Topics on Branching Brownian motion

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Contents

Disclaimer: This set of notes is not in its final form and still contains many typo, errors and much room for improvement. I have also not properly acknowledged the sources material which include

1. Zhan Shi *Random Walks and Trees* Guanajuato lecture notes. ESAIM: Proceedings 31 (2011) 1-39. `http://www.esaim-proc.org/articles/proc/abs/2011/01/proc113101/proc113101.html`

2. Ofer Zeitouni, *Gaussian Fields, Notes for lectures.* `http://cims.nyu.edu/~zeitouni/notesGauss.pdf`

3. A. Bovier, *From spin glasses to branching Brownian motion – and back?*, to appear in the Proceedings of the 2013 Prague Summer School on Mathematical Statistical Physics, M. Biskup, J. Cerny, R. Kotecky, eds. `https://www.dropbox.com/s/3qhxvkrbljb9qw0/pragueschool.pdf`

Chapter 1

Branching particle systems: basic setup

Informally, we want to describe in the greatest possible generality a model in which particles are independent (no interaction), move in space according to some (Markovian) process, and branch. The most convenient way to define this class of models is to define it as a random variables with values in the space for *marked* trees.

In a nutshell, a *dyadic* branching Brownian motion in $\mathbb{R}$ can be thought of as follows: A particle starts at $x \in \mathbb{R}$ and moves according to a Brownian motion. After a random time distributed as an exponential with parameter β, the particle splits and is replaced by two daughter particles, which in turn start to move as independent Brownian motions and who also split after independent exponential lifetimes, and so on.... At time t we call $\mathcal{N}_t$ the set of particles alive at time t and for $u \in \mathcal{N}_t$ we let $X_u(t)$ be the position of particle u. It should be clear from the *absence of memory property* of exponentials that $(X_u(t), u \in \mathcal{N}_t)$ is a Markov process.

It should be clear that there is nothing special about the fact that the movement is Brownian (any Markov process would do) and the dyadic character, we could instead chose to have particles produce i.i.d. number of offsprings upon dying.

In this section we are going to lay down the basic definitions and notations necessary to manipulate branching particle systems.

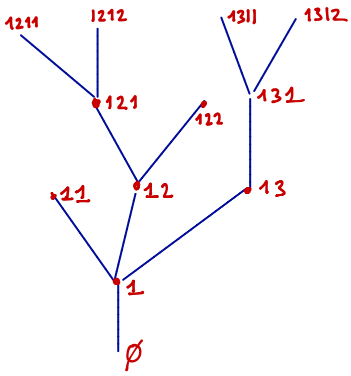
1.1 Galton-Watson trees and Galton-Watson processes

We will be interested in (finite or infinite) rooted ordered trees, which are called plane trees in combinatorics. We set $\mathbb{N} = \{1, 2, \ldots\}$ and by convention $\mathbb{N}^0 = \{\emptyset\}$. We introduce the sets

$$\mathcal{U} = \cup_{n=0}^{\infty} \mathbb{N}^n, \qquad \bar{\mathcal{U}} = \mathcal{U} \cup \mathbb{N}^{\infty}.$$

An element of $\mathcal{U}$ is thus a finite sequence $u = (u_1, \ldots, u_n)$ of integers and we set $|u| = n$ so that $|u|$ represents the "generation" of u. We think of u as the labels of nodes in a tree: i.e. $u = (3, 1, 2)$ is the second child of the first child of the third child of the root. If $u \in \mathcal{U}$ with $|u| = n \geq m$ or $u \in \bar{\mathcal{U}}$ we define

$$u_{|m} = (u_1, \dots, u_m).$$



If $u = (u_1, \dots, u_m)$ and $v = (v_1, \dots, v_n)$ belong to $\mathcal{U}$, we write $uv = (u_1, \dots, u_m, v_1, \dots v_n)$. In particular $u\emptyset = \emptyset u = u$.

The mapping $p : \mathcal{U} \backslash \emptyset \to \mathcal{U}$ is defined by $p(u_1, \dots, u_n) = (u_1, \dots, u_{n-1})$ (i.e. it defines the parent of u).

Definition 1. *A (locally finite, rooted) plane tree τ is a subset of $\mathcal{U}$ such that*

1. $\emptyset \in \tau$.
2. $u \in \tau \backslash \{\emptyset\} \Rightarrow p(u) \in \tau$.
3. *For every $u \in \tau$ there exists an integer $A_u \geq 0$ such that for every $j \in \mathbb{N}, uj \in \tau$ if and only if $j \leq A_u$.*

Let us denote by $\mathcal{T}$ the set of all (locally finite, rooted) plane trees.

The number $A_u := A_u(\tau)$ is the "number of children" of u in τ.

Let μ be an offspring distribution. This means that μ is a probability on $\mathbb{Z}^+ = \{0, 1, \dots\}$ and call m its mean

$$m := \sum_{k=0}^{\infty} k\mu(k).$$

To define a Galton-Watson tree with offspring distribution μ, we let $(A_u, u \in \mathcal{U})$ be a collection of independent random variables with law μ indexed by the set $\mathcal{U}$. Denote by $\mathbf{T}$ the random subset of $\mathcal{U}$ defined by

$$\mathbf{T} = \{u = (u_1, \ldots, u_n) \in \mathcal{U} : u_j \leq A_{p(u_{|j})}, \text{ for every } 1 \leq j \leq n\}.$$

The Galton-Watson process with reproduction law μ is a discrete time Markov process with values in $\mathbb{Z}_+$ which is defined by the recursive equation

$$Z_0 = 1; \qquad \forall n \geq 0,\ Z_{n+1} = \sum_{i=1}^{Z_n} X_{i,n} \tag{1.1}$$

where the $X_{i,n}$ are i.i.d. variables with law μ.

Proposition 2. $\mathbf{T}$ *is a.s. a tree (the Galton-Watson tree with reproduction law μ). Moreover, if*

$$Z_n = \#\{u \in \mathbf{T} : |u| = n\},$$

then $(Z_n, n \geq 0)$ is a Galton-Watson process with offspring distribution μ and initial value $Z_0 = 1$.

We denote by $\mathbb{P}_\mu$ the law of $\mathbf{T}$ on $\mathcal{T}$.

From now on $\mathbf{T}$ is a μ-Galton-Watson tree, $(Z_n, n \geq 0)$ is the assocated Galton-Watson proces, A is a variable with law μ (i.e. an offspring variable). To avoid trivial cases we always assume $\mu(0) + \mu(1) < 1$.

1.2 spatial trees

A spatial tree is a tree $\tau \in \mathcal{T}$ enriched with the following extra structure: for each $u \in \tau$ we associate a life-time $\sigma_u \geq 0$. For each u this allows us to define the *birth-time* of u by $b_u = \sum_{v<u} \sigma_v$ and its death time $d_u = b_u + \sigma_u$. Furthermore, for each $u \in \tau$ there is a map $Y_u : \mathbb{R}^+ \to E$ where E is the space in which particles are living.

Formally, a marked tree is the triplet

$$\mathbf{t} = (\tau, \sigma, Y) = (\tau, \{\sigma_u, (Y_u(s), s \geq 0), u \in \tau\}).$$

We let $\mathcal{N}(t) \subset \mathcal{U}$ be the set of particles that are alive at time t,

$$\mathcal{N}(t) := \{u \in \tau : b_u \leq t < d_u\}.$$

For each u in $\mathcal{N}(t)$ we define inductively the position in E of the particle u at time t as

$$X_u(t) := Y_u(t - b_u) + X_{p(u)}(b_u -)$$

where recall that $X_{p(u)}(d_u)$ is just the position of death of the parent of u.

We extend the notion of position for $u \in \mathcal{N}(t)$ to include the ancestors of u, so if $v \in \mathcal{N}(s)$ for some $s < t$ and v is an ancestor of u, then we set $X_u(s) := X_v(s)$.

Example 3. *Take τ to be the binary tree (i.e. $\tau = \cup_{n\in\mathbb{N}}\{1,2\}^n$), $\sigma_u = 1/|u|$ and* $Y_u(s) = \begin{cases} 2^{-|u|}s \text{ if last digit is } 1 \\ -2^{-|u|}s \text{ if last digit is } 2 \end{cases}$.

We call $\mathbf{t}$ the set of all marked trees. We are now going to define various probability distributions on this space.

1.3 Dyadic branching Brownian motion

The standard branching Brownian motion (or dyadic branching Brownian motion) is obtained with the following choice:

1. $A_u = 2, \forall u \in \mathcal{U}$ (that is τ is the regular binary tree $\tau = \cup_{n\in\mathbb{N}}\{1,2\}^n$)
2. the σ_u are i.i.d. with mean one exponential distribution.
3. The Y_u are standard Brownian motions.

Definition 4. *Writing $\omega = (\tau, (\sigma_u, Y_u)_{u\in\tau})$ we then define*

$$X(t) = X(t,\omega) = (X_u(t), u \in \mathcal{N}_t)$$

to be the branching Brownian motion process. The natural filtration of this process is

$$\mathcal{F}_t = \sigma\{X(s), s \le t\} = \sigma\{X(s), \mathcal{N}_s\}.$$

The law of this process is denoted by $\mathbb{P}$ or $\mathbb{P}_x$ when we need to emphasize that its initial state is a single particle sitting at $x \in E$.

The branching Brownian motion process is sometime defined as measure-valued process

$$\mu_t(\cdot) = \sum_{u\in\mathcal{N}_t} \delta_{X_u(t)}(\cdot)$$

which is also $\mathcal{F}_t$-adapted but contains less information than $X(t)$ (why?).

Remark 5. *Observe that the following random variables are measurable with respect to $\mathcal{F}_t$*

- *All the σ_v such that $v < u$ for some $u \in \mathcal{N}(t)$ (i.e. the life-times of particles that are already dead by time t)*
- *Any $X_u(s)$ for $s \le t$ and $u \in \mathcal{N}(t)$ (i.e. the positions of the ancestors of particles alive at time t)*

The following proposition is is clear from the *absence of memory* property of exponentials and the Markov property of Brownian motions.

Proposition 6. *The branching Bronian motion $X(t)$ (or μ_t) is strongly Markovian.*

1.4 Branching random walk

Let us now see a second example of random marked tree which will be a related but different model.

A (random) point-measure $\Theta(\cdot)$ on E is a discrete point-mass measure on E

$$\Theta(\cdot) = \sum_{i=1}^{N} \delta_{x_i}(\cdot)$$

where N (random) is (a.s.) finite and the x_i are (random) points of E.

Definition 7. *A (discrete-time) branching random walk with reproduction-displacement mechanism Θ is a random marked tree such that*

- *τ is a Galton-Watson process with reproduction mechanism given by $N = \int_E d\Theta$ (the number of atoms in θ)*
- *the life-times σ_u are all equal to 1 (deterministic),*
- *For each $u \in \tau$, the point process $\sum_{i=1}^{A_u} \delta_{Y_u(0)}(\cdot)$ is an independent copy of the point process θ (i.e. offsprings are born at distances from the parent which are given by an independent copy of the point measure Θ).*
- *The maps $s \to Y_u(s)$ are constant (i.e. particles are born at a distance of their parent given by $Y_u(0)$ and then don't move).*

Observe that the Y_u are not necessarily independent one from another when they have the same parent at the previous generation.

As for the Brownian motion, we like to think about branching random walks as Markovian processes. At each generation, each particle reproduces independently of the others, and the position of the offsprings relative to their parent form i.i.d. copies of the point-measure Θ.

1.5 First properties of the dyadic branching Brownian motion

Proposition 8.

$$\mathbb{E}[N(t)] = e^t, \qquad \forall t \geq 0$$

In fact $(N(t), t \geq 0)$ is a pure birth process called a Yule process.

The branching Brownian motion is a cloud of particles which is growing in size and shape. At time t there is of order e^t particles and each of these particle u is at a position $X_u(t)$ which is a centered Gaussian variable with variance t (like a Brownian motion). However, the $X_u(t)$ are not independent, in fact their correlation structure is given by their genealogical history.

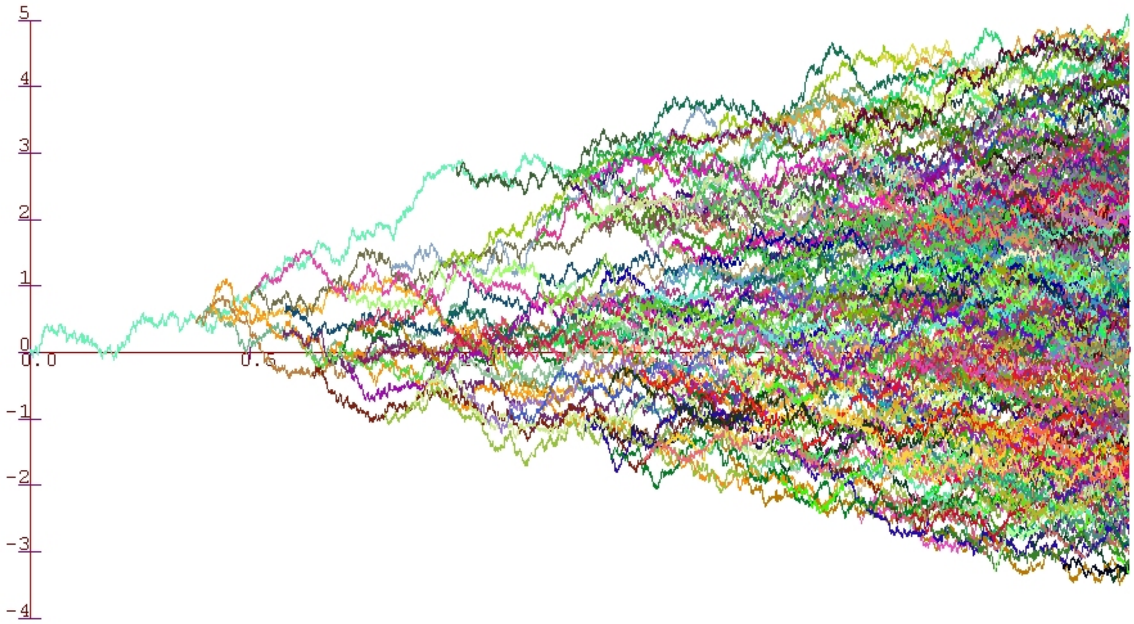


Figure 1.1: A realization of a dyadic branching Brownian motion (Image courtesy of Matthew Roberts)

Proposition 9. *Let $\tau_{u,v}$ designate the death time of $u \wedge v$ the most recent common ancestor of $u, v \in \mathcal{N}_t$ we have*

$$\mathbb{E}[X_u(t)X_v(t)|\tau_{u,v}] = \tau_{u,v}.$$

Chapter 2

The F-KPP equation and McKean representation

2.1 FKPP equation

The *F-KPP* or *KPP* or *Kolmogorov* equation is a semilinear heat equation of the form

$$u_t = \frac{1}{2}u_{xx} + g(u) \tag{2.1}$$

where the forcing term g is assumed to be in $C^1[0,1]$ and to satisfy the conditions

$$g(0) = g(1) = 0, g(u) > 0,\ u \in (0,1) \tag{2.2}$$

and

$$g'(0) = \beta > 0,\ g'(u) \leq \beta,\ u \in (0,1]. \tag{2.3}$$

This equation was first considered in 1937 by R.A. Fisher in *The advance of advantageous genes*; and by Kolmogorov, Petrovsky and Piskunov *Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique*. Over the years, this *reaction-diffusion* equation has been studied by many authors, both probabilistically and analytcally (see Kolmogotv et al., Fisher, Skorokhod, McKean, Bramson, Neveu, Aronson and Weinberger, Harris, Kyprianou to name just some of them.)

This equation is ubiquitous in the study of reaction-diffusion phenomena and front propagation. It appears in models related to fields as diverse as ecology, population genetics, combustion, epidemiology, etc... It is one of the simplest example of a semilinear parabolic equation which admits *traveling wave solutions* (more on this later) and its study is a very active field of research in the p.d.e. community.

The prototypical example of forcing term that we are going to consider is

$$g(u) = \beta u(1-u).$$

The case $g(u) = u(1-u)^2$ appears naturally in the context of a genetic model for the spread of an advantageous gene through a population. More generally, if $g(u)/u$ is monotone decreasing then $u(t,x)$ may be considered as the density of a population of individuals with exponential growth near 0 and which saturates at $u = 1$.

The Kolmogorov equation is sufficiently well behaved so that there is no difficulty in establishing existence and unicity of the solution under measurable initial data. One may next enquire about the asymptotic behavior of solutions as $t \to \infty$.

2.2 Maximum of the BBM

Here is a first connection between the branching Brownian motion and the Kolmogorov equation. Let X be a BBM with reproduction law $(p_k)_{k\geq 0}$ and branching rate β (i.e. σ_u is exponential parameter β for each u). Let

$$M(t) = \max_{u\in\mathcal{N}_t} X_u(t).$$

Theorem 10. *Let* $u(t,x) := \mathbb{P}_0[M(t) \leq x]$. *Then* u *satisfies*

$$\begin{cases} u_t = \frac{1}{2}u_{xx} + \beta(f(u) - u) \\ u(0,x) = \mathbf{1}_{\{x\geq 0\}} \end{cases} \tag{2.4}$$

where $f(u) := \sum_{k=0}^{\infty} p_k u^k$.

The initial condition $u(0,x) = \mathbf{1}_{\{x\geq 0\}}$ is sometimes called the Heavyside initial condition.

We are going to see two proofs of this. The first one although slightly informal could be made rigorous with little effort.

Proof. For simplicity we assume that $\beta = 1$ and $p_2 = 1$ (dyadic case). We want to compute $u(t+dt,x) - u(t,x)$ up to terms of order dt where dt is small. The idea is to decompose according to what happen to the branching Brownian motion in the initial interval $[0,\, \mathrm{d}t]$ of time.

- With probability $(1-\,\mathrm{d}t) + o(\,\mathrm{d}t)$ it doesn't branch and conditionally to this event
$$\mathbb{P}[M(t+\,\mathrm{d}t) \leq x] = \mathbb{P}[M(t) \leq x - B_{\mathrm{d}t}] = u(t, x - B_{\mathrm{d}t})$$
where B is a Brownian motion.

- With probability $\,\mathrm{d}t + o(\,\mathrm{d}t)$ there is exactly one branching event and conditionally to this event
$$\mathbb{P}[M(t+\,\mathrm{d}t) \leq x] = (\mathbb{P}[M(t) \leq x - B_{\mathrm{d}t}])(\mathbb{P}[M(t) \leq x - B'_{\mathrm{d}t}]) = (\mathbb{P}[M(t) \leq x])^2 + o(1)$$
where B and B' are correlated Brownian motions.

Thus

$$\mathbb{P}[M(t+\,\mathrm{d}t)\le x]-\mathbb{P}[M(t)\le x]=(1-\,\mathrm{d}t)\mathbb{P}[M(t)\le x-B_{\mathrm{d}t}]+\,\mathrm{d}tu^2(t,x)+o(\,\mathrm{d}t)-u(t,x)$$
$$=[\mathbb{E}(u(t,x-B_{\mathrm{d}t}))-u(t,x)]+\,\mathrm{d}t[u^2(t,x)-u(t,x)]+o(\,\mathrm{d}t)$$

where in the last line we have absorbed the term $\mathrm{d}t\times(u(t,x-B_{\mathrm{d}t})-u(t,x))$ in the $o(\,\mathrm{d}t)$ term. Recall that if g is a smooth enough function, then $v(t,x)=\mathbb{E}_x[g(B_t)]$ solve the heat equation $v_t=\frac{1}{2}v_xx$. Thus, writing $g(z)=u(t,z)$ we see that

$$\mathbb{E}[u(t,x-B_{\mathrm{d}t})]=\mathbb{E}_x[g(B_{\mathrm{d}t})]$$

and therefore

$$\lim_{\mathrm{d}t\to 0}\frac{\mathbb{E}(u(t,x-B_{\mathrm{d}t}))-u(t,x)}{\mathrm{d}t}=\frac{\partial^2}{\partial x^2}u(x,t).$$

Hence

$$\lim_{\mathrm{d}t\to 0}\frac{u(t+\,\mathrm{d}t,x)-u(t,x)}{\mathrm{d}t}=\frac{\partial^2}{\partial x^2}u(x,t)+[u^2(t,x)-u(t,x)].$$

□

Unless specified otherwise, we will consider the following version of the FKPP equation. recall that $f(s)=\mathbb{E}(s^A)$ is the generating function of the offspring distribution. We pick a non-linear term g which as follows

$$\frac{\partial u}{\partial t}=\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+\beta(f(u)-u). \tag{2.5}$$

In the special case where the branching is dyadic this becomes

$$\frac{\partial u}{\partial t}=\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+\beta(u^2-u). \tag{2.6}$$

2.3 McKean representation

The following Theorem, due to McKean, 1975 [24], gives a representation of solutions to the F-KPP equation (2.5) in term of the BBM.

We will need to deal with functionals of the BBM which can be expressed in terms of functions of the position of the particles at time t. Thus, we would like to be able to tell when such a functional is a martingale. This is very easy once we write down the generator.

For now, let us look at the BBM as a Markov process in the state space

$$S:=\cup_{n\in\mathbb{N}}\big(\{n\}\times\mathbb{R}^n\big),$$

Consider functions $F:\mathbb{R}^+\times S\mapsto\mathbb{R}$, $F(t,n,x)=F(t,n,(x_1,\dots,x_n))$ which are C^2 in the space variables. Of course, the second argument of F being the dimension of x the third argument, it is completely redundant and is written only for the sake of clarity. We write $\hat{x}_i=(x_1,x_2\dots,x_i,x_i,\dots x_n)$ for the vector obtained from x by repeating once the i-th coordinate.

For simplicity, we continue to restrict ourselves to the case of the dyadic branching Brownian motion.

Theorem 11. *The dyadic branching brownian motion with branching rate $\beta > 0$ is Fellerian and its generator is*

$$\mathcal{G}F(t,n,x) := \sum_{i=1}^{n} \frac{1}{2}\frac{\partial^2}{\partial x_i^2} F(t,n,x) + \sum_{i=1}^{n} \beta[F(t,n+1,\hat{x}_i) - F(t,n,x)]$$

The following is classical:

Proposition 12. *If $F : [0,\infty) \times S \mapsto \mathbb{R}$ is $C^{1,2}$ in t and x respectively and*

$$\left(\mathcal{G} + \frac{\partial}{\partial t}\right) F \equiv 0 \tag{2.7}$$

then $(F(t,\#\mathbb{N}_t, X(t)), t \geq 0)$ is a local martingale.

The next Theorem is often called the McKean representation. It says that solutions of the FKPP equation can be viewed as an expectation with respect to the BBM. It is at heart a Feynman-Kac type of result.

Theorem 13. *If $u : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}$ satisfies $u \in [0,1]$ and solves the FKPP equation with initial condition g*

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \beta(f(u) - u) \\ u(0,x) = g(x) \end{cases} \tag{2.8}$$

then u has the representation

$$u(t,x) = \mathbb{E}_x\left[\prod_{u \in \mathcal{N}_t} g(X_u(t))\right]. \tag{2.9}$$

Proof. Suppose u satisfies the FKPP equation (2.8). Then it is easily checked that the function

$$F(s,X(s)) := \prod_{u \in \mathcal{N}_s} u(t-s, X_u(s)), \qquad s \leq t$$

satisifies the condition of Proposition 12 and is thus a local martingale. But since $u \in (0,1)$ it is bounded and is thus a true martingale. We conclude that

$$u(t,x) = E_x[F(0,X(0))] = E_x(F(t,X(t))) = \mathbb{E}_x\left[\prod_{u \in \mathcal{N}_t} g(X_u(t))\right]$$

which is the McKean representation.

Indeed (let's check this easily checkable fact)

$$\begin{aligned} \mathcal{G}F(s,X(s)) &= \sum_{u \in \mathcal{N}_s} \prod_{v \neq u} u(t-s, X_v(s)) \frac{\partial^2}{\partial x^2}(t-s, X_u(s)) \\ &\quad + \sum_{u \in \mathcal{N}_s} F(s,X(s))(u(t-s,X_u(s)) - 1) \end{aligned}$$

while

$$\frac{\partial}{\partial t}F(s,X(s)) = -\Big[\sum_{u\in\mathcal{N}_s}\prod_{v\neq u}u(t-s,X_v(s))\Big\{\frac{\partial^2}{\partial x^2}u(t-s,X_u(s)) + (u^2(t-s,X_u(s))-u(t-s,X-u(s)))\Big\}\Big]$$

which are the same up to the minus sign. □

Chapter 3

Position of the rightmost particle

We have seen $M(t) = \max_{u \in \mathcal{N}_t} X_u(t)$ the position of the rightmost particle can be studied through the analysis of the KPP equation. In this section we start our exploration of the extremal point process of the branching Brownian motion by looking at the asymptotic behavior of $M(t)$.

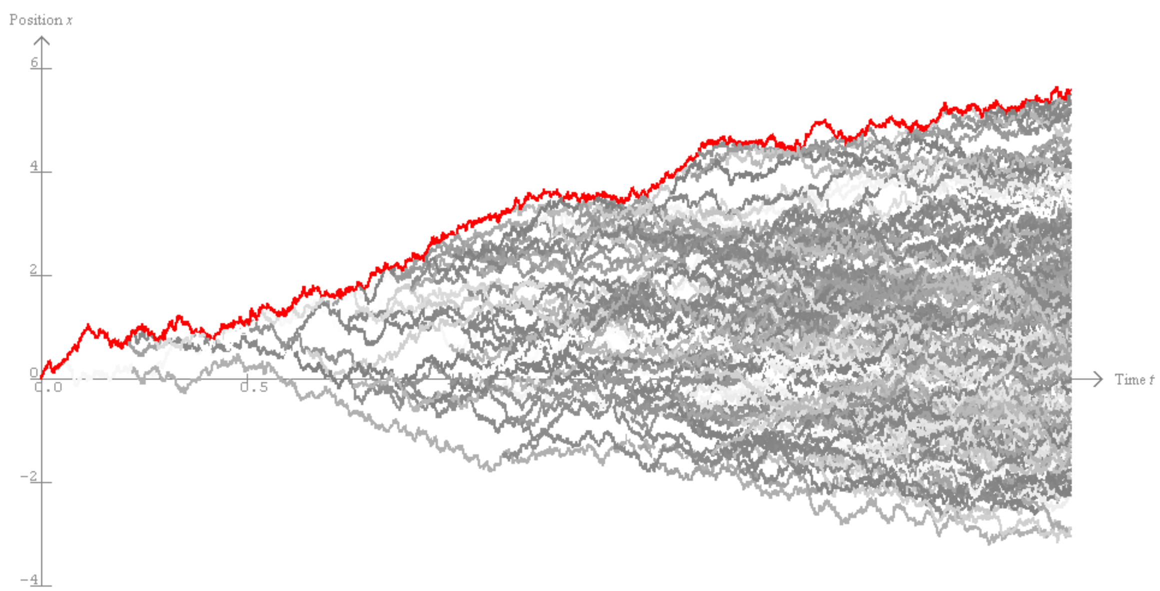


Figure 3.1: Position of M_t (Image by M. Roberts)

3.1 Kolmogorov's and Bramson's result

As we have seen in the previous chapter, $u(t,x) = \mathbb{P}_0(M(t) \leq x)$ solves the F-KPP equation with initial condition $u(0,x) = \mathbf{1}_{\{x>0\}}$. The following is one of the result proved in the original paper of Kolmogorov *et al.*

Theorem 14 (Kolmogorov, Petrovski and Piskunov, 1937)**.** *There exists a map $t \mapsto m_t$ such that*

$$u(t, m_t + x) \to w(x) \quad \text{uniformly in } x \in \mathbb{R} \text{ as } t \to \infty$$

where w solves

$$\frac{1}{2}w'' + \sqrt{2}w' + w(w-1) = 0. \tag{3.1}$$

check sign of nonlinearity *Furthermore, $m_t = \sqrt{2}t + o(t)$*

Remark 15. *Observe that this says that $M(t) - m_t$ converges in distribution to a variable whose cumulative distribution function is given by w.*

Any function w that solves (3.1) is called a traveling wave solution of (2.1) with speed $\sqrt{2}$ since it is ealy checked that then

$$u(t,x) = w(x - \sqrt{2}t)$$

is a solution of (2.1). More generally, if w_λ solves

$$\frac{1}{2}w'' + \lambda w' + w(w-1) = 0. \tag{3.2}$$

then

$$u(t,x) = w(x - \lambda t)$$

is also a solution. w_λ is a traveling wave solution with speed λ. The terminology comes from the fact that the fixed shape front w_λ is traveling at constant speed λ. Kolmogorov et al. also show that traveling waves exist for all λ such that $\lambda \geq \sqrt{2}$ and that for each such λ the solutions of (3.2) are unique up to an additive constant in the argument (i.e. if w_λ is solution, so is $w_\lambda(\cdot + k)$). We will come back to this in section 5.

This result was greatly improved upon by Bramson in two steps, first in 1978 and then in 1983. He showed that

Theorem 16 ([7] Bramson, 1983)**.** *For all initial condition $u(0,x) = g(x)$ increasing sufficiently fast*[1] *(including the Heaviside initial condition) then*

$$u(t, x + m_t) \to w(x) \quad \text{uniformly in } x \in \mathbb{R} \text{ as } t \to \infty \tag{3.3}$$

where $m_t = \sqrt{2}t\frac{3}{2\sqrt{2}}\log t + C + o(1)$

Remark 17. *In particular this means that we can choose $m_t = \sqrt{2}t - 3 \cdot 2^{-3/2}\log t$ in the convergence (3.3). The constant C in m_t depends on the precise shift of w which is chosen.*

We will also show that

Theorem 18. *Almost surely, $\lim_{t\to\infty} \frac{M_t}{t} = \sqrt{2}$ and $\lim_{t\to\infty} M_t - \sqrt{2}t = -\infty$.*

[1] This will be made precise later

3.2 First moment calculations

Let us look at a first moment argument and see if we can obtain the first order of the position of the rightmost particle, i.e $M_t \sim \sqrt{2}t$.

I want to determine a value $c > 0$ which is critical for $\mathbb{P}(\exists u \in \mathcal{N}(t) : X_u(t) > ct)$.

The following bound for the tail distribution of a Gaussian variable is classical and will be used repeatedly in the following

Exercise 19. *Show that*

$$\frac{e^{-x^2/2}}{\sqrt{2\pi}}\Big(\frac{1}{x} - \frac{1}{x^3}\Big) \le \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-y^2/2}\, \mathrm{d}y \le \frac{e^{-x^2/2}}{x\sqrt{2\pi}}. \tag{3.4}$$

Using the linearity of expectation and (3.4) we see that

$$\begin{aligned}\mathbb{P}(\exists u \in \mathcal{N}(t) : X_u(t) > ct) &\le \mathbb{E}\Big[\sum_{u\in\mathcal{N}(t)} \mathbf{1}_{\{X_u(t)>ct\}}\Big] \\ &= e^t \mathbb{P}[B(t) > ct] \\ &\le e^t \frac{e^{-c^2 t/2}}{ct^{1/2}\sqrt{2\pi}} = e^{t(1-c^2/2)} \frac{1}{ct^{1/2}\sqrt{2\pi}}\end{aligned}$$

which tends to 0 as soon as $c \ge \sqrt{2}$ and

$$\mathbb{E}\Big[\sum_{u\in\mathcal{N}(t)} \mathbf{1}_{\{X_u(t)>ct\}}\Big] \ge e^{t(1-c^2/2)} \frac{1}{\sqrt{2\pi}}\Big(\frac{1}{ct^{1/2}} - \frac{1}{c^3 t^{3/2}}\Big).$$

which tends to infinity as soon as $c < \sqrt{2}$. So if we can show that this expectation "doesn't lie", i.e. that it is not dominated by rare events in which we have massive number of particles above ct then we should be able to show that $c = \sqrt{2}$. This is what we are going to do in this section.

Observe first that Kingman's ergodic subadditive theorem implies that $M_n/n \to c$ for some finite constant. **develop**

3.3 Warm up: the independent case

It will be instructive to start by looking at the following simpler case. Suppose that $(\tilde{X}_u, u \in \mathcal{U})$ is a collection of *independent* Brownian motions, and that $\mathcal{N}_t$ is the population at time t of a branching Brownian motion. Let us call $\tilde{M}_t := \max\{\tilde{X}_u(t), u \in \mathcal{N}_t\}$ the maximum of $N(t) := \#\mathcal{N}_t$ independent Gaussian variables with the same variances as the $X_u(t)$.

Remember that $(N(t), t \ge 0)$ is a Yule process and that it is known that $Z(t) = e^{-t}N(t)$ is a (positive) martingale which converges almost surely and in L^1 to a limit Z which is furthermore an exponential variable with mean 1. We are going to show the following

Proposition 20.

$$\mathbb{P}(\tilde{M}_t \leq \tilde{m}_t + x) \to \tilde{w}(x) = \mathbb{E}\big[e^{-cZe^{-\sqrt{2}x}}\big]$$

where $\tilde{m}_t = \sqrt{2}t - \frac{1}{2\sqrt{2}}\log t$ *and* Z *is the almost sure limit of the martingale* $e^{-t}N(t)$.

Remark 21. *This proves the convergence in distribution of* $\tilde{M}_t - \tilde{m}_t$

Proof. The estimate (3.4) implies that for any $a_t = o(t^{1/2})$

$$\mathbb{P}(\tilde{X}(t) > \sqrt{2}t - a_t) = \mathbb{P}(t^{-1/2} > \sqrt{2t} - a_t/t^{1/2}) \sim \frac{1}{\sqrt{4\pi}} t^{-1/2} e^{-t+\sqrt{2}a_t + o(1)}$$

so by choosing $a_t = \frac{1}{2\sqrt{2}}\log t - x$ we have that

$$\mathbb{P}(\tilde{X}(t) > \sqrt{2}t - \frac{1}{2\sqrt{2}}\log t + x) \sim ce^{-t}e^{-\sqrt{2}x}.$$

Thus,

$$\begin{aligned}
\mathbb{P}\big(\tilde{M}_t \leq \tilde{m}_t + x\big) &= \mathbb{P}\big(\forall u \in \mathcal{N}_t : \tilde{X}_u(t) \leq \tilde{m}_t + x\big) \\
&= \mathbb{E}\big[(1 - \mathbb{P}(\tilde{X}(t) > \tilde{m}_t + x))^{\#\mathcal{N}_t}\big] \\
&= \mathbb{E}\big[(1 - ce^{-t}e^{-\sqrt{2}x}(1 + o_t(1)))^{e^t Z(1+o_t(1))}\big]
\end{aligned}$$

where Z is the limit of the martingale $Z(t) := e^{-t}\#\mathcal{N}_t$ and is an exponential mean one variable. Finally we conclude that

$$\mathbb{P}\big(\tilde{M}_t \leq \tilde{m}_t + x\big) \sim_{t\to\infty} \mathbb{E}\big[\exp\big\{ - cZe^{-\sqrt{2}x}\big\}\big].$$

□

Remark 22. *Observe that the tail of* $\tilde{M}_t - \tilde{m}_t$ *is doubly exponential to the left and exponential to the right. This asymmetry is typical of Gumbel variables.*

What we are now going to show is a similar Theorem for $M(t)$

Theorem 23. *Take* $m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t$*, then* $M(t) - m_t$ *converges in distribution and there exists a random variable* Z *such that*

$$\lim_{t\to\infty} \mathbb{P}(M(t) \leq m_t + y) = \mathbb{E}\big[\exp(-cZe^{-\sqrt{2}y})\big]$$

3.4 A first attempt at the moments method for the law of large numbers

Looking in more details at the first moment argument above we see that we can easily get an upper bound out of it. Indeed, for each $\epsilon > 0$ we can find $c(\epsilon) > 0$ such that

$$\mathbb{P}(\exists u \in \mathcal{N}(t) : X_u(t) > (1+\epsilon)\sqrt{2}t) \leq e^{-c(\epsilon)t}$$

for t large enough. Thus, we conclude (with Borel Cantelli) that for any fixed $\delta > 0$

$$\limsup_n \frac{M_{n\delta}}{n\delta} \leq \sqrt{2}$$

This easily implies that $\limsup_t \frac{M_t}{t} \leq \sqrt{2}$ (do it!).

What would be the natural approach for the lower bound? One might want to define

$$\underline{Z}(t) := \sum_{u \in \mathcal{N}_t} \mathbf{1}_{\{X_u(t) > (1-\epsilon)\sqrt{2}t\}}$$

and show that with high probability $\underline{Z}(t) > 1$. To do this the classical tool is the second moment method. Given a $\mathbb{N}$-valued random variable Z

$$\mathbb{E}(Z) = \mathbb{E}(Z\mathbf{1}_{\{Z\geq 1\}}) \leq \mathbb{E}(Z^2)^{1/2}\mathbb{P}(Z \geq 1)^{1/2}$$

so that

$$\mathbb{P}(Z \geq 1) \geq \frac{(\mathbb{E}Z)^2}{\mathbb{E}(Z^2)}.$$

Now, in the cases where we have independent Gaussian variables, as in the warm-up, we have that

$$\begin{aligned}\mathbb{P}(\tilde{M}_t \geq (1-\epsilon)\sqrt{2}t) &\geq \frac{e^{2t}\mathbb{P}(\tilde{X}(t) \geq (1-\epsilon)\sqrt{2}t)^2}{\mathbb{E}(\sum_{u,v\in\mathcal{N}_t} \mathbf{1}_{\{\tilde{X}_u(t)>(1_\epsilon)\sqrt{2}t\}} \mathbf{1}_{\{\tilde{X}_v(t)>(1-\epsilon)\sqrt{2}t\}})} \\ &= \frac{e^{2t}\mathbb{P}(\tilde{X}(t) \geq (1-\epsilon)\sqrt{2}t)^2}{e^t\mathbb{P}(\tilde{X}(t) \geq (1-\epsilon)\sqrt{2}t) + \mathbb{E}[N_t(N_t-1)]\mathbb{P}(\tilde{X}(t) \geq (1-\epsilon)\sqrt{2}t)^2}\end{aligned}$$

Recall that $\mathbb{E}[N_t(N_t-1)] = e^{2t} - 2e^t$ and that for each $\epsilon > 0$ there exist $c(\epsilon) > 0$ such that

$$e^t\mathbb{P}(\tilde{X}(t) \geq (1-\epsilon)\sqrt{2}t) \geq e^{c(\epsilon)t}$$

we see that

$$\mathbb{P}(\tilde{M}_t \geq (1-\epsilon)\sqrt{2}t) \geq \frac{1}{\frac{e^{2t}-2e^t}{e^{2t}} + e^{-c(\epsilon)t}} \geq 1 - e^{-c'(\epsilon)t}$$

which implies that

$$\liminf_t \frac{\tilde{M}_t}{t} \geq \sqrt{2}.$$

This line of reasoning fails when we try it with M_t. The reason is that in this case $\mathbb{E}[\sum_{u,v\in\mathcal{N}_t} \mathbf{1}_{\{X_u(t)\geq(1-\epsilon)\sqrt{2}t\}}\mathbf{1}_{\{X_v(t)\geq(1-\epsilon)\sqrt{2}t\}}]$ is much larger due to correlations.

Here is a "back of the envelope" calculation that should help convince you: Let us call G the number of particles near λt at time t (with $\lambda > 0$). Because the cost for a Brownian motion to get to λt is $e^{-\frac{\lambda^2}{2}t}$ we have that

$$\mathbb{E}[G] \approx e^{t-\frac{\lambda^2}{2}t}.$$

Now we want to consider $\mathbb{E}[G^2]$ which is the expectation of the number of pairs of particles near λt at time t. We do a violent lower bound

$$\begin{aligned}
\mathbb{E}[G^2] &\geq \mathbb{E}[\#\text{ pairs of particles near } \lambda t \text{ whose last common ancestor was near } \tfrac{2}{3}\lambda t \text{ at time } \tfrac{t}{2}] \\
&= \mathbb{E}[\sum_{w\in\mathcal{N}_t} \mathbf{1}_{\{X_w(t/2)\approx\frac{2}{3}\lambda\frac{t}{2}\}} \sum_{\substack{u\wedge v=w\\ u,v\in\mathcal{N}_t}} \mathbf{1}_{\{X_u(t)=\lambda t\}}\,\mathbf{1}_{\{X_v(t)=\lambda t\}}] \\
&\approx e^{t/2-\frac{1}{2}(\frac{4}{3}\lambda^2)\frac{t}{2}} \times \left(e^{\frac{t}{2}-\frac{1}{2}(\frac{2}{3}\lambda)^2\frac{t}{2}}\right)^2 \\
&= e^{\frac{3}{2}t-\frac{2}{3}\lambda^2 t}
\end{aligned}$$

which means that if $\lambda > \sqrt{\frac{3}{2}}$, then $\mathbb{E}[G^2] \gg \mathbb{E}[G]^2$.

The following picture illustrates why the moment method doesn't work here.

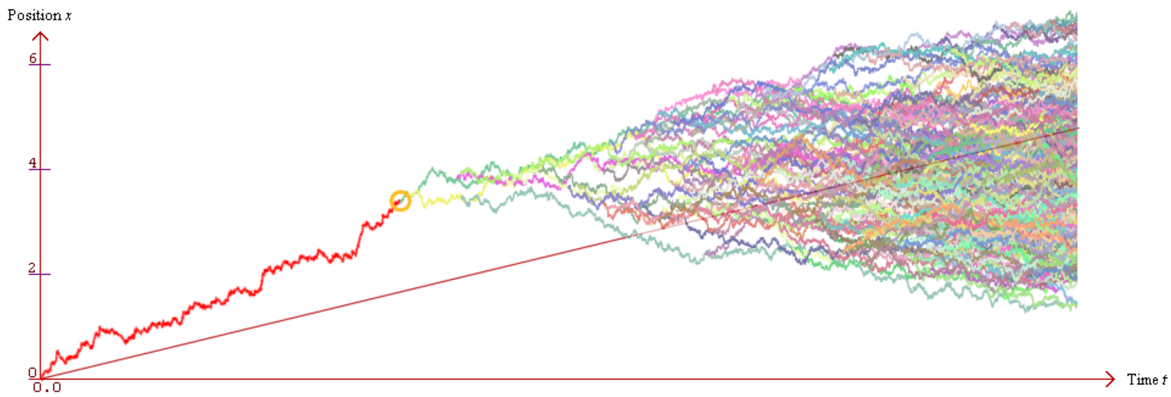


Figure 3.2: If one particle manages to rise high above the line $s \mapsto \sqrt{2}\frac{s}{t}$, it then generates a massive number of particles above the desired final position. (Image by M. Roberts)

3.5 Bessel-3 process

Let us recall some very basic facts about the Bessel-3 process. If $W_t, t \geq 0$ is a Brownian motion in $\mathbb{R}^3$, then its modulus $|W_t|, t \geq 0$ is called a Bessel-3 process.

Suppose that $B_t, t \geq 0$ is a Brownian motion in $\mathbb{R}$ started from $B_0 = x$ under P_x: then

$$\zeta_t := \frac{B_t}{x}\mathbf{1}_{\{B_s>0, \forall s \leq t\}}$$

is a non-negative, mean one martingale under P_x, so we can define a new probability measure Q_x by

$$\frac{\mathrm{d}Q_x}{\mathrm{d}P_x}\Big|_{\mathcal{G}_t} := \zeta_t$$

where $\mathcal{G}_t$ is the natural filtration of B_t. Then $B_t; t \geq 0$ is a Bessel process started from x under Q_x. The density of a Bessel process satisfies

$$Q_x(B_t \in \mathrm{d}z) = \frac{z}{x\sqrt{2\pi t}}\left(e^{-(z-x)^2/2t} - e^{-(z+x)^2/2t}\right)\mathrm{d}z$$

and it can be checked that

$$Q_x(\cdot) = \lim_{t\to\infty} P_x(\cdot|\tau_0 \geq t) = P_x(\cdot|\tau_0 = \infty)$$

where $\tau_0 = \inf\{t : B_t = 0\}$ (the Bessel process is a Brownian motion conditioned to never hit 0).

The following estimate for the density of a Bessel is going to be crucial.

Exercise 24 (Estimate for the density of a Bessel process). *For any $x, z = o(\sqrt{t})$*

$$Q_x(B_t \in \mathrm{d}z) \sim \sqrt{\frac{2}{\pi}}\frac{z^2}{t^{3/2}}.$$

as $t \to \infty$.

3.6 The many-to-one Lemma

The many-to-one Lemma is a simple and well known tool in branching processes.

Lemma 25 (Many-to-one Lemma (1st version)). *For any $t \geq 0$, for any measurable function $F : C_{[0,t]} \mapsto \mathbb{R}$ we have*

$$\mathbb{E}[\sum_{u\in N(t)} F(X_u(s), 0 \leq s \leq t)] = e^t\mathbb{E}[F(B_s, 0 \leq s \leq t)].$$

where B_s is simply a standard Brownian motion under $\mathbb{P}$.

The proof is straightforward as this is simply the independence of the particles trajectories and the branching history coupled with the linearity of the expectation.

Examples of functionals we may want to look at are

$$F(X(s), s \leq t) = \mathbf{1}_{\{X(t)\geq x\}}, \quad F(X(s), s \leq t) = \mathbf{1}_{\{X(s)\geq x \forall s\leq t\}}, \quad F(X(s), s \leq t) = t^2 e^{\int_0^t X(s)\,\mathrm{d}s}.$$

Suppose now that we are given a curve $f : [0, \infty) \to \mathbb{R}, f \in C^2$ such that $f(0) = 0$ and we want to compute

$$\mathbb{E}[\sum_{u \in \mathcal{N}_t} F(X_u(s), s \le t) \mathbf{1}_{\{X_u(s) < \alpha + f(s), \forall s \le t\}}]$$

where $\alpha > 0$ s fixed. Observe that the summands are again just path functionals of the X_u. If we wanted a Brownian particle B to follow f we would apply the usual Girsanov martingale transform

$$g_t = e^{\int_0^t f'(s) \,\mathrm{d}B_s - \int_0^t (f'(s))^2 \,\mathrm{d}s}$$

i.e. under $\hat{P}_f$ defined by $\mathrm{d}\hat{P}_f / \mathrm{d}P|_{\mathcal{G}_t} = g_t$ the process $\hat{B}_t := \alpha + f(t) - B_t$ is a Brownian motion started from α. Let us now do a further change of measure and define a new probability Q by

$$\frac{\mathrm{d}Q}{d\hat{P}_f}\Big|_{\mathcal{F}_t} = \frac{\hat{B}_t}{\alpha} \mathbf{1}_{\{\hat{B}_s > 0 \forall s \le t\}}$$

then $\hat{B}$ is a Bessel-3 process under Q started from α.

Combining the many-to-one Lemma with these two changes of measure we get that for a path-functional F

Lemma 26. *Let* $\zeta(t) := \frac{1}{\alpha}(\alpha + f(t) - B_t) e^{\int_0^t f'(s) \,\mathrm{d}B_s - \int_0^t (f'(s))^2 \,\mathrm{d}s} \mathbf{1}_{\{B_s \le \alpha + f(s), s \le t\}}$. *Then*

$$\mathbb{E}[\sum_{x \in \mathcal{N}_t} F(X_u(s), s \le t) \mathbf{1}_{\{X_u(s) \le \alpha + f(s), s \le t\}}] = e^t \mathbb{Q}[\frac{1}{\zeta(t)} F(B_s)]$$

where under $\mathbb{Q}$, $\hat{B}_t = \alpha + f(t) - B_t, t \ge 0$ *is a Bessel process.*

3.7 M. Roberts' "simple path"

We are now going to follow Matthew Roberts recent paper [28] and some developments from Zeitouni's lecture notes to obtain Theorem 23 which is part of Bramson's results.

The main idea is that in order to avoid the second moment problem we are going to count only the particles that stay below a certain line.

3.7.1 Bounds on the tail of $M(t)$

Let

$$\beta := \sqrt{2} - \frac{3}{2\sqrt{2}} \frac{\log t}{t} \tag{3.5}$$

and define

$$H(y,t) := \#\{u \in \mathcal{N}_t : X_u(s) \le \beta s + 1 \, \forall s \le t, \beta t + y - 1 \le X_u(t) \le \beta t + y\} \tag{3.6}$$

Lemma 27 (First moment for H)**.** *For $t \geq 1$ and $y \in [0, \sqrt{t}]$*

$$\mathbb{E}[H(y,t)] \asymp e^{-\sqrt{2}t}.$$

Proof. We apply our many to one Lemma for counting particles staying under a curve with $f(s) = \beta s$ and $\alpha = 1$ to obtain

$$\begin{aligned}
\mathbb{E}[H(y,t)] &= e^t \mathbb{Q}\big[\frac{1}{\zeta(t)} \mathbf{1}_{\{\beta t + y - 1 \leq B_t \leq \beta t + y\}}\big] \\
&= e^t \mathbb{Q}\big[\frac{y e^{-\beta B_t + \beta^2 t/2}}{\beta t + y - \mathcal{B}_t} \mathbf{1}_{\{\beta t + y - 1 \leq B_t \leq \beta t + y\}}\big] \\
&\asymp y e^{t - \beta^2 t/2} \mathbb{Q}[\beta t + y - 1 \leq B_t \leq \beta t + y] \\
&\asymp y t^{3/2} e^{-\sqrt{2}y}\ \mathbb{Q}[1 \leq \beta t + y + 1 - B_t \leq 2].
\end{aligned}$$

Since $\beta t + 1 - B_t$ is a Bessel under $\mathbb{Q}$ we have that

$$\mathbb{Q}(1 \leq \beta t + y + 1 - B_t \leq 2) \asymp \int_1^2 \frac{z^2}{t^{3/2}}\, \mathrm{d}z \asymp t^{-3/2}.$$

□

We sketch the rest of the proof. Roberts then use a *many-to-two Lemma* to prove that there exists a constant such that

$$\mathbb{E}[H(y,t)^2] \leq c\mathbb{E}[H(y,t)]$$

and from there the second moment method allows us to conclude that

$$\mathbb{P}(H(y,t) \neq 0) \geq c' y e^{-\sqrt{2}y}.$$

this gives us a lower bound for $m_{1/2}(t)$.

For the upper bound, Roberts use a similar technique. Let us introduce the following set Γ which has almost the same definition as H

$$\Gamma = \#\{u \in \mathcal{N}_t : X_u(s) \leq f(s)\, \forall s \leq t, \beta t + y - 1 \leq X_u(t) \leq \beta t + y\}$$

where now

$$f(s) = \begin{cases} \beta s + y + \frac{3}{2\sqrt{2}}\ \log(s+1) & \text{if } s \leq t/2 \\ \beta s + y + \frac{3}{2\sqrt{2}}\ \log(t-s+1) & \text{if } t/2 \leq s \leq t \end{cases}$$

Using the same type of calculations (albeit more involved) as above, Roberts shows that

$$\mathbb{E}[\Gamma] \leq c(y+2)^4 e^{-\sqrt{2}y}.$$

Interestingly, we don't need a second moment here (as we are going for an upper bound). Instead we are going to estimate the probability that a particle cross our curve $f(s)$ before time t. Let's call τ the first time a particle crosses $f(s)$. Roberts shows that there exists a constant c such that

$$\mathbb{P}(\tau < t) \leq c(y+2)^4 e^{-\sqrt{2}y}.$$

To see this, it is enough to show that

$$\mathbb{E}[\Gamma | \tau < t] \ \geq c'$$

for some constant $c' > 0$, i.e. that once we hit $f(s)$ the hard work is done and we will have particles ending up near $\beta t + y$. Thus

$$\mathbb{P}(\tau < t) \leq \frac{\mathbb{E}[\Gamma]\mathbb{P}(\tau < t)}{\mathbb{E}[\Gamma \mathbf{1}_{\{\tau < t\}}]} = \frac{\mathbb{E}[\Gamma]}{\mathbb{E}[\Gamma | \tau < t]} \leq \frac{c_1}{c_2} \ (y+2)^4 e^{-\sqrt{2}y}.$$

By Markov inequality this implies

$$\mathbb{P}(M(t) \geq \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + y) \leq \mathbb{P}(\Gamma \geq 1) + \mathbb{P}(\tau < t) \leq c(y+2)^4 e^{-\sqrt{2}y}$$

for some $c > 0$.

3.7.2 Result about $\mathbb{E}[M(t)]$ and the median

Proposition 28. *The median $m_{1/2}(t)$ of $M(t)$ satisfies*

$$m_{1/2}(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + O(1) \qquad \text{as } t \to \infty.$$

Proof. To summarize what we have learned in the previous section : we now know that there are two constants c, c' such that for $0 \leq y \leq \sqrt{t}$,

$$ce^{-\sqrt{2}t} \leq \mathbb{P}(M_t > \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + y) \leq c'(y+2)^4 e^{-\sqrt{2}t}. \tag{3.7}$$

Define $m_\delta(t) := \inf\{x : \mathbb{P}(M(t) > x) \leq \delta\}$. Then, by choosing δ small enough the above implies that

$$m_\delta(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + O(1).$$

Let us show quickly that $M(t) - m_\delta(t)$ is tight which is enough to yield the desired conclusion. Fix $\epsilon > 0$ and choose L such that

$$\mathbb{E}[(1-\delta)^{-N(L)}] < \epsilon/2$$

and then a such that

$$\mathbb{P}(\min_{u \in \mathcal{N}_t} X_u(t) < -a) < \epsilon/2.$$

For a particle $u \in \mathcal{N}_L$ and $t > L$ note as usual $M_t^{(u)}$ the position of the maximal descendent of u at time t. Then

$$\begin{aligned}
&\mathbb{P}(M(t) \leq m_\delta(t-L) - a) \\
&\leq \mathbb{P}(\min_{u\in\mathcal{N}_t} X_u(t) < -a) + \mathbb{P}(\min_{u\in\mathcal{N}_t} X_u(t) \geq -a, \max_{u\in\mathcal{N}_L} M_t^{(u)} < m_\delta(t-L) - a) \\
&\leq \mathbb{P}(\min_{u\in\mathcal{N}_t} X_u(t) < -a) + \mathbb{E}[\mathbb{P}(M(t-L) < m_\delta(t-L))^{N(L)}] \\
&\leq \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

□

Observe that the upper bound implies that

$$\mathbb{E}[M(t)] \leq \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + O(1).$$

A matching lower bound can be obtained provided we show that

$$\lim_{z\to-\infty} \limsup_{t\to\infty} \int_{-\infty}^{z} \mathbb{P}(M_t \leq m_t + y)\, \mathrm{d}y.$$

This is the approach take in Zeitouni's lecture notes to obtain

Proposition 29.

$$\mathbb{E}[M(t)] = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + O(1) \qquad \text{as } t \to \infty.$$

3.8 Convergence in law of $M(t) - m_t$.

3.8.1 An exact equivalent for the tail of $M(t)$

The first step is to reinforce the bounds

$$ce^{-\sqrt{2}t} \leq \mathbb{P}(M_t > \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + y) \leq c'(y+2)^4 e^{-\sqrt{2}t}$$

into the following asymptotic equivalent:

Lemma 30. *Remember that $m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t$. There exists a constant c such that*

$$\mathbb{P}(M(t) > m_t + y) \sim cye^{-\sqrt{2}y}, \ \text{as } y \to \infty.$$

Proof. We only sketch the proof here.

□

3.8.2 Convergence of $M(t) - m_t$

Write $\hat{M}_t = M(t) - m_t$. Then

$$\begin{aligned}\mathbb{P}(\hat{M}_t \leq y) &= \mathbb{E}[\mathbb{P}(\hat{M}_t \leq y|\mathcal{F}_s)] \\ &= \mathbb{E}[\prod_{u\in\mathcal{N}_s} \mathbb{P}(M^u_{t-s} \leq m_t + y - X_u(s)|X_u(s))] \\ &= \mathbb{E}[\prod_{u\in\mathcal{N}_s} (1 - \mathbb{P}(M^u_{t-s} > m_{t-s} + y - (X_u(s) - \sqrt{2}s) + o(1)|X_u(s))]\end{aligned}$$

where we use that $m_t = m_{t-s} + \sqrt{2}s + o(1)$.

With probability tending to one, $M_s - \sqrt{2}s \to -\infty$ (this is Kolmogorov result, it can also be shown directly with martingale techniques, so that $y - (X_u(s) - \sqrt{2}s) \gg 1$ for all $u \in \mathcal{N}_s$ and hence we can use our tail estimate to obtain

$$\begin{aligned}\mathbb{P}(\hat{M}_t \leq y) &\sim_t \mathbb{E}\big[\prod_{u\in\mathcal{N}_s} (1 - c(y + \sqrt{2}s - X_u(s))e^{-\sqrt{2}(y+\sqrt{2}s-X_u(s))}\big] \\ &\sim_s \mathbb{E}\big[\exp(-cZ_s e^{-\sqrt{2}y})\big]\end{aligned}$$

where, in the last line, we have used the notation

$$Z_s := \sum_{u\in\mathcal{N}_s} (\sqrt{2}s - X_u(s))e^{-\sqrt{2}(\sqrt{2}s-X_u(s))}.$$

We have shown that

$$\lim_{t\to\infty} \mathbb{P}(M(t) - m_t \leq y) \sim \mathbb{E}[\exp(-cZ_s e^{-\sqrt{2}y})] \text{ as } s \to \infty.$$

However, the left-hand side **does not** depends on s and hence the right hand side must have a limit when $s \to \infty$. This implies that $Z_s \to Z$ in distribution (Z can't degenerate because of a priori bounds). We conclude that

$$\mathbb{P}(M(t) - m_t \leq y) \to \mathbb{E}\big[\exp(-cZe^{-\sqrt{2}y})\big].$$

Remark 31. *Observe that the same argument shows that*

$$\lim_t \mathbb{P}(\hat{M}(t) \leq y|\mathcal{F}_s) = \exp(-cZ_s e^{-\sqrt{2}y})$$

but that we can't then take a limit in s on both side.

Chapter 4

Spines, martingales and probability changes

Spine decompositions and their relation to probability tilting for branching processes is one of these ideas that have discovered several time by different group of people under different guises. One should however certainly credit the 1998 Paper of Lyons, Pemantle and Peres for bringing it in sharper focus. It is a tool that has since been used with great efficiency by people like Kyprianou, Harris, Roberts, etc...

The idea is that we are going to distinguish one particular line of descent from the root, so at each time there will be a *tagged* particle that we will call the spine. Once we have enlarged our probability space to take this extra-information into account this helps tremendously to simplify the probability tilting method pioneered by Chauvin and Rouault.

4.1 Dyadic Brownian motion with spine

Definition 32. *A spatial tree with a spine is a pair* $(\mathbf{t}, \xi)$ *where* $\mathbf{t} = (\tau, \sigma, Y)$ *is a spatial tree and* ξ *is a subset of* $\mathcal{U}$ *with the following properties*

1.
$$\#\{\xi \cap \mathcal{N}(t)\} \leq 1, \forall t \geq 0$$

2. $u \in \xi \Rightarrow v \in \xi$ *for each* $v < u$.

The space of marked trees with spines is denoted by $\mathcal{T}^*$*. In other words, a spine is a distinguished line of descent in a tree, finite or infinite. If* $v \in \xi \cap N(t)$ *we write* $\xi_t = v$ *for the label of the spine and* $\Xi(t) = X_{\xi_t}(t)$ *for the position of the spine particle.*

Let us now define, as simply as we can, a law $\tilde{\mathbb{P}}$ on $\mathcal{T}^*$ such that if the pair $(\mathbf{t}, \xi)$ has law $\tilde{\mathbb{P}}$ then $\mathbf{t}$ is a dyadic Brownian motion.

Definition 33. *Let* $\mathbf{t}$ *be a dyadic Brownian motion with law* $\mathbb{P}$. *Then conditionally on* $\mathbf{t}$ *construct inductively* ξ *as follows. For all* $t < d_\emptyset$ *let* $\xi_t = \emptyset$. *Pick* U_1 *uniformly among the offsprings of* $\emptyset$ *and for all* $t \in [d_\emptyset, d_{U_1})$ *let* $\xi_t = U_1$, *then* ξ_t *becomes a children of* U_1 *picked uniformly at random,* U_2 *and so on .... The law of the pair* $\mathbf{t}^* = (\mathbf{t}, \xi)$ *is denoted by* $\tilde{\mathbb{P}}$ *(or* $\tilde{\mathbb{P}}_x$ *to emphasize the starting point).*

As for the Brownian motion, we think of $(X(t), \xi(t))_{t\geq 0}$ as the *process* version of $\mathbf{t}^*$ and we call $\tilde{\mathcal{F}}_t$ its natural filtration. Observe that $\mathcal{F}_t \subset \tilde{\mathcal{F}}_t$. We further introduce

$$\mathcal{G}_t = \sigma(\Xi(s), s \leq t), t \geq 0$$

the natural filtration of Ξ and

$$\tilde{\mathcal{G}}_t = \sigma(\Xi(s), s \leq t) \vee \sigma(\sigma_u, u < \xi_t), t \geq 0$$

which is $\tilde{\mathcal{G}}$ augmented with the information of the branching times along ξ.

The following simple proposition tells us that one can instead chose to first decide the trajectory of the spine $(\Xi(t), t \geq 0)$ and then immigrate normal BBM on this spine.

Proposition 34 (Spine decomposition). *Let* $\Xi(t), t \geq 0$ *be a standard Brownian motion, let* $\pi = (t_1, t_2, \ldots)$ *be a rate 1 Poisson point process on* $\mathbb{R}_+$. *At each time* t_i *a new particle is created along the spine at* $\Xi(t_i)$. *The label of* Ξ *before the split is* ξ_{t_i-} *and* ξ_{t_i} *is obtained by appending "1" or "2" at the end with probability 1/2. The new non-spine particle starts a new independent dyadic branching Brownian motion (without spine) shifted by time, space and label. The law of the object created* $\mathbf{t}^* = (\mathbf{t}, \xi)$ *is again* $\mathbb{P}$.

We now make a couple of observations which will be useful later and illustrates the use of our notations.

First observe that

$$\mathbb{P}(u \in \xi \cap \mathcal{N}_t | \mathcal{F}_t) = \mathbf{1}_{\{u \in \mathcal{N}_t\}} \prod_{v<u} \frac{1}{A_v} = \mathbf{1}_{\{u \in \mathcal{N}_t\}} 2^{-|u|},$$

where the last equality is because we are restricting ourselves to the dyadic case.

We begin with the following useful Lemma

Lemma 35. *If* Y *is an* $\tilde{\mathcal{F}}_t$*-measurable random variable, then we can write*

$$Y = \sum_{u \in \mathcal{N}_t} Y_u \mathbf{1}_{\xi_t = u}$$

where each Y_u *is* $\mathcal{F}_t$*-measurable.*

Proof. This is essentially a consequence of the fact that if $Y \in \tilde{\mathcal{F}}_t$ then there exists a measurable map $F : \mathcal{U} \times \mathcal{T} \to \mathbb{R}$ such that

$$Y = F(\xi_t, \mathbf{t}_t)$$

where $\mathbf{t}_t = (X(s), s \leq t)$ is all the information in $\mathbf{t}$ available at time t. Thus we have that

$$Y = \sum_{u \in \mathcal{N}_t} F(u, \mathbf{t}_t) \mathbf{1}_{\{\xi_t = u\}}$$

and clearly $Y_u = F(u, \mathbf{t}_t)$ is $\mathcal{F}_t$-measurable. □

The expectation under $\tilde{\mathbb{P}}$ of a variable $Y \in \tilde{\mathcal{F}}_t$ can therefore be related to expectations under $\mathbb{P}$ of its decomposition.

Definition 36. *Let $Y \in \tilde{\mathcal{F}}_t$ have decomposition $Y = \sum_{u \in \mathcal{N}_t} F(u, \mathbf{t}_t) \mathbf{1}_{\{\xi_t = u\}}$. Then*

$$\tilde{\mathbb{P}}[Y] = \mathbb{P}\left[\sum_{u \in N(t)} Y_u \prod_{v < u} \frac{1}{A_v}\right] = \mathbb{P}\left[\sum_{u \in N(t)} Y_u 2^{-|u|}\right]. \tag{4.1}$$

4.2 The many-to-one Lemma

The following elementary result (of which more general versions will be given later) will be used repeatedly.

Lemma 37 (Many-to-one Lemma (1st version))**.** *For any $t \geq 0$, for any function $F : C_{[0,t]} \mapsto \mathbb{R}$ we have*

$$\mathbb{E}[\sum_{u \in N(t)} F(X_u(s), 0 \leq s \leq t)] = e^t \tilde{\mathbb{E}}[F(\Xi(s), 0 \leq s \leq t)].$$

Proof.

$$\begin{aligned}
\mathbb{E}[\sum_{u \in N(t)} F(X_u(s), 0 \leq s \leq t)] &= \mathbb{E}[\mathbb{E}[\sum_{u \in N(t)} F(X_u(s), 0 \leq s \leq t) | \# N(t)]] \\
&= \mathbb{E}[\# N(t) \tilde{\mathbb{E}}[F(\Xi(s), 0 \leq s \leq t)]] \\
&= e^t \tilde{\mathbb{E}}[F(\Xi(s), 0 \leq s \leq t)]
\end{aligned}$$

□

Remark 38. *It is clear that if branching happens at rate β and instead of dyadic branching each splitting produce an i.i.d. number according to some random variable A with $\mathbb{E}(A) = 1 + m$ then the formula becomes*

$$\mathbb{E}[\sum_{u \in N(t)} F(X_u(s), 0 \leq s \leq t)] = e^{m\beta t} \tilde{\mathbb{E}}[F(\Xi(s), 0 \leq s \leq t)]$$

In particular this lemma can be applied for functions of the form $F(X_u(s), 0 \leq s \leq t) = f(X_u(t))$.

A slightly more general version can be formulated as follows

Lemma 39 (Many-to-one Lemma (2nd version))**.** *For any $t \geq 0$, and a variable $Y \in \tilde{\mathcal{F}}_t$ with decomposition $Y = Y_\xi = \sum_{u \in N(t)} \mathbf{1}_{\{\xi_t = u\}} Y_u$ we have*

$$\mathbb{E}[\sum_{u \in \mathcal{N}_t} Y_u] = \tilde{\mathbb{E}}[Y_\xi \prod_{u < \xi_t} A_u] = \mathbb{E}[Y_{\xi_t} 2^{|\xi_t|}]$$

Proof.

$$\begin{aligned} \tilde{\mathbb{E}}[Y_\xi \prod_{u<\xi_t} A_u] &= \tilde{\mathbb{E}}[\sum_{u \in \mathcal{N}_t} Y_u \mathbf{1}_{\{\xi_t = u\}} \big(\prod_{v<u} A_v\big)] \\ &= \tilde{\mathbb{E}}\Big[\sum_{u \in \mathcal{N}_t} Y_u \tilde{\mathbb{P}}(\xi_t = u | \mathcal{F}_t) \big(\prod_{v<u} A_v\big)\Big] \\ &= \mathbb{E}[\sum_{u \in \mathcal{N}_t} Y_u] \end{aligned}$$

□

4.3 Additive martingales

The many-to-one Lemma is useful to construct martingales. Given a functional of continuous paths $F : t, C_{[0,t)} \to \mathbb{R}, (x(s), s \leq t) \mapsto F[(x(s), s \leq t)]$ let us define the quantities

$$\begin{aligned} \zeta(t) &:= F\big(\Xi(s), s \leq t\big) \\ \zeta_u(t) &:= F\big(X_u(s), s \leq t\big) \text{ for } u \in \mathcal{N}_t \\ Z(t) &:= e^{-t} \sum_{u \in \mathcal{N}_t} \zeta_u(t). \end{aligned}$$

Observe that Z is $\mathcal{F}_t$-adapted and that ζ is $\mathcal{G}_t$ adapted (it's a Brownian functional).

Proposition 40. *If ζ is a $\mathcal{G}_t$-martingale, then Z is an $\mathcal{F}_t$-martingale.*

Remark 41. *It can be shown that this is in fact an equivalence.*

Proof. For $u \in \mathcal{N}_t$ let us write F_u for the functional

$$F_u\big(f(r), r \leq s\big) = F\big(\phi(r), r \leq t + s\big)$$

where

$$\phi(r) = \begin{cases} X_u(r) \text{ if } & r \leq t \\ X_u(t) + f(r-t) \text{ if } & r > t \end{cases}$$

Suppose first that $\zeta(t)$ is a $\mathcal{G}_t$ martingale. Then

$$\begin{aligned}\mathbb{E}\big[e^{-(t+s)} \sum_{v\in\mathcal{N}_{t+s}} \zeta_v(t+s)\big|\mathcal{F}_t\big] &= e^{-t}\sum_{u\in\mathcal{N}_t}\zeta_u(t)\mathbb{E}\big[e^{-s}\sum_{u\leq v\in\mathcal{N}_{t+s}}\frac{F_u(X_v(t+r), r\leq s)}{\xi_u(t)}\big|\mathcal{F}_t\big]\\ &= e^{-t}\sum_{u\in\mathcal{N}_t}\zeta_u(t)\mathbb{E}\big[\frac{F_u(\Xi(r), r\leq s)}{\xi_u(t)}\big|\mathcal{F}_t\big]\\ &= e^{-t}\sum_{u\in\mathcal{N}_t}\zeta_u(t)\end{aligned}$$

where we have used that conditionally on $\mathcal{F}_t$, each particle $u\in\mathcal{N}_t$ starts an independent BBM from its position for which we can apply the many-to-one Lemma. The last line is just the property that ζ being a martingale, its expectation stays constant. Thus we have shown that Z is an $\mathcal{F}_t$ martingale. □

The case where F depends only on the current position is particularly simple

Proposition 42. *Let $h:\mathbb{R}\mapsto\mathbb{R}$ be a C^2 function, then*

$$W(t) := \sum_{u\in N(t)} h(X_u(t)+ct),\ \ t\geq 0$$

is a local $\mathcal{F}_t$-martingale if and only if h solves

$$\frac{1}{2}h'' + ch' + h = 0. \tag{4.2}$$

Equation (4.2) will appear again as the linearized version off the so-called *KPP traveling wave* equation.

Example 43 (Exponential additive martingales)**.** *For $\lambda\in\mathbb{R}$ we define*

$$W_\lambda(t) := \sum_{u\in N(t)} e^{-\lambda(X_u(t)+c_\lambda t)} \tag{4.3}$$

for $t\geq 0, \lambda\geq 0$ and with $c_\lambda = \lambda/2 + m\beta/\lambda$. Then the process $(W_\lambda(t), t\geq 0)$ is a martingale.

Proof. It suffices to observe that $e^{-m\beta t}e^{-\lambda(\xi_t+c_\lambda t)} = e^{-\lambda\xi_t-\frac{\lambda^2}{2}t}$ is a Brownian martingale. □

Remark 44. *W_λ is clearly a positive martingale and therefore $W_\lambda(t)\to W_\lambda(\infty)$ almost surely. The question is now to see for which parameters λ the variable $W_\lambda(\infty)$ is non trivial. Observe that in the $\lambda=0$ case the spatial character has no importance and W_λ only depends on the size of the population.*

Example 45 (The derivative martingale)**.** *The process*

$$Z(t) := \sum_{u \in \mathcal{N}_t} (\sqrt{2}t - X_u(t))e^{\sqrt{2}(X_u(t)-\sqrt{2}t)}, \qquad t \geq 0$$

is a martingale.

Observe that $Z(s)$ exactly the process which appears in the expression we gave for $\lim_t \mathbb{P}[M(t) - m_t > y|\mathcal{F}_s]$ so we already know that $Z(t)$ converges in distribution to a non-degenerate variable Z.

Proof. The fact that it is a martingale is clear from Proposition 42. Here

$$h(x) = -xe^{\sqrt{2}x} \text{ and } Z(t) = \sum_{u \in \mathcal{N}_t} h(X_u(t) - \sqrt{2}t)$$

so that $h'(x) = -e^{\sqrt{2}x}(x\sqrt{2} + 1)$ and $h''(x) = -2e^{\sqrt{2}x}(x + \sqrt{2})$. .

Later we will present a result of Lalley and Sellke 1987 [22] (see also Neveu [27]) which shows that $Z(t)$ converge almost surely to a positive random variable.

Proposition 46 (Lalley and Sellke, 1987)**.**

$$Z(t) \to Z > 0, \text{ as } t \to \infty \ \mathbb{P} - \text{almost surely.}$$

□

To finish observe that if we define

$$\tilde{\zeta}(t) := \zeta(t)e^{-t}2^{|\xi(t)|}$$

then we have that

Proposition 47. *$\tilde{\zeta}$ is a $\tilde{\mathcal{G}}_t$-martingale if and only if ζ is a $\mathcal{G}_t$-martingale and*

$$Z(t) = \tilde{\mathbb{E}}[\tilde{\zeta}(t)|\mathcal{F}_t] \ ; \ \zeta_t = \tilde{\mathbb{E}}[\tilde{\zeta}(t)|\mathcal{G}_t].$$

4.4 Changing probability with an additive martingale.

Suppose we chose F a path functional as above such that ζ is a $\mathcal{G}_t$-martingale, positive with mean 1. Then Z is also mean 1 and positive and we can define a new probability measure $\mathbb{Q}$ on $\mathcal{T}$ by the relation

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = Z(t).$$

In this section we are going to describe the law of X under $\mathbb{Q}$. The trick we are going to use is the following. We are first going to do a change of probability on $\mathcal{T}^*$ by

$$\frac{d\tilde{\mathbb{Q}}}{d\tilde{\mathbb{P}}}|_{\mathcal{F}_t} = \tilde{\zeta}(t).$$

If we can describe the law of $\mathbf{t}^*$ under $\tilde{\mathbb{Q}}$, then because $\mathbb{P} = \tilde{\mathbb{P}}|_{\mathcal{F}_\infty}$ and $Z(t) = \mathbb{E}[\tilde{\zeta}(t)|\mathcal{F}_t]$ we will have that $\mathbb{Q} = \tilde{\mathbb{Q}}|_{\mathcal{F}_\infty}$. Otherwise said, if we know what the variable $\mathbf{t}^* = (\mathbf{t}, \xi)$ looks like under $\tilde{\mathbb{Q}}$ then we simply forget the spine and the marked tree we obtain has law $\mathbb{Q}$ (i.e. $\mathbf{t} \sim \mathbb{Q}$).

The following Theorem is adapted from Chauvin and Rouault [?]

Theorem 48. *Under $\tilde{\mathbb{Q}}$ the BBM with spine $\mathbf{t}^* = (\mathbf{t}, \xi)$ has the following description:*

- *The path of the spine $t \mapsto \Xi(t)$ is governed by the law $\hat{\mathbf{P}}$*

$$\left.\frac{d\hat{\mathbf{P}}}{d\mathbf{P}}\right|_{\mathcal{G}_t} := \frac{\zeta(t)}{\zeta(0)},$$

  where $\mathbf{P}$ the law of a standard Brownian motion.

- *The spine splits at an accelerated rate 2β.*

- *At each branching along the spine, the new spine particle is chosen uniformly among the 2 children.*

- *At each branching event along the spine, the non-spine particle starts an independent branching Brownian motion with law $\mathbb{P}$ (no spine) shifted in time, space and label.*

The law of the first coordinate $\mathbf{t}$ of $\mathbf{t}^*$ has law $\mathbb{Q}$.

Remark 49. *If the BBM has branching rate β and is not dyadic but produces a random number A of offsprings at each branching with $\mathbb{E}[A] = 1+m$ then under $\tilde{\mathbb{Q}}$ the branching along the spine is at rate $\beta(m+1)$ and the numbers of offsprings produces at each branching along the spine $(\hat{A}_1, \hat{A}_2, \ldots)$ is an i.i.d. sequence of variables which are size-biased versions of A, i.e.*

$$\mathbb{Q}(\hat{A} = k) = \frac{k\mathbb{P}(A = k)}{1+m}.$$

Proof. Once we note that $\tilde{\zeta}(t) = \zeta(t)e^{-t}2^{|\xi(t)|}$ is the product of two independent martingales, one which only depends on the path Ξ and one which only depend on the Poisson process $|\xi(t)|$ we see that the conclusion follows trivially once we remember the following fact about change of probability for Poisson processes: Let

$\mathbb{L}^{(\alpha)}$ be the law of a Poisson process $(n_t t \geq 0)$ with rate $\alpha > 0$ adapted to some filtration $\{\mathcal{G}_t, t \geq 0\}$ and let $\mathbb{L}_t^{(\alpha)}$ be its restriction to $\mathcal{G}_t$. We have

$$\frac{\mathrm{d}\mathbb{L}_t^{(\beta(m+1))}}{\mathrm{d}\mathbb{L}^{(\beta)_t}}(\pi) = e^{-\beta m t}(m+1)^{n_t}$$

for all $t > 0$.

□

Let us give a particular example.

Example 50. *Suppose we use $\zeta(t) = e^{-\lambda \Xi(t) - \lambda^2 t/2}$. In this case recall that $Z(t)$ is simply the additive exponential martingale W_λ introduced above. ζ is a simple Girsanov transform and thus under $\hat{\mathbf{P}}$ it is well known that Ξ is a Brownian motion with drift $-\lambda$.*

Thus, under $\tilde{\mathbb{Q}}$, $\mathbf{t}^*$ is obtained by first letting Ξ be a Brownian motion with drift $-\lambda$ and conditionally on Ξ, we immigrate on Ξ at rate 2 standard branching Brownian motions with the usual law $\mathbb{P}$.

Exercise 51.

1. *Show that t for any $y > 0$ the process*

$$Z_y(t) := \sum_{u \in \mathcal{N}_t} (\sqrt{2}t + y - X_u(t)) e^{\sqrt{2}(X_u(t) - \sqrt{2}t)}, \qquad t \geq 0$$

is a martingale.

2. *Show that*

$$\bar{Z}_y(t) := \sum_{u \in \mathcal{N}_t} \frac{\sqrt{2}t + y - X_u(t)}{y} \mathbf{1}_{\{X_u(s) \leq \sqrt{2}s + y, \forall s \leq t\}} e^{\sqrt{2}(X_u(t) - \sqrt{2}t)}, \qquad t \geq 0$$

is a positive, mean one martingale (Hint: use Proposition 40).

3. *Describe the law of $(X(t), t \geq 0)$ under Q defined by*

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \bar{Z}_y(t).$$

Recall that $Z(t) = e^{-m\beta t} \sum_{u \in N(t)} \zeta_u(t)$. The next Lemma shows that the weights $e^{-m\beta t}\zeta_u(t)/Z(t)$ correspond to the probability under $\tilde{\mathbb{Q}}$ that u is the spine.

Lemma 52. *For all $u \in \mathcal{U}$ and $t \geq 0$*

$$\tilde{\mathbb{Q}}[\xi_t = u | \mathcal{F}_t] = \frac{e^{-m\beta t}\zeta_u(t)}{Z(t)} \mathbf{1}_{\{u \in N(t)\}}.$$

Proof. For any $F \in \mathcal{F}_t$

$$\begin{aligned}\tilde{\mathbb{Q}}(\{\xi_t = u\} \cap F) &= \tilde{\mathbb{P}}[\mathbf{1}_{\{\xi_t=u\}}\mathbf{1}_{\{F\}}2^{|\xi_t|}e^{-t}\zeta(t)] \\ &= \tilde{\mathbb{P}}[\mathbf{1}_{\{\xi_t=u\}}\mathbf{1}_{\{F\}}\sum_{w\in\mathcal{N}(t)} 2^{|w|}e^{-t}\zeta_w(t)\mathbf{1}_{\{w=\xi_t\}}] \\ &= \tilde{\mathbb{P}}[\mathbf{1}_{\{u\in\mathcal{N}(t)\}}\mathbf{1}_{\{F\}}2^{|u|}e^{-t}\zeta_u(t)\mathbf{1}_{\{u=\xi_t\}}] \\ &= \mathbb{P}[\mathbf{1}_{\{u\in\mathcal{N}(t)\}}\mathbf{1}_{\{F\}}e^{-t}\zeta_u(t)] \\ &= \mathbb{Q}\left[\frac{1}{Z(t)}\mathbf{1}_{\{u\in\mathcal{N}(t)\}}\mathbf{1}_{\{F\}}e^{-t}\zeta_u(t)\right].\end{aligned}$$

□

Observe that if $\mathbb{Q}$ and $\mathbb{P}$ are equivalent, then almost sure events under $\mathbb{Q}$ are also almost sure under $\mathbb{P}$. If we use $W_{-\lambda}$ to define $\mathbb{Q}$, then under $\mathbb{Q}$, almost surely $\liminf M_t/t \geq \liminf \Xi(t)/t = \lambda$. Thus, if we know for which values of λ the new probability $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ we gain information on the position of the rightmost particle (among other things).

4.5 Some results about change of measures

We recall some elementary results about change of probability. These are written here in discrete time, but the result adapt without any change to the continuous time setting. Let $\mathcal{F}_n$ be a filtration and let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures on $(\Omega, \mathcal{F}_\infty)$. Assume that for any $n, \mathbb{Q}_{|\mathcal{F}_n}$ is absolutely continuous with respect to $\mathbb{P}_{|\mathcal{F}_n}$ with density $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_n} = X_n$ and call $X := \limsup X_n$.

Proposition 53. *The process $(X_n, n \geq 0)$ is a $\mathbb{P}$-martingale and $X_n \to X$ $\mathbb{P}$-a.s. with $X < \infty$ $\mathbb{P}$-a.s. Furthermore,*

$$\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[X\mathbf{1}_A] + \mathbb{Q}(A \cap \{X = \infty\}), \forall A \in \mathcal{F}_\infty.$$

Proof. Let $A \in \mathcal{F}_n$. then

$$\mathbb{E}_{\mathbb{P}}[X_{n+1}\mathbf{1}_{\{A_n\}}] = \mathbb{Q}(A_n) = \mathbb{E}_{\mathbb{P}}[X_n\mathbf{1}_{\{A_n\}}]$$

Therefore $\mathbb{E}_{\mathbb{P}}[X_{n+1}\mathcal{F}_n] = X_n$ is a non-negative $\mathbb{P}$-martingale and thus converges almost surely to $X < \infty$..

Assume first that $\mathbb{Q} << \mathbb{P}$ and let $\eta := \frac{d\mathbb{Q}}{d\mathbb{P}}$. By the same argument as above, we see that

$$X_n = \mathbb{E}_{\mathbb{P}}[\eta|\mathcal{F}_n]$$

$\mathbb{P}-$almost surely. Levy's Martingale convergence[1] Theorem tells us that $X_n \to \eta, \mathbb{P}$-a.s. so $\eta = X$ $\mathbb{P}$-a.s. Thus, for any $A \in \mathcal{F}_\infty$ we have

$$\mathbb{Q}(A, X < \infty) = \mathbb{Q}(A, \eta < \infty) = \mathbb{Q}(A) = \mathbb{P}(X\mathbf{1}_{\{A\}})$$

[1] if η is $\mathbb{P}$ integrable $\mathbb{E}_{\mathbb{P}}(\eta|\mathcal{F}_n)$ converges a.s. and in $L^1(\mathbb{P})$ to $\mathbb{E}_{\mathbb{P}}(\eta|\mathcal{F}_\infty)$

whereas

$$\mathbb{Q}(A \cap \{X < \infty\}) = 0.$$

So we have the desired conclusion.

In the general case, ket $\rho = \frac{1}{2}(\mathbb{P} + \mathbb{Q})$ so that $\mathbb{P} << \rho$ and $\mathbb{Q} << \rho$. Applying the proof above to

$$r_n = \frac{d\mathbb{P}}{d\rho}|_{\mathcal{F}_n}, \quad s_n = \frac{d\mathbb{Q}}{d\rho}|_{\mathcal{F}_n}.$$

Let $r := \limsup r_n$, $s = \limsup s_n$. According to the proof above $r_n \to r = \frac{d\mathbb{P}}{d\rho}$ and $s_n \to s = \frac{d\mathbb{Q}}{d\rho}$.Since $r_n + s_n = 1$ ρ-almost surely

$$\frac{s}{r} = \frac{\lim s_n}{\lim r_n} = \lim \frac{s_n}{r_n} = \lim X_n = X.$$

In particular

$$\{r = 0\} = \{x = \infty\}, \{r > 0\} = \{\xi < \infty\}, \mathbb{Q}\text{-a.s..}$$

Let $A \in \mathcal{F}_\infty$. We have

$$\mathbb{Q}(A) = \int_a s d\rho = \int_A s\mathbf{1}_{\{r>0\}} d\mathbb{Q} + \int_A s\mathbf{1}_{\{r=0\}} d\mathbb{Q}.$$

Since $\int_A s\mathbf{1}_{\{r>0\}} d\mathbb{Q} = \int_A rX\mathbf{1}_{\{x<\infty\}} d\mathbb{Q} = \mathbb{E}_{\mathbb{P}}[X\mathbf{1}_{\{X<\infty\}}\mathbf{1}_{\{A\}}]$ (because we know that $X < \infty$ $\mathbb{P}$-a.s.) and $\int_A s\mathbf{1}_{\{X=\infty\}} d\mathbb{Q} = \mathbb{Q}(A \cap \{X = \infty\})$ this yields the conlcusion. □

Proposition 54.

$$\mathbb{Q} \ll \mathbb{P} \Leftrightarrow X < \infty \ \mathbb{Q} \ a.s. \ \Leftrightarrow \mathbb{E}(X) = 1 \quad (4.4)$$

$$\mathbb{Q} \perp \mathbb{P} \Leftrightarrow X = \infty \ \mathbb{Q} \ a.s. \ \Leftrightarrow \mathbb{E}(X) = 0 \quad (4.5)$$

Proof. Exercice □

4.6 Additive martingale convergence

Recall from (4.3) that

$$W_\lambda(t) = \sum_{u \in N(t)} e^{-\lambda(X_u(t) + c_\lambda t)}$$

with $c_\lambda = \lambda/2 + m\beta/\lambda = \lambda/2 + 1/\lambda$ is an additive martingale and that under $\tilde{\mathbb{Q}}_\lambda$ the spine follows a Brownian motion with drift $-\lambda$ (since the corresponding $\mathcal{G}$ martingale is $\zeta(t) = e^{-\lambda\xi(t) - \lambda^2 t/2}$). Let us define $\lambda^* := \sqrt{2\beta m} = \sqrt{2}$. Observe that $\lambda \mapsto c_\lambda$ attains its maximum on $(-\infty, 0)$ at $-\lambda^*$ and its minimum on $(0, \infty)$ at λ^*.

Theorem 55. *The limit $W_\lambda := \lim_{t\to\infty} W_\lambda(t)$ exists $\mathbb{P}$-almost surely and*

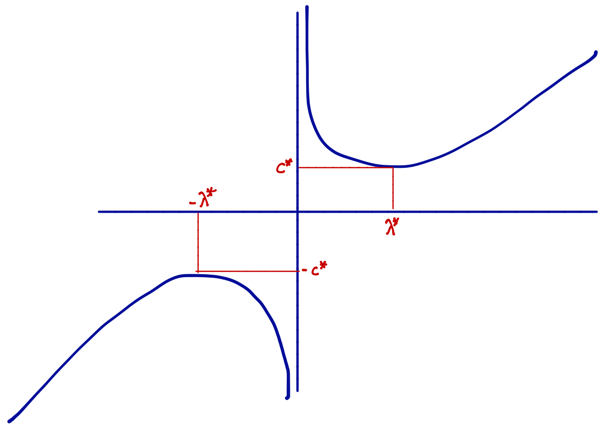


Figure 4.1: The function $\lambda \mapsto c_\lambda$

(i) If $|\lambda| \geq \lambda^*$ then $W(\lambda) = 0$ $\mathbb{P}$-almost surely.

(ii) If $|\lambda| \in [0, \lambda^*)$ then $W(\lambda)$ is a $L^1(\mathbb{P})$ limit and $\mathbb{P}(W_\lambda > 0) = 1$.

Remark 56. *When the BBM is not dyadic one must further distinguish between the cases $\mathbb{E}[A \log^+ A] = \infty$ (in which case $W_\lambda(\infty) = 0$ even if $|\lambda| < \lambda^*$) and $\mathbb{E}[A \log^+ A] < \infty$ (then it's like in the dyadic case).*

Before proving this Theorem we first give a decomposition and a 0-1 law result. Observe that

$$\begin{aligned} W_\lambda(t+s) &= \sum_{u \in N(t)} \sum_{u \leq v \in N(t+s)} e^{-\lambda(X_v(t+s)+c_\lambda(t+s))} \\ &= \sum_{u \in N(t)} e^{-\lambda(X_u(t)+c_\lambda t)} \sum_{u \leq v \in N(t+s)} e^{-\lambda((X_v(t+s)-X_u(t))+c_\lambda s)} \\ &= \sum_{u \in N(t)} e^{-\lambda(X_u(t)+c_\lambda t)} W^u_\lambda(s) \end{aligned}$$

where the $W^u_\lambda(s)$ are i.i.d. copies of $W_\lambda(s)$. By taking $s \to \infty$ we get

$$W_\lambda = \sum_{u \in N(t)} e^{-\lambda(X_u(t)+c_\lambda t)} W^u_\lambda \tag{4.6}$$

where the W^u_λ are i.i.d. copies of W_λ started from one particle at the origin.

Proposition 57. *Define* $p := \mathbb{P}(W_\lambda = 0)$. *Then* p *can either be* $q = \mathbb{P}(\exists t < \infty : \mathcal{N}_t = \emptyset\}$ *or 1.*

In the dyadic case, obviously $q = 0$.

Proof. Clearly, as the BBM is homogeneous in space, W_λ under $\mathbb{P}^x$ has the same distribution as $e^{-\lambda x} W_\lambda$ under $\mathbb{P}^0$. Thus, $p = \mathbb{P}^x(W_\lambda = 0)$ does not depend on x.

Observe that using (4.6)

$$\begin{aligned} p = \mathbb{P}(W_\lambda = 0) &= \mathbb{P}(\sum_{u \in N(t)} e^{-\lambda(X_u(t) + c_\lambda t)} W_\lambda^u = 0) \\ &= \mathbb{P}\left(\forall u \in \mathcal{N}_t : W_\lambda^u = 0\right) \\ &= \mathbb{E}(p^{\# N(t)}) \end{aligned}$$

Using the Markov and branching property, we easily see that $(p^{\# N(t)}, t \geq 0)$ is a bounded martingale (Exercise: do it). If $p < 1$ the limit will be 0 a.s. on the event of non-extinction contradicting the martingale convergence theorem unless $p = q$. Therefore we either have $p = q$ or $p = 1$. □

Proof of Theorem 55. **Case (i)** $\lambda \geq \lambda^*$. Observe first that $-\lambda + c_\lambda \leq 0$ (it is 0 when $\lambda = \lambda^*$). Thus $\Xi(t) + c_\lambda t$ is a $\tilde{\mathbb{Q}}_\lambda$ Brownian motion with non-positive drift. As

$$W_\lambda(t) \geq \exp(-\lambda(\Xi(t) + c_\lambda t))$$

it follows immediately that $\limsup W_\lambda(t) = \infty$, $\tilde{\mathbb{Q}}_\lambda$-a.s. and hence by Proposition 54 $W_\lambda = 0$ $\mathbb{P}$-a.s.

Case (ii) $\lambda < \lambda^*$.

$(\Xi(t) + c_\lambda t, t \geq 0)$ is a $\tilde{\mathbb{Q}}_\lambda$ Brownian motion with strictly positive drift.

We are now going to use a *spinal* decomposition of $W_\lambda(t)$. The variable $W_\lambda(t)$ is a sum over all particles alive at time t. We separate them according to when they have branched off from the spine ξ and write

$$\begin{aligned} W_\lambda(t) &= \sum_{u < \xi_t} \sum_{u < v \in \mathcal{N}_t} e^{-\lambda(X_v(t) + c_\lambda t)} \\ &= \sum_{u < \xi_t} e^{-\lambda(\Xi(d_u) + c_\lambda d_u)} \sum_{u < v \in \mathcal{N}_t} e^{-\lambda([X_v(t) - \Xi(d_u)] + c_\lambda [t - d_u])} \end{aligned}$$

where recall that the collection $(d_u, u < \xi_t)$ is simply the successive times of branching along ξ before t. For $u < \xi_t$ the collection $(X_v(s + d_u) - \Xi(d_u), u < v \in \mathcal{N}_{d_u + s})$ is a standard branching Brownian motion, and thus, for each $u < \xi_t$, conditionally on d_u and $\Xi(d_u)$ the process $\sum_{u < v \in \mathcal{N}_t} e^{-\lambda([X_v(d_u + s) - \Xi(d_u)] + c_\lambda s)}$ is a martingale. We conclude that

$$\tilde{\mathbb{Q}}(W_\lambda(t) | \tilde{\mathcal{G}}_\infty) = \sum_{u < \xi_t} e^{-\lambda(\Xi(d_u) + c_\lambda d_u)} + e^{-\lambda(\Xi(t) + c_\lambda t)}.$$

Using Fatou's Lemma and the strong law of large numbers

$$\begin{aligned}\tilde{\mathbb{Q}}(\liminf_t W_\lambda(t)|\mathcal{G}) &\le \limsup \tilde{\mathbb{Q}}(W_\lambda(t)|\mathcal{G})\\ &\le \limsup \sum_{u<\xi_t} e^{-\lambda(\xi_{d_u}+c_\lambda d_u)} + \limsup e^{-\lambda(\xi_t+c_\lambda t)}\\ &< \infty \ \tilde{\mathbb{Q}}_\lambda - \text{a.s.}\end{aligned}$$

where we have used that there exists $k > 0$ such that $\tilde{\mathbb{Q}}_\lambda$-almost surely, for t large enough $\lambda(\xi_t + c_\lambda t) > kt$.

Hence $\liminf W_\lambda < \infty$ $\tilde{\mathbb{Q}}_\lambda$-a.s. and thus $\mathbb{Q}_\lambda$-a.s. In a moment we will show that $1/W_\lambda(t)$ is a $\tilde{\mathbb{Q}}_\lambda$-martingale (in the present case it actually is a martingale). As it's positive it converges $\tilde{\mathbb{Q}}_\lambda$ almost surely to its lim inf. By the dichotomy we now know that $\mathbb{E}_\mathbb{P}[W_\lambda(\infty)] = 1$ so that $\mathbb{P}(W_\lambda(\infty) = 0) = 0$.

To conclude, let us just show that $1/W_\lambda(t)$ is a $\tilde{\mathbb{Q}}_\lambda$-martingale. Suppose $A \in \mathcal{F}_t$. Then

$$\begin{aligned}\mathbb{Q}[\frac{\mathbf{1}_{\{A\}}}{W_\lambda(t)}\mathbb{P}(W_\lambda(t+s) > 0|\mathcal{F}_t)] &= \mathbb{P}[\frac{\mathbf{1}_{\{A\}}}{W_\lambda(t)}\mathbb{P}(W_\lambda(t+s) > 0|\mathcal{F}_t)\cdot W_\lambda(t)]\\ &= \mathbb{P}(A \cap \{W_\lambda(t+s) > 0\})\\ &= \mathbb{Q}[\frac{1}{W_\lambda(t+s)}\mathbf{1}_{\{A\}}].\end{aligned}$$

□

4.7 First application : The speed of the rightmost particle

Recall that $\lambda^* = \sqrt{2\beta m}$ and let $c^* := c_{\lambda^*} = \lambda^*$.

Theorem 58. *Suppose $A \equiv 1$. The extremal particle has asymptotic speed $\sqrt{2\beta}$, i.e. if we define $M(t) := \sup_{u\in N(t)} X_u(t)$*

$$\frac{M(t)}{t} \to \sqrt{2\beta}, \ \mathbb{P}\text{-}a.s.$$

and furthermore

$$M(t) - c^* t \to -\infty.$$

Proof. We use the additive martingale convergence Theorem 55. All the martingales

$$W_\lambda(t) = \sum_{u\in N(t)} e^{-\lambda(X_u(t)+c_\lambda t)}$$

converge. Furthermore, $W_\lambda(t) \to 0$ $\mathbb{P}$-a.s. as soon as $|\lambda| \geq \lambda^*$. Note that $e^{\lambda^*(M(t)+c_{-\lambda^*}t)} \leq W_{-\lambda^*}(t) \to 0$ as soon as $|\lambda| \geq \lambda^*$. Thus $M(t) + c_{-\lambda^*}t \to -\infty$ and we just need to observe that $c_{-\lambda^*} = -c^*$ to conclude that

$$M(t) - c^* t \to -\infty.$$

and furthermore $\limsup \frac{M(t)}{t} \leq c^*$.

We just need to prove the converse bound to conclude. When $\lambda \in (-\lambda^*, 0]$, the probabilities $\mathbb{Q}_\lambda$ and $\mathbb{P}$ are equivalent. But since under $\mathbb{Q}_\lambda$ the process $\Xi(t)$ is a BM with drift $-\lambda$ we see that $\liminf M(t)/t \geq |\lambda|, \mathbb{Q}_\lambda$-a.s. and thus $\mathbb{P}$-a.s. as well. As λ is arbitrary in $(-\lambda^*, 0]$ we obtain $\liminf M(t)/t \geq \lambda^*, \mathbb{P}$-a.s.

□

Chapter 5

Traveling waves

A striking feature of the KPP equation is that it is one of the simplest example of a partial differential equation which admits a *traveling wave solution*, that is

$$u_t = \frac{1}{2}u_{xx} + u(u-1) \tag{5.1}$$

has solutions of the form $u(t,x) = w(x-ct)$. If such a solution exists it means that $u(x,t)$ is always a translate of the function w which we can thus see as a *front* with constant shape moving with the velocity c on the real line.

Proposition 59. *Let $w : \mathbb{R} \to [0,1]$ be C^2. Then $u(t,x) = w(x-ct)$ is a solution of* (5.1) *if and only if*

$$0 = \frac{1}{2}w'' + cw' + w(w-1). \tag{5.2}$$

One of the most striking result in Kolmogorov et al. original 1937 paper is the following:

Theorem 60. *Equation* (5.1) *has a monotone traveling wave w_c of speed c if and only if $|c| \geq \sqrt{2}$. Furthermore, the traveling wave solution of speed c is unique up to a shift in its argument and if $c > 0$ (resp. $c < 0$) it is increasing (resp. decreasing) with $w_c(-\infty) = 0, w_c(\infty) = 1$ (resp. $w_c(-\infty) = 1, w_c(\infty) = 0$).*

The traveling wave with speed $\sqrt{2}, w = w_{\sqrt{2}}$ is often called the *critical* traveling wave. Remember that we also know from KPP, Bramson that

Theorem 61. *If u is solution of* (5.1) *with initial condition $u(0,x) = \mathbf{1}_{\{x \geq 0\}}$, then $u(t, x+m_t) \to w(x)$ uniformly in x as $t \to \infty$*

If we admit this, the results of Chapter 3 shows that

$$w(x) = \mathbb{E}\big[\exp\big(-cZe^{-\sqrt{2}x}\big)\big]$$

and

$$1 - w(x) \sim cxe^{-\sqrt{2}x}.$$

5.1 Traveling waves and multiplicative martingales

We now give a similar result for *product martingales* of the form

$$M(t) = \prod_{u \in N(t)} \phi(X_u(t) + ct), t \geq 0$$

where ϕ is a C^2 map $\mathbb{R} \mapsto [0,1]$.

Proposition 62. *Let $\phi : \mathbb{R} \mapsto \mathbb{R}$ be a C^2 function. It forms a product (local) martingale with (speed) parameter c,*

$$M_\phi(t) = \prod_{u \in N(t)} \phi(X_u(t) + ct), t \geq 0$$

if and only if ϕ solves (2.5)

$$\frac{1}{2}\phi'' + c\phi' + \beta(f(\phi) - \phi) = 0.$$

If in addition ϕ takes its values in $[0,1]$ then M_ϕ is a true martingale.

Proof of Proposition 62. Write $F(t, X(t)) = M_\phi(t) = \prod_{u \in N(t)} \phi(X_u(t) + ct)$. It is easily seen that if ϕ solves (2.5) then $\left(\mathcal{G} + \frac{\partial}{\partial t}\right) F \equiv 0$ and thus by Proposition 12 $(M_\phi(t), t \geq 0)$ is a local martingale.

Suppose now that there exists y such that $\frac{1}{2}\phi''(y) + c\phi'(y) + \beta(f(\phi) - \phi)(y) > 0$. Then this implies that for $t \geq 0, x \in \mathbb{R}$ such that $x + ct = y$

$$\left(\mathcal{G} + \frac{\partial}{\partial t}\right) F(t,x) = \frac{1}{2}\phi''(y) + c\phi'(y) + \beta(f(\phi) - \phi)(y) > 0.$$

This would imply that

$$\lim_{\epsilon \to 0} \frac{\mathbb{E}[F(t+\epsilon, X(t+\epsilon)) | X(t) = \{x\}] - F(t,x)}{\epsilon} > 0$$

which means that $F(t, X(t))$ cannot be a (local) martingale since For any Borel set $A \subseteq \mathbb{R}$ we have $\mathbb{P}(t, X(t) \in A) > 0$.

For instance, suppose that $\frac{1}{2}\phi''(y) + c\phi'(y) + \beta(f(\phi) - \phi)(y) = \eta > 0$ and define a stopping time τ as follows:

$$\tau = \inf\{t > 0 : \#N(t) > 2\} \wedge \inf\{t > 0 : (\mathcal{G} + \partial_t)F(t, X(t)) < \eta/2\}.$$

Now under $\mathbb{P}_y, \tau > 0$ and $\mathbb{E}(\tau) > 0$ (properties of BM and continuity of ϕ, ϕ' and ϕ''). Thus

$$\mathbb{E}_y(F(\tau, X(\tau)) - F(0,y) = \mathbb{E}\left(\int_0^\tau (\mathcal{G} + \partial_t)F(t, X(t))\, \mathrm{d}t\right) > \mathbb{E}_y(\tau)\eta/2 > 0.$$

□

5.2 Existence of traveling waves at supercriticality

Definition 63. *For $\lambda \in \mathbb{R}$ we define*

$$M_\lambda(t) := \mathbb{E}\left[e^{-W_\lambda(\infty)} | \mathcal{F}_t\right], \ t \geq 0. \tag{5.3}$$

The process $(M_\lambda(t), t \geq 0)$ is a martingale, bounded in $[0,1]$ and thus unifomly integrable with $M_\lambda(t) \to M_\lambda(\infty) := e^{-W_\lambda(\infty)}$ almost surely and in L_1. Define

$$w_\lambda(x) := \mathbb{E}_x\left[e^{-W_\lambda(\infty)}\right] = \mathbb{E}_0\left[e^{-e^{-\lambda x} W_\lambda(\infty)}\right],$$

and observe that when $\lambda > 0, x \mapsto w_\lambda(x)$ is monotone and increases from 0 to 1 (when $W_\lambda(\infty)$ is almost surely finite and not almost surely 0 when $|\lambda| < \lambda*$, which means $|c_\lambda| > c*$).

Remember that we have he following decomposition

$$W_\lambda(\infty) = \sum_{u \in N(t)} e^{-\lambda(X_u(t) + c_\lambda t)} W_\lambda^{(u)}(\infty)$$

where the variables $(W_\lambda^{(u)}(\infty), u \in N(t))$ are iid, independent of $N(t)$ and distributed as $W_\lambda(\infty)$. Thus

$$\begin{aligned} M_\lambda(t) &= \mathbb{E}\left[e^{-W_\lambda(\infty)} | \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\prod_{u \in N(t)} e^{-e^{-\lambda(X_u(t)+c_\lambda t)} W_\lambda^{(u)}(\infty)} | \mathcal{F}_t\right] \\ &= \prod_{u \in N(t)} \mathbb{E}\left[e^{-e^{-\lambda(X_u(t)+c_\lambda t)} W_\lambda^{(u)}(\infty)} | \mathcal{F}_t\right] \\ &= \prod_{u \in N(t)} w_\lambda(X_u(t) + c_\lambda t) \end{aligned}$$

is a product martingale with speed c_λ. To sum up

Theorem 64 (Existence)**.** *When $|c| > c^*$ the mapping,*

$$w_\lambda(x) = \mathbb{E}^x e^{-W_\lambda(\infty)}$$

where λ is picked so that $c = c_\lambda$ is a monotone traveling wave solution of (2.5) with speed c.

Note that w_λ is increasing from 0 to 1 if $c > 0$ and decreasing from 1 to 0 otherwise.

5.3 Non-existence of traveling waves at subcriticality

Take $c < c^*$. Remember that if $L(t) = \min_{u \in N(t)} X_u(t)$, by symetry

$$\lim_{t \to \infty} L(t) + ct = -\infty.$$

Suppose that ϕ_c is a non-trivial TW of speed c with $\phi_c \in [0,1]$, $\phi_c(-\infty) = 0$ and $\phi_c(+\infty) = 1$. We know that

$$M(t) := \prod_{u \in N(t)} \phi_c(X_u(t) + ct)$$

is a product martingale. Since it is bounded it is UI and converges a.s. and in L^1. But

$$\prod_{u \in N(t)} \phi_c(X_u(t) + ct) \leq \phi_c(L(t) + ct) \to 0$$

which is a contradiction. Hence, no bounded traveling waves from 0 to 1 exists at this speed.

Chapter 6

Extremal point process: The delay method

6.1 The centered maximum can't converge in an ergodic sense

Recall that by Kolmogorov result we know that if

$$u(t,x) = \mathbb{P}(M(t) \le x)$$

then

$$u(t, x + m_t) \to w(x) \text{ uniformly in } x \text{ as } t \to \infty$$

where w is the unique (up to a shift in the argument) solution of

$$0 = \frac{1}{2}w'' + \sqrt{2}w' + w(w-1).$$

Furthermore, we have seen that there exists c such that

$$1 - w(y) \sim c y e^{-\sqrt{2}y} \text{ as } y \to \infty.$$

This means in particular that $M(t) - m_t$ converges in distribution. But does it converge in an ergodic sense, i.e. is it true that for all $x \in \mathbb{R}$

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{1}_{\{M(s)-m_s \le x\}} \, \mathrm{d}s \to w(x) \text{ a.s.?}$$

A simple argument shows that this cannot be the case. Start two independent branching Brownian motions, one form 0 and one from x and note $M^x(t)$ fro the maximum of the one started form x.

Then, by invariance by translation

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{1}_{\{M^x(s)-m_s \le x\}} \, \mathrm{d}s = \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{1}_{\{M(s)-m_s \le 0\}} \, \mathrm{d}s = w(0) a.s.$$

But there is a strictly positive probability that before any branching event, the two initial particles meet and realize a successful coupling (after this meeting time the two process stays identical) which implies that there is a positive probability for

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{1}_{\{M^x(s)-m_s\le x\}}\ \mathrm{d}s = \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{1}_{\{M(s)-m_s\le x\}}\ \mathrm{d}s$$

or $w(x) = w(0)$, which is a contradiction.

6.2 Convergence of the derivative martingale and the centered maximum

We now show the results that were announced earlier

Proposition 65 (Convergence of the derivative martingale)**.** *Recall that the derivative martingale is the process*

$$Z(t) := \sum_{u\in\mathcal{N}_t} (\sqrt{2}t - X_u(t))e^{\sqrt{2}(X_u(t)-\sqrt{2}t)}$$

Then, almost surely

$$Z(t) \to Z > 0.$$

Proof. From the multiplicative martingale principle we know that

$$W^x(t) := \prod_{u\in\mathcal{N}_t} w(\sqrt{2}t - X_u(t) + x)$$

is an $\mathcal{F}_t$-martingale which is positive and bounded. It thus converges almost surely and L^1 to its limit

$$W^x := \lim_t W^x(t) \in [0,1]$$

and $\mathbb{E}W^x = w(x)$. Since $\min_{u\in\mathcal{N}_t} \sqrt{2}t - X_u(t) + x \to +\infty$ almost surely, the large time behavior of W^x is related to the asymptotic of w (this is essentially the same calculation we already did). As $t \to \infty$

$$\begin{aligned}\log W^x(t) &= \sum_{u\in\mathcal{N}_t} \log w(\sqrt{2}t - X_u(t) + x)\\ &\sim \sum_{u\in\mathcal{N}_t} -c(\sqrt{2}t - X_u(t) + x)\exp(\sqrt{2}X_u(t) - 2t - \sqrt{2}x)\\ &\sim -cZ(t)e^{-\sqrt{2}x} - cxW_{-\sqrt{2}}(t)e^{-\sqrt{2}x}\end{aligned}$$

Remember that we know that $W_{-\sqrt{2}}(t) \to 0$ almost surely. Thus

$$\lim_t Z(t) = \big(-e^{-\sqrt{2}x}/c\big)\log W^x$$

This proves that

$$w(x) = \mathbb{E}\big[\exp\big(-cZ(\infty)e^{-\sqrt{2}x}\big)\big]$$

which we already knew and that

$$Z(t) \to Z(\infty) > 0 \text{ a.s.}$$

Now, suppose $P(Z(\infty) = \infty) > 0$, and choose x such that $\mathbb{E}[W^x] > 1 - \mathbb{P}(Z(\infty) = \infty)/2$. Then

$$1 - \mathbb{P}(Z(\infty) = \infty)/2 < \mathbb{E}[W^x] = \mathbb{E}[W^x \mathbf{1}_{\{W_x > 0\}}] \leq \mathbb{P}(W^x > 0) = 1 - \mathbb{P}(Z(\infty) = \infty)$$

which is a contradiction.

And $Z(\infty) > 0$ is almost the same: suppose $\mathbb{P}(Z(\infty) = 0) > 0$, and choose x such that $\mathbb{E}[W^x] < \mathbb{P}(Z(\infty) = 0)/2$ (we can do this since $\mathbb{E}[W^x] = w(x) \to 0$ as $x \to -\infty$). Then

$$\mathbb{P}(Z(\infty) = 0)/2 > \mathbb{E}[W^x] \geq \mathbb{E}[W^x \mathbf{1}_{\{W_x = 1\}}] = \mathbb{P}(W^x = 1) = \mathbb{P}(Z(\infty) = 0)$$

□

Now that we know that the derivative martingale converges to a non-degenerate positive limit we can state lalley and Sellke's main result

Theorem 66 (Lalley and Sellke, 1987 [22]). *There exists a constant $c > 0$ such that for each $x \in \mathbb{R}$*

$$\lim_{s\to\infty} \lim_{t\to\infty} \mathbb{P}\big(M(t+s) - m_{t+s} \leq x \big| \mathcal{F}_s\big) = \exp\big\{-cZe^{-\sqrt{2}x}\big\} \text{ a.s.} \tag{6.1}$$

Consequently, the critical traveling wave has representation

$$w(x) = \mathbb{E}\exp\big\{-cZe^{-\sqrt{2}x}\big\} \tag{6.2}$$

and in particular

$$1 - w(x) \sim cxe^{-\sqrt{2}x} \quad \text{as} \quad x \to \infty.$$

One can use the same proof as in Section 3. For self-containedness we replicate it here.

Proof. As noted, $m_{t+s} = m_t + \sqrt{2}s + o_t(1)$. So we can write

$$\mathbb{P}\big(M(t+s) - m_{t+s}\ lex \big| \mathcal{F}_s\big) = \prod_{u\in\mathcal{N}_t} u(t, x + m_{t+s} - X_u(s))$$

where $u(t,x) = \mathbb{P}(M(t) \leq x)$ is the solution of the KPP equation with Heaviside initial data. Combining all this together

$$\lim_{t\to\infty} \mathbb{P}\big(M(t+s) - m_{t+s}\ lex \big| \mathcal{F}_s\big) = \prod_{u\in\mathcal{N}_t} w(x + \sqrt{2}s - X_i(s)) = W^x(s).$$

This yields the claimed result. □

6.3 Heuristic meaning of Lalley and Sellke's result

What does this result *mean*. Observe that if you treat Z as a known, fixed quantity (which you asymptotically can by conditioning on $\mathcal{F}_s$ with s big enough), then we have the representation

$$\mathbb{P}(M(t) - m_t \le x) \sim e^{-cZe^{-\sqrt{2}x}} = \exp\big\{ - e^{-\sqrt{2}(x-2^{-1/2}\log(cZ))}\big\}$$

which we can rewrite as

$$\mathbb{P}(\sqrt{2}(M(t) - m_t) - log(cZ) \le x) \sim e^{-e^{-x}}.$$

On the right hand side we recognize the distribution function of a Gumbel variable (a distribution which occurs often in extreme values). So the meaning of Theorem 66 is that $M(t) - m_t$ builds up an initial delay of size $+2^{-1/2}\log cZ$ from the fluctuation sof the firstfew particles. After some time, there are enough particles that the law of large numbers starts to act and then $M(t)$ starts to fluctuate around $m_t + 2^{-1/2}\log cZ$ with Gumbel fluctuations.

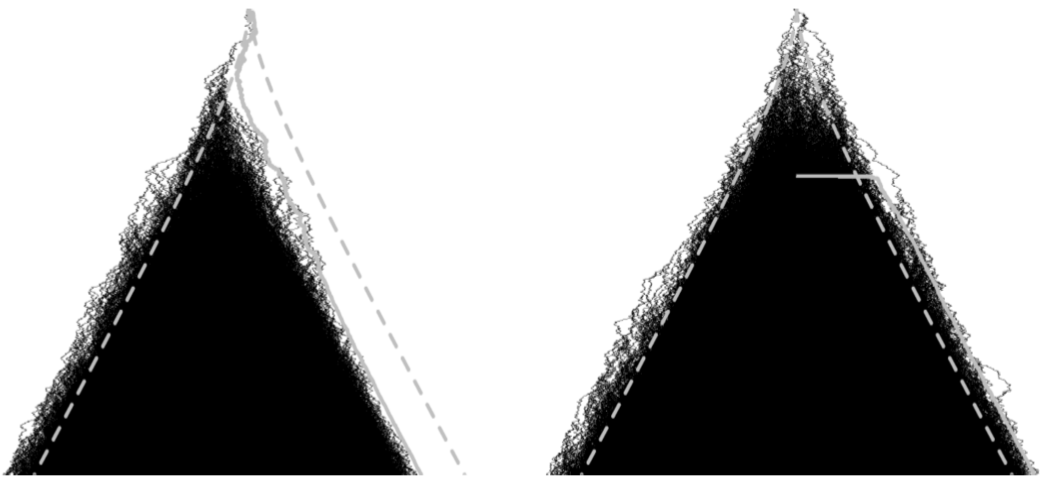


Figure 6.1: On the left realisation, $M(t) - m_t$ builds a negative initial delay, while on the right, by chance the initial particles go far to the right and build a positive initial advantage (image courtesy of Eric Brunet).

In fact Lalley and Sellke conjecture that *seen from* $m_t + 2^{-1/2}\log cZ$, the point process $\sum_{u\in\mathcal{N}_t} \Delta_{X_u(t)}(\cdot)$ converges in distribution.

6.4 Brunet and Derrida's delays method

At the heart of Brunet and Derrida delay method (see [8]) we find the other part of Bramson result. Since we need it here, we will start by stating a more precise version, but if you don't want to read the following, all you have to remember from Bramson is the following "meta-theorem".

Theorem 67. *If u solves*

$$u_t = \frac{1}{2}u_{xx} + u(u-1)$$

with initial condition $u(0,x) = g(x)$ such that $g = \mathbb{R} \to [0,1]$ with $1-g(x) = o(e^{-\sqrt{2}x})$ then there exists a constant c_g such that

$$u(t, x + m_t) \to w(x + c_g)$$

where $m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t$ and w is distribution function of the limit of $M(t) - m_t$.

Let us now give a more precise statement. We consider the equation

$$u_t = \frac{1}{2}u_{xx} + f(u)$$

with

$$f(0) = f(1) = 0, \;\; f(u) > 0 \text{ for } 0 < u \le 1$$

and

$$f'(0) = 1, f'(u) \le 1 \text{ for } 0 < u \le 1$$

and we also assume that $1 - f'(u) = O(u^\rho)$ for some $\rho > 0$. Also $w^c(x)$ denotes the travelling wave with speed $c > \sqrt{2}$. We set

$$c = c_\lambda = \frac{1}{\lambda} + \frac{\lambda}{2}.$$

Theorem 68. *Assume that the initial data is measurable with $0 \le u(0,x) \le 1$ for all x. If $c > \sqrt{2}$ then*

$$u(t, x + m(t)) \to w^c(x)$$

uniformly in x as $t \to \infty$ for some choice of $m(t)$ if and only if for some (and equivalently, all) $h > 0$

$$\lim_{t\to\infty} \frac{1}{t}\log\left[\int_t^{t(1+h)} u(0,y)dy\right] = -\lambda$$

and for some $\eta > 0, M > 0$ and $N > 0$

$$\int_x^{x+N} u(0,y)dy > \eta \text{ for } x \le -M.$$

If $c = \sqrt{2}$, then

$$u(t, x + m(t)) \to w^*(x)$$

uniformly in x as $t \to \infty$ for some choice of $m(t)$ if and only if for some (and equivalently, all) $h > 0$

$$\limsup_{t\to\infty} \frac{1}{t}\log\left[\int_t^{t(1+h)} u(0,y)dy\right] = -\sqrt{2}.$$

The other very important result obtained by Bramson concerns the centering term $m(t)$.

Theorem 69. *Suppose that $u(0, y) = \mathbb{I}_{y\leq 0}$ is the heavyside initial condition. Then we can chose*

$$m(t) = 2^{1/2}t - 3.2^{-3/2}\log t.$$

Define $h(y) = u(0, y)e^{\sqrt{2}y}$. If $h(y) = y^\alpha$ for $y \geq 1$ with $\alpha > -2$ then we can chose

$$m(t) = 2^{1/2}t - (\alpha - 1).2^{-3/2}\log t$$

whereas when $\alpha \leq -2$

$$m(t) = 2^{1/2}t - 3.2^{-3/2}\log t$$

still holds.

Remark 70.
1. *This means that if we define $m_a(t) = \inf\{x > 0 : u(t, x) = a\}$ then for any initial condition $u_0, x) = \phi(x)$ that decays faster than $e^{-2^{1/2}x}x^-2$ we have $m_a(t) = 2^{1/2}t - 3.2^{-3/2}\log t + \delta(a, \phi, f)$ where we call δ the delay.*
2. *Proof : Feynman-Kac integral with sample path estimates for Brownian motion.*

6.5 Laplace transforms

As we have seen, the initial data $u(0, x) = h(x) = \mathbf{1}_{\{x\geq 0\}}$ leads (through McKean's represntation) to

$$\begin{aligned} u(t, x + m_t) &= \mathbb{E}_{x+m_t}[\prod_{u\in\mathcal{N}_t} h(X_u(t))] \\ &= \mathbb{E}_0[\prod_{u\in\mathcal{N}_t} h(X_u(t) + x + m_t)] \\ &= \mathbb{E}_0[\prod_{u\in\mathcal{N}_t} h(x + m_t - X_u(t))] \\ &= \mathbb{E}_0[\prod_{u\in\mathcal{N}_t} \mathbf{1}_{\{X_u(t)\leq x+m_t\}}] \\ &= \mathbb{P}_0(M(t) \leq x + m_t) \to w(x) \end{aligned}$$

as $t \to \infty$ uniformly in x.

Let us now employ a slightly different starting condition. For $\lambda > 0$ fixed, let

$$h_1(x) = e^{-\lambda}\mathbf{1}_{\{x<0\}} + \mathbf{1}_{\{x\geq 0\}}.$$

The same argument yields that

$$\begin{aligned} u(t, x + m_t) &= \mathbb{E}_{x+m_t}[\prod_{u\in\mathcal{N}_t} h_1(X_u(t))] \\ &= \mathbb{E}_0[\prod_{u\in\mathcal{N}_t} h_1(x + m_t - X_u(t))] \\ &= \mathbb{E}_0[\exp\{-\lambda N_{[x,\infty)}(t)\}] \end{aligned}$$

where for $A \subset \mathbb{R}$

$$N_A(t) = \#\{u \in \mathcal{N}_t : X_u(t) - m_t \in A\}.$$

Thus, Bramson's result for initial datum h_1 reads

$$\mathbb{E}_0[\exp\{-\lambda N_{[x,\infty)}(t)\}] \to w(x + c_{h_1})$$

uniformly in x. This show that the variable $N_{[x,\infty)}(t)$ converges in distribution as $t \to \infty$.

For the convergence in distribution of the point process of particles centered by m_t

$$\bar{\mu}_t(\cdot) := \sum_{u \in \mathcal{N}_t} \delta_{X_u(t)-m_t}(\cdot)$$

we need a bit more. We need that the *joint*-Laplace transforms of the number of particles in disjoint Borel sets converge. Fir $\lambda, \mu > 0$ and $x_1 < x_2 \in \mathbb{R}$ consider

$$h_2(x) := \begin{cases} e^{-\mu x} \text{ if } \ x \le x_1 \\ e^{-\lambda x} \text{ if } \ x \in [x_1, x_2] \\ 1 \text{ if } \ x > x_2 \end{cases}$$

Now, if u is the solution of KPP with h_2 as initial datum, then the McKean representation tells us that

$$u(t, m_t + x) = \mathbb{E}_0[e^{-\mu N_{[x-x_1,\infty)}(t)} e^{-\lambda N_{[x-x_2,x-x_1)}}] \to w(x + c_{h_2}).$$

We can see that it is possible obtain in this way the convergence of the joint Laplace transform of the number of particles in any finite collection of disjoint Borel sets centered around m_t.

We conclude that

Theorem 71 (Brunet and Derrida, 2011)**.** *The extremal point process centered by m_t, i.e. $\bar{\mu}_t(\cdot)$ converges in distribution.*

Remark 72. *In fact, Brunet and Derrida show the convergence of the point process seen from $M(t)$, the right-most particle.*

6.6 Superposability

Now that we know that $\bar{\mu}_t(\cdot)$ converges in distribution, we want to know what the limit looks like.

Let us adopt a couple of notations. Given a point process $\mu(\cdot)$ on $\mathbb{R}$ and $\alpha \in \mathbb{R}$ we define the shift operator T_α by

$$T_\alpha \mu(\cdot) = \mu(\cdot - \alpha)$$

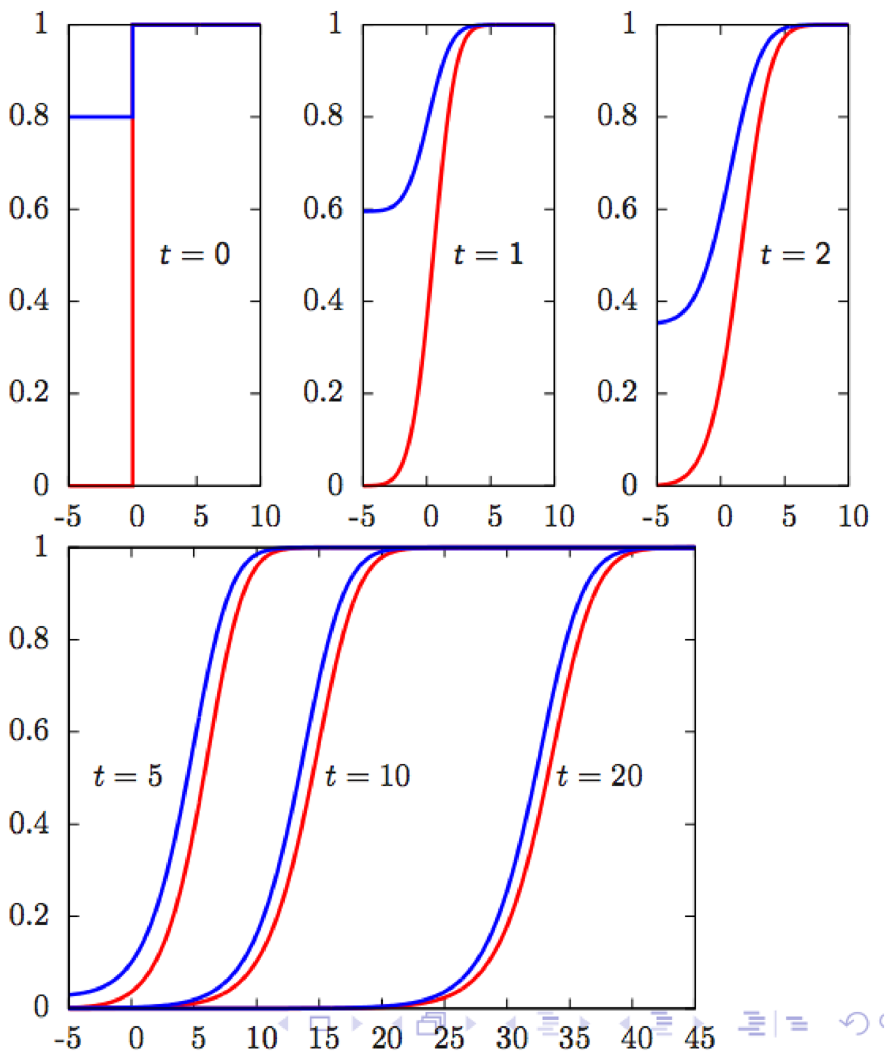


Figure 6.2: Solution of KPP equation with initial data h and h_1.

i.e. all the atoms x_i of μ are shifted by α to $x_i + \alpha$. We will also want to consider the particles of the branching Brownian motion from right to left $X_1(t) \geq X_2(t) \geq \ldots \geq X_{N(t)}(t)$. Lalley and Sellke's result is that $X_1(t) - m_t - 2^{-1/2} \log cZ$ converges in distribution to a Gumbel. By looking at suitable Laplace transforms as above this can be strengthened to obtain the convergence of the whole point process μ_t recentered

$$T_{(-m_t - \frac{\log(cZ)}{\sqrt{2}})}\mu_t \to \mathcal{L} \qquad \text{in distribution as } t \to \infty$$

where is the limit point process.

Now take two branching Brownian motions an denote by Z and Z' respectively the limit of their derivative martingale. The union of both process is again a BBM with derivative martingale limit $Z'' = Z + Z'$. Applying the above mentioned convergence result to both BBMs as well as to their union we get that for almost all realizations of Z and Z', $T_{\frac{\log(c(Z+Z'))}{\sqrt{2}}}\mathcal{L}$ is equal in law to $T_{\frac{\log(cZ)}{\sqrt{2}}}\mathcal{L} + T_{2\frac{\log(cZ')}{\sqrt{2}}}\mathcal{L}'$ where $\mathcal{L}$ and $\mathcal{L}'$ are two independent copies of the limit point process $\mathcal{L}$. For simplicity let us forget about the factors $2^{-1/2}$ and c to obtain

$$\mathcal{L} =_{\text{dist}} T_{\log\left(\frac{Z}{Z+Z'}\right)}\mathcal{L} + T_{\log\left(\frac{Z'}{Z+Z'}\right)}\mathcal{L}'$$

Since Z and Z' can take any values (by varying the starting points of the BBMs for instance) we see that for any a and b such that $e^a + e^b = 1$ we have

$$\mathcal{L} =_{\text{dist}} T_a\mathcal{L} + T_b\mathcal{L}'$$

where $\mathcal{L}$ and $\mathcal{L}'$ are two independent copies of $\mathcal{L}$. Point processes with this property are called *superposable* by Brunet and Derrida or *exponentially 1-stable* by Maillard.

What kind of point processes have this property?

Exercise 73. *1. Show that Poisson point processes with intensity $e^{-x}\,dx$ are superposable.*

2. Let (D_i) be a sequence of i.i.d. copies of a point process D, and let $(x_i)_{i\in\mathbb{N}}$ be the atoms of a point process with intensity $e^{-x}\,dx$. Show that

$$\mathcal{L} = \sum_i T_{\xi_i} D_i \tag{6.3}$$

is superposable.

Brunet and Derrida asked if *every* superposable point process is in fact a decorated exponential point process of the form (6.3). Maillard was able to show that this is indeed the case

Theorem 74. *Every exponentially* 1*-stable point process is of the form* (6.3) *for some decoration measure D.*

It turns out that in addition to the simple proof of Maillard, this result can also be obtained from a classic result known as *LePage decomposition of a stable point process.*

Chapter 7

The extremal point process of the branching Brownian motion

The study of the extremal point process of branching Brownian motion has been a very active area of research recently. In particular a full convergence result which include the description of the decoration measure has been obtained independently by Arguin, Bovier and Kistler [3, 4, 5] on the one hand, and by Aïdekon, Berestycki, Brunet and Shi [2] on the other. The same type of result was obtained by Madaule for branching random walks. One reason this type of results is interesting is that it was conjectured, and indeed it has been confirmed to a large exttent, that the extrema of the Gaussian free field behave very much as the extrema of the branching brownian motions.

7.1 The setup

It is convenient when describing the extremal point process to get rid of the $\sqrt{2}t$ in the position of the rightmost particle simply by adding a drift to the movements of the particles that tilts the cone shape of a branching Brownian motion in a space-time plane. Furthermore, instead of considering the rightmost particles, we are going to look at the leftmost ones. The process being symmetrical, this is of course arbitrary.

More precisely, the system starts with a single particle at the origin which follows a Brownian motion with drift ϱ and variance $\sigma^2 > 0$. Branching is dyadic and occur at rate $\beta > 0$.

We are going to abuse the notation $\mathscr{N}(t)$ in order to designate both the collection of particles alive at time t and the point measure of the particle positions. Call $N(t) = \#\mathscr{N}(t)$ the number of particles alive at time t and $X_1(t) \leq X_2(t) \leq \cdots \leq X_{N(t)}(t)$ their positions enumerated from left to right.

We will work impose the followin conditions: for all $t > 0$,

$$\mathbb{E}\Big[\sum_{i=1,\ldots,N(t)} \mathrm{e}^{-X_i(t)}\Big] = 1, \qquad \mathbb{E}\Big[\sum_{i=1,\ldots,N(t)} X_i(t)\mathrm{e}^{-X_i(t)}\Big] = 0. \tag{7.1}$$

In this context the many-to-one Lemma tells us that for any measurable function F and each $t > 0$,

$$\mathbb{E}\Big[\sum_{i=1,\dots,N(t)} F(X_{i,t}(s),\, s \in [0,\, t])\Big] = \mathrm{e}^{\beta t}\,\mathbb{E}\Big[F(\sigma B_s + \varrho s,\, s \in [0,\, t])\Big],$$

where, for each $i \in \{1, \dots, N(t)\}$ we let $X_{i,t}(s)$, $s \in [0,\, t]$ be the position, at time s, of the ancestor of $X_i(t)$ and B is a standard Brownian motion. Thus the equations (7.1) become $\varrho = \beta + \frac{\sigma^2}{2}$ and $\varrho = \sigma^2$. Hence the usual conditions amount to supposing $\varrho = \sigma^2 = 2\beta$. **In these notes we always assume $\beta = 1$, $\rho = 2$ and $\sigma = \sqrt{2}$.**

7.2 Bramson and Lalley-Sellke in the new setup

Since we have changed the model slightly, we should start by reformulating Bramson's result in this context.

There exists a constant $C_B \in \mathbb{R}$ and a real valued random variable W such that

$$X_1(t) - m_t \overset{\text{law}}{\to} W, \qquad t \to \infty, \tag{7.2}$$

where

$$m_t := \frac{3}{2}\log t + C_B \tag{7.3}$$

and furthermore the distribution function of W is a solution to the critical Fisher-KPP travelling wave equation.

The derivative martingale becomes simply

$$Z(t) := \sum_{i=1,\dots,N(t)} X_i(t)\mathrm{e}^{-X_i(t)}. \tag{7.4}$$

and we have seen that

$$Z := \lim_{t\to\infty} Z(t) \tag{7.5}$$

exists and is strictly positive, finite with probability 1. The main result of Lalley's and Sellke's paper is then that $\exists C > 0$ such that

$$\lim_{s\to\infty}\lim_{t\to\infty} \mathbb{P}(X_1(t+s) - m(t+s) \geq x | \mathscr{F}_s) = \exp\left(-CZ\mathrm{e}^{x}\right)$$

where $\mathscr{F}_t$ is the natural filtration of the branching Brownian motion. As a consequence,

$$\mathbb{P}(W \leq x) \;\sim\; C\,|x|\mathrm{e}^{x}, \qquad x \to -\infty. \tag{7.6}$$

7.3 Main results

We consider the point process of the particles seen from the Lalley and Sellke referential and enumerated from the leftmost:

$$\bar{\mathscr{N}}(t) := \mathscr{N}(t) - m_t + \log(CZ) = \{X_i(t) - m_t + \log(CZ),\ 1 \le i \le N(t)\}.$$

Theorem 75. *As $t \to \infty$ the pair $\{\bar{\mathscr{N}}(t), Z(t)\}$ converges jointly in distribution to $\{\mathscr{L}, Z\}$ where $\mathscr{L}$ and Z are independent and $\mathscr{L}$ is obtained as follows.*

(i) Define $\mathscr{P}$ a Poisson point process on $\mathbb{R}$, with intensity measure $\mathrm{e}^x\,\mathrm{d}x$.

(ii) For each atom x of $\mathscr{P}$, we attach a point process $x + \mathscr{Q}^{(x)}$ where $\mathscr{Q}^{(x)}$ are independent copies of a certain point process $\mathscr{Q}$.

(iii) $\mathscr{L}$ is then the superposition of all the point processes $x + \mathscr{Q}^{(x)}$, i.e., $\mathscr{L} := \{x + y :\ x \in \mathscr{P},\ y \in \mathscr{Q}^{(x)}\}$.

Exercise 76. *Show that one can deduce from the above Theorem the following results which concern the point process of positions seen from the leftmost particle*

$$\mathscr{N}'(t) := \{X_i(t) - X_1(t),\ 1 \le i \le N(t)\}.$$

As $t \to \infty$ the point process $\mathscr{N}'(t)$ converges in distribution to the point process $\mathscr{L}'$ obtained by replacing the Poisson point process $\mathscr{P}$ in step (i) above by $\mathscr{P}'$ described in step (i)' below:

(i)' Let $\mathbf{e}$ be a standard exponential random variable. Conditionally on $\mathbf{e}$, define $\mathscr{P}'$ to be a Poisson point process on $\mathbb{R}_+$, with intensity measure $\mathbf{e}\mathrm{e}^x \mathbf{1}_{\mathbb{R}_+}(x)\,\mathrm{d}x$ to which we add an atom in 0.

The decoration point process $\mathscr{Q}(x)$ remains the same.

We next give a precise description of the *decoration* point process $\mathscr{Q}$ which can be considered to be the main result of [2] (note that [5] give their own, quite different description of the decoration). For each $s \le t$, let $X_{1,t}(s)$ be the position at time s of the ancestor of $X_1(t)$, i.e., $s \mapsto X_{1,t}(s)$ is the path followed by the leftmost particle at time t. We define

$$Y_t(s) := X_{1,t}(t-s) - X_1(t), \qquad s \in [0,t]$$

the time reversed path back from the final position $X_1(t)$. For each $t > 0$ and for each path $X := (X(s),\ s \in [0,\ t])$ that goes from the ancestor to a particle in $\mathscr{N}(t)$, let us write (τ_i^X) for the successive splitting times of branching along the trajectory X (enumerated backward), $\mathscr{N}_{t,X}^{(i)}$ for the set of all particles at time t which are descended from the one particle which has branched off X at time τ_i^X relative to the

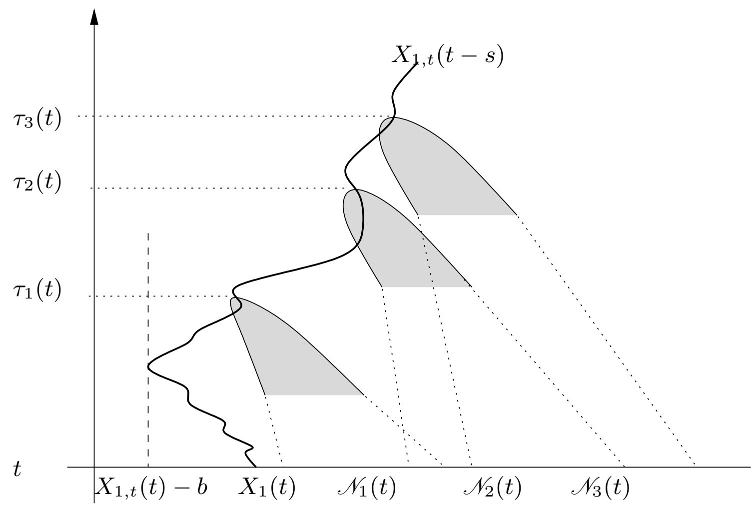


Figure 7.1: $(Y, \mathscr{Q})$ is the limit of the path $s \mapsto X_{1,t}(t-s) - X_1(t)$ and of the points that have branched recently off from $X_{1,t}$.

final position $X(t)$ (see figure 7.3). More precisely, if we define $\tau_{X,j}(t)$ to be the time at which the particle $X_j(t)$ has branched off the path of X during $[0,t]$ we have

$$\mathscr{N}_{t,X}^{(i)} := \{X_j(t) - X(t),\ \tau_{X,j}(t) = \tau_i^X\}.$$

We then define

$$\mathscr{Q}(t,\zeta) := \bigcup_{\tau_i^{X_1(t)} > t-\zeta} \mathscr{N}_{t,X_1(t)}^{(i)}$$

i.e., the set of particles at time t which have branched off $X_{1,t}(s)$ after time $t-\zeta$.

We will first show that $\{(Y_t(s), s \in [0,t]); \mathscr{Q}(t,\zeta)\}$ converges jointly in distribution (by first letting $t \to \infty$ and then $\zeta \to \infty$) towards a limit $\{(Y(s), s \geq 0); \mathscr{Q}\}$ where the second coordinate is our point process $\mathscr{Q}$ which is described by growing conditioned branching Brownian motions born at a certain rate on the path Y. We first describe the limit $\{(Y(s), s \geq 0), \mathscr{Q}\}$ and then we state the precise convergence result.

The following family of processes indexed by a real parameter $b > 0$ plays a key role in this description. Let $B := (B_t,\, t \geq 0)$ be a standard Brownian motion and let $R := (R_t,\, t \geq 0)$ be a three-dimensional Bessel process started from $R_0 := 0$ and independent from B. Let us define $T_b := \inf\{t \geq 0 : B_t = b\}$. For each $b > 0$, we define the process $\Gamma^{(b)}$ as follows:

$$\Gamma_s^{(b)} := \begin{cases} B_s, & \text{if } s \in [0,\, T_b], \\ & \\ b - R_{s-T_b}, & \text{if } s \geq T_b. \end{cases} \tag{7.7}$$

Let us define

$$G_t(x) := \mathbb{P}_0(X_1(t) \leq x) = \mathbb{P}_{-x}(X_1(t) \leq 0)$$

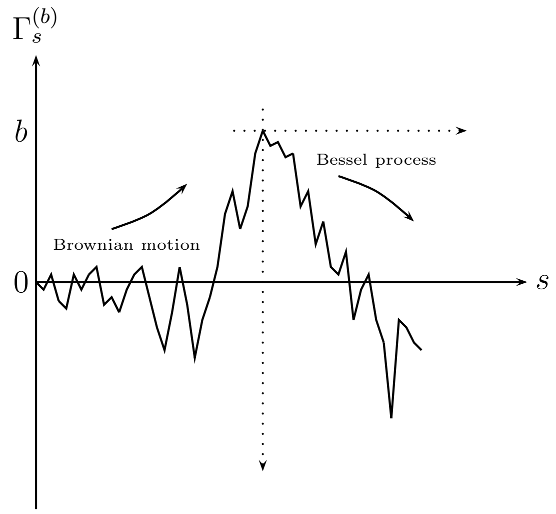


Figure 7.2: the process $\Gamma^{(b)}$

the probability of presence to the left of x at time t, where we write $\mathbb{P}_x$ for the law of the branching Brownian motion started from one particle at x. Hence, by (7.2) we see that $G_t(x+m_t) \to \mathbb{P}(W \le x)$.

We can now describe the law of the backward path Y. Let b be a random variable with values in $(0, \infty)$ whose density is given by $\mathbb{P}(\sigma b \in \mathrm{d}x) = \frac{f(x)}{c_1}\,\mathrm{d}x$ where

$$f(x) := \mathbb{E}\Big[\mathrm{e}^{-2\int_0^\infty G_v(\sigma\Gamma_v^{(x)})\,\mathrm{d}v}\Big] \tag{7.8}$$

and

$$c_1 := \int_0^\infty \mathbb{E}\Big[\mathrm{e}^{-2\int_0^\infty G_v(\sigma\Gamma_v^{(a)})\,\mathrm{d}v}\Big]\,\mathrm{d}a.$$

Conditionally on b, the minimum of Y/σ is $-b$ and the path Y has a density with respect to the law of $-\Gamma^{(b)}$ which is given by

$$\frac{1}{f(b)}\mathrm{e}^{-2\int_0^\infty G_v(\sigma\Gamma_v^{(b)})\,\mathrm{d}v} \tag{7.9}$$

i.e.,

$$\mathbb{P}(Y \in A) = \frac{1}{f(b)}\mathbb{E}\left[\mathrm{e}^{-2\int_0^\infty G_v(\sigma\Gamma_v^{(b)})\,\mathrm{d}v}\mathbf{1}_{-\sigma\Gamma^{(b)}\in A}\right].$$

Now, conditionally on the path Y, we let π be a Poisson point process on $[0,\infty)$ with intensity $2\big(1-G_t(-Y(t))\big)\,\mathrm{d}t = 2\big(1-\mathbb{P}_{Y(t)}(X_1(t) < 0)\big)\,\mathrm{d}t$. For each point $t \in \pi$ start an independent branching Brownian motion $(\mathscr{N}^*_{Y(t)}(u), u \ge 0)$ at position $Y(t)$ conditioned to $\min \mathscr{N}^*(t) > 0$. Then define $\mathscr{Q} := \cup_{t\in\pi}\mathscr{N}^*_{Y(t)}(t)$.

Theorem 77. *The following convergence holds jointly in distribution.*

$$\lim_{\zeta\to\infty}\lim_{t\to\infty}\{(Y_t(s), s \in\in [0,t]);\ \mathscr{Q}(t,\zeta);\ X_1(t)-m_t\} = \{(Y(s), s \ge 0);\ \mathscr{Q};\ W\},$$

where the random variable W is independent of the pair $((Y(s), s \geq 0), \mathscr{Q})$, and $\mathscr{Q}$ is the point process which appears in Theorem 75.

Observe that the parameter ζ only matters for the decoration point process in the second coordinate.

7.4 A Laplace transform result

Using the less trivial version of the many-to-one result yields a first result which turns out to be an important element in the proofs.

Theorem 78 characterizes the joint distribution of the path $s \mapsto X_{1,t}(s)$ that the particle which is the leftmost at time t has followed, of the point processes of the particles to its right and the times at which these particles have split in the past in terms of a Brownian motion functional.

For any positive measurable functional F and any positive measurable function $f : [0, t] \to \mathbb{R}_+$, for $n \in \mathbb{N}, (\alpha_1, \dots, \alpha_n) \in \mathbb{R}_+^n$ and $A_1, \dots, A_n$ a collection of Borel subsets of $\mathbb{R}_+$ define

$$I(t) := \mathbb{E}\Big\{ F(X_{1,t}(s),\, s \in [0,\, t]) \,\exp\Big(-\sum_i f(t-\tau_i^{X_{1,t}}) \sum_{j=1}^{n} \alpha_j \#[\mathscr{N}^{(i)}_{t,X_1(t)} \cap (X_1(t)+A_j)] \Big) \Big\},$$

For each $r \geq 0$ and every $x \in \mathbb{R}$ recall that $G_r(x) = \mathbb{P}\{X_1(r) \leq x\}$, and further define

$$\overline{G}_r^{(f)}(x) \quad := \quad \mathbb{E}\Big[\mathrm{e}^{-f(r)\sum_{j=1}^{n} \alpha_j \#[\mathscr{N}(r) \cap (x+A_j)]}\, \mathbf{1}_{\{X_1(r) \geq x\}} \Big].$$

Hence, when $f \equiv 0$ we have $\overline{G}_r^{(f)}(x) = 1 - G_r(x)$.

Theorem 78. *We have*

$$I(t) = \mathbb{E}\Big[\mathrm{e}^{\sigma B_t}\, F(\sigma B_s,\, s \in [0,\, t])\, \mathrm{e}^{-2\int_0^t [1-\overline{G}^{(f)}_{t-s}(\sigma B_t - \sigma B_s)]\,\mathrm{d}s} \Big], \tag{7.10}$$

where B in the expectation above is a standard Brownian motion. In particular, the path $(s \mapsto X_{1,t}(s), 0 \leq s \leq t)$ is a standard Brownian motion in a potential:

$$\mathbb{E}\Big[F(X_{1,t}(s),\, s \in [0,\, t]) \Big] = \mathbb{E}\Big[\mathrm{e}^{\sigma B_t}\, F(\sigma B_s,\, s \in [0,\, t])\, \mathrm{e}^{-2\int_0^t G_{t-s}(\sigma B_t - \sigma B_s)\,\mathrm{d}s} \Big]. \tag{7.11}$$

The proof relies on the use of the following version of the many-to-one principle. We are going to use the *critical* additive martingale (which converges to zero) The process

$$M_t := \sum_{i \leq N(t)} \mathrm{e}^{-X_i(t)}, \qquad t \geq 0.$$

Let $\mathbb{Q}$ be the probability measure on $\mathscr{F}_\infty$ such that, for each $t \geq 0$,

$$\mathbb{Q}_{|\mathscr{F}_t} = M_t \bullet \mathbb{P}_{|\mathscr{F}_t} .$$

Under $\mathbb{Q}$ the particle with label ξ_s at time s branches at (accelerated) rate 2 and gives birth to normal branching Brownian motions (without spine) with distribution $\mathbb{P}$, whereas the process of the position of the spine $(\Xi(s),\, s \in [0,\, t])$ is a driftless Brownian motion of variance $\sigma^2 = 2$. Furthermore, for each $t \geq 0$ and each $i \leq N(t)$,

$$\mathbb{Q}\{\Xi_t = i \,|\, \mathscr{F}_t\} = \frac{\mathrm{e}^{-X_i(t)}}{M_t} .$$

For each $i \leq N(t)$ consider Ψ_i a random variable which is measurable in the filtration of the branching Brownian motion up to time t (i.e., it is determined by the history of the process up to time t) and suppose that we wish to compute $\mathbb{E}_{\mathbb{P}}[\sum_{i\leq N(t)} \Psi_i]$. Then, thanks to the above, we have

$$\mathbb{E}_{\mathbb{P}}\Big[\sum_{i\leq N(t)} \Psi_i\Big] = \mathbb{E}_{\mathbb{Q}}\Big[\frac{1}{M_t}\sum_{i\leq N(t)} \Psi_i\Big] = \mathbb{E}_{\mathbb{Q}}\Big[\mathrm{e}^{\Xi(t)}\Psi_{\xi_t}\Big]. \tag{7.12}$$

Proof. For the sake of brevity let us only treat the case where $f \equiv 0$. Letting $X_{i,t}(s)$ be the position of the ancestor at time s of the particle at $X_i(t)$ at time t, we have

$$I(t) = \mathbb{E}\Big[\sum_{i\leq N(t)} \mathbf{1}_{\{i=1\}}\, F(X_{i,t}(s),\, s \in [0,\, t])\Big],$$

Using the many-to-one principle and the change of probability presented in equation (7.12) we see that

$$\begin{aligned} I(t) &= \mathbb{E}_{\mathbb{Q}}\Big[\mathrm{e}^{\Xi(t)}\, \mathbf{1}_{\{\Xi_t=1\}}\, F(\Xi(s),\, s \in [0,\, t])\Big] \\ &= \mathbb{E}_{\mathbb{Q}}\Big[\mathrm{e}^{\Xi(t)}\, F(\Xi(s),\, s \in [0,\, t]) \prod_k \mathbf{1}_{\{\min \mathscr{N}_k^{(\xi_t)} > 0\}}\Big] \end{aligned}$$

where we recall that by convention, for a point measure $\mathscr{N}$, $\min \mathscr{N}$ is the infimum of the support of $\mathscr{N}$.

Conditioning on the σ-algebra generated by the spine (including the successive branching times) we obtain

$$I(t) = \mathbb{E}_{\mathbb{Q}}\Big[\mathrm{e}^{\Xi(t)}\, F(\Xi(s),\, s \in [0,\, t]) \prod_i \big(1 - G^{(f)}_{t-\tau_i^{(\xi_t)}(t)}(\Xi(t) - \Xi(\tau_i^{(\xi_t)}(t)))\big)\Big],$$

where, recall that for any $r \geq 0$ and any $x \in \mathbb{R}$,

$$G_r^{(f)}(x) := \mathbb{E}\Big[\mathbf{1}_{\{\min \mathscr{N}(r) \leq x\}}\Big]. \tag{7.13}$$

Since $(\tau_i^{(\Xi_t)}(t), i \geq 0)$ is a rate 2 Poisson process under $\mathbb{Q}$, we arrive at:[1]

$$
\begin{aligned}
I(t) &= \mathbb{E}_{\mathbb{Q}}\Big[\mathrm{e}^{\Xi(t)}\, F(\Xi(s),\, s \in [0,\, t])\, \mathrm{e}^{-2\int_0^t [G_{t-s}^{(f)}(\Xi(t)-\Xi(s))]\, \mathrm{d}s}\Big] \\
&= \mathbb{E}\Big[\mathrm{e}^{\sigma B_t}\, F(\sigma B_s,\, s \in [0,\, t])\, \mathrm{e}^{-2\int_0^t [G_{t-s}^{(f)}(\sigma B_t - \sigma B_s)]\, \mathrm{d}s}\Big], \qquad (7.14)
\end{aligned}
$$

where, in the last identity, we used the fact that $(\Xi(s),\, s \in [0,\, t])$ under $\mathbb{Q}$ is a centered Brownian motion (with variance $\sigma^2 = 2$). This yields Theorem 78. □

7.5 Localization result for the path of the leftmost particle

A second important ingredient of the proof is the following result which give precise informations about the path followed by the leftmost particle. This result was first obtained by Arguin, Bovier and Kistler in [3].

When applying the many-to-one principle as in (7.12), if the functional Ψ_ξ only depends on the path of $\Xi(s)$ then the last expectation is simply the expectation of a certain event for the standard Brownian motion. For instance, suppose that we want to check if there exists a path $(X_{i,t}(s), s \in [0, t])$ with some property in the tree. Let A be a measurable subset of continuous functions $[0, t] \mapsto \mathