P}(T_n = \emptyset \text{ for some } n)$. Let $p_t(k) = \mathbb{P}(Z_1 = k)$. For $z \in [0, 1]$, let

$$\varphi_t(z) = \sum_{k=0}^{\infty} p_t(k) z^k.$$

Let $q_{t,*} = \min\{q \in [0,1] : \varphi_t(q) = q\}$, which is the probability that a Galton–Watson branching process goes extinct if each individual independently has k offspring with probability $p_t(k)$.

Let

$$q_* = \sup_{t>0} q_{t,*}.$$

We claim that for all $t > 0$ and all $n \geq 0$, we have $q_{t,n} \leq q_*$. We prove this claim by induction on n. Because $q_{t,0} = 0$ for all $t > 0$, the claim is clear when $n = 0$. Suppose the claim holds for some n. Then by the induction hypothesis,

$$\mathbb{P}(T_{n+1} = \emptyset | T_1 = \{s_1, \ldots, s_k\}) = \prod_{j=1}^{k} q_{t-s_j,n} \leq q_*^k.$$

Taking expectations of both sides gives

$$q_{t,n+1} \leq \sum_{k=0}^{\infty} p_t(k) q_*^k = \varphi_t(q_*).$$

Because $\varphi_t(q_{t,*}) = q_{t,*}$ and $\varphi_t(1) = 1$, the fact that $z \mapsto \varphi_t(z)$ is nondecreasing and convex implies that if $z \geq q_{t,*}$, then $\varphi_t(z) \leq z$. Therefore, since $q_* \geq q_{t,*}$, we have $\varphi_t(q_*) \leq q_*$. Thus, $q_{t,n+1} \leq q_*$, and the claim follows by induction.

The claim implies that $q_t \leq q_*$ for all $t > 0$. If $0 < t \leq A$, then $p_t(1) = 1$ and thus $q_{t,*} = 0$. If $t \geq A$, then by Lemma 19 and Eqs. (56) and (57),

$$1 - q_{t,*} \geq \frac{2(\mathbb{E}[Z_1] - 1)}{\mathbb{E}[Z_1(Z_1 - 1)]} \geq \frac{2(\mathbb{E}[Z_1] - 1)}{\mathbb{E}[Z_1^2]} \geq C.$$

It follows that $1 - q_* \geq C$, and therefore $1 - q_t \geq C$ for all $t \geq A$.

Thus, there is a constant C such that, for all $t \geq A$, the probability that $T_n \neq \emptyset$ for all n is at least C. However, if $T_n \neq \emptyset$ for all n, then eventually some particle must reach $L(s)$ for some $s \in [t - A, t - A/3]$. The probability that a particle reaching $L(s)$ for some $s \in [t - A, t - A/3]$ survives until time t is bounded below by a constant. The result follows. □

Proof of Theorem 2 We first obtain an upper bound for the extinction time. Let $\beta > 0$, and let $t_+ = t + \beta x^2$ where $t = \tau x^3$. For $0 \leq s \leq t_+$, let $L_+(s) = c(t_+ - s)^{1/3}$. Consider the process in which particles are killed at time s if they reach $L_+(s)$. The probability that the original process survives until time t_+ is bounded above by the probability that a particle is killed at $L_+(s)$ for some $s \in [0, t_+]$. Note that $L_+(0) - x = c(t_+^{1/3} - t^{1/3}) \asymp \beta x^2 t^{-2/3} \asymp \beta$. Therefore, as soon as x is large enough so that $t \geq A_0$ we can apply Corollary 17 to bound the probability that the original process survives until time t_+ by

$$C e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L_+(0)}\right) t_+^{1/3} e^{-(3\pi^2 t_+)^{1/3}}. \tag{58}$$

Observe that furthermore

$$\sin\left(\frac{\pi x}{L_+(0)}\right) \leq \frac{\pi(L_+(0) - x)}{L_+(0)} \leq C\beta t_+^{-1/3}.$$

and

$$\exp(\sqrt{2}x - (3\pi^2 t_+)^{1/3}) = \exp(-(3\pi^2)^{1/3}(t_+^{1/3} - t^{1/3})) \leq e^{-C'\beta},$$

for some positive constant C'. Therefore, the probability in (58) is at most $C\beta e^{-C'\beta}$, which is less than $\varepsilon/2$ for sufficiently large β. For such β, we have

$$\mathbb{P}_x(\zeta < t_+) \geq 1 - \frac{\varepsilon}{2} \tag{59}$$

for sufficiently large x.

To obtain the lower bound on the extinction time, let $t_- = t - \beta x^2$. For $0 \le s \le t_-$, let $L_-(s) = c(t_- - s)^{1/3}$. For $y > 0$ and $\xi > 0$, let $K_\xi(y)$ denote the number of particles that would be killed, if particles were killed upon reaching $x - y$ before time ξ. By Lemma 18, we can choose y and ξ sufficiently large and $\gamma > 0$ sufficiently small that $y \ge 2c^3\beta + 1$ and

$$\mathbb{P}(K_\xi(y) > \gamma y^{-1} e^{\sqrt{2}y}) > 1 - \frac{\varepsilon}{4}. \tag{60}$$

Observe that for sufficiently large x,

$$t_- - \xi = t - \beta x^2 - \xi \ge \frac{t}{2}, \tag{61}$$

which means for all $u \in (0, \xi)$,

$$x - L_-(u) = c[t^{1/3} - (t - \beta x^2 - u)^{1/3}] \le \frac{c}{3}\left(\frac{t}{2}\right)^{-2/3} (\beta x^2 + \xi) \le c^3\beta$$

for sufficiently large x. Because $y \ge c^3\beta + 1$, it follows that

$$x - y \le L_-(u) - 1 \tag{62}$$

for all $u \in (0, \xi)$, if x is sufficiently large.

Now suppose a particle reaches $x - y$ at time $u \in (0, \xi)$. In view of (62), we can apply Corollary 17 to see that the probability that a descendant of this particle reaches $L(s)$ for some $s \in [u, u + (t_- - u)/2]$ is at least

$$Ce^{\sqrt{2}(x-y)} \sin\left(\frac{\pi(x-y)}{L_-(u)}\right) (t_- - u)^{1/3} \exp(-(3\pi^2(t_- - u))^{1/3}). \tag{63}$$

Using that $y \ge 2c^3\beta$ and that $\sin(x) \ge 2x/\pi$ for $0 \le x \le \pi/2$,

$$\begin{aligned}\sin\left(\frac{\pi(x-y)}{L_-(u)}\right) &= \sin\left(\frac{\pi(L_-(u) - x + y)}{L_-(u)}\right) \ge \frac{2(L_-(u) - x + y)}{L_-(u)} \\ &\ge \frac{2(y - c^3\beta)}{ct^{1/3}} \ge \frac{y}{ct^{1/3}}.\end{aligned} \tag{64}$$

Also, for sufficiently large x, we have $t^{1/3} - (t - \beta x^2 - u)^{1/3} \ge (1/3)t^{-2/3} \cdot \beta x^2 = (c^2/3)\beta$, and so

$$\begin{aligned}\exp(-(3\pi^2(t_- - u))^{1/3}) &= \exp(-(3\pi^2 t)^{1/3}) \exp((3\pi^2)^{1/3}[t^{1/3} - (t - \beta x^2 - u)^{1/3}]) \\ &\ge \exp(-(3\pi^2 t)^{1/3}) \exp((3\pi^2)^{1/3} c^2\beta/3).\end{aligned} \tag{65}$$

Recall also that

$$e^{\sqrt{2}x} e^{-(3\pi^2 t)^{1/3}} = 1. \tag{66}$$

By (61), (64), (65), and (66), for sufficiently large x, the probability in (63) is at least

$$Cye^{-\sqrt{2}y}e^{(3\pi^2)^{1/3}c^2\beta/3}, \tag{67}$$

where the constant C does not depend on β. By Proposition 20, the probability that a descendant of this particle survives until time t_- is also bounded below by (67), with a different positive constant C. Therefore, conditional on the event that $K_\xi(y) > \gamma y^{-1}e^{\sqrt{2}y}$, the probability that some particle survives until t_- is at least

$$1-(1-Cye^{-\sqrt{2}y}e^{(3\pi^2)^{1/3}c^2\beta/3})^{\gamma y^{-1}e^{\sqrt{2}y}}.$$

Using the inequality $1-a \le e^{-a}$ for $a \in \mathbb{R}$, we see that this expression is bounded below by

$$1-\exp(-C\gamma e^{(3\pi^2)^{1/3}c^2\beta/3})$$

and therefore is at least $1-\varepsilon/4$ if β is chosen to be large enough. Combining this result with (60) gives that for such β,

$$\mathbb{P}_x(\zeta > t_-) \ge 1-\frac{\varepsilon}{2} \tag{68}$$

for sufficiently large x. The result follows from (59) and (68). □

Proof of Theorem 1 First, suppose that $t \ge \max\{A_0, 2A\}$, where A_0 is the constant from Corollary 17 and A is the constant from Proposition 20. Suppose also that $0 < x < ct^{1/3}-1$. For $0 \le s \le t$, let $L(s) = c(t-s)^{1/3}$. Consider a modification of the branching Brownian motion in which particles, in addition to getting killed at the origin, are killed if they reach $L(s)$ for some $s \in [0,t]$. Let R_1 be the number of particles that are killed at $L(s)$ for some $s \in (0,t)$, and let R_2 be the number of particles that are killed at $L(s)$ for some $s \in (0,t/2)$. By Corollary 17,

$$\mathbb{P}_x(R_1 > 0) \le Ce^{\sqrt{2}x}\sin\left(\frac{\pi x}{L(0)}\right)t^{1/3}e^{-(3\pi^2 t)^{1/3}}. \tag{69}$$

In this modified process, all particles disappear before time t. Therefore, the only way to have $\zeta > t$ is to have, in the modified process, a particle killed at $L(s)$ for some $s \in (0,t)$. The upper bound in (2) thus follows from the upper bound in (69).

Likewise, Corollary 17 implies that

$$\mathbb{P}_x(R_2 > 0) \ge Ce^{\sqrt{2}x}\sin\left(\frac{\pi x}{L(0)}\right)t^{1/3}e^{-(3\pi^2 t)^{1/3}}.$$

By Proposition 20, a particle that reaches $L(s)$ at time $s \in (0,t/2)$ has a descendant alive at time t with probability greater than C. This implies the lower bound in (2).

Next, suppose $0 < t < \max\{A_0, 2A\}$ and $0 < x < ct^{1/3} - 1$. Let $(B(s), s \geq 0)$ be standard Brownian motion with $B(0) = x$. The probability that the branching Brownian motion survives until time t is bounded below by $P(B(s) > 0 \text{ for all } s \in [0, t])$ and is bounded above by $e^t P(B(s) > 0 \text{ for all } s \in [0, t])$. Because both x and t are bounded above by a positive constant, both of these expressions are of the order x, as are the expressions on the left-hand side and the right-hand side of (2). Consequently, (2) holds when $0 < t < \max\{A_0, 2A\}$.

Finally, (3) follows from (2) by fixing $x > 0$ and letting $t \to \infty$. □

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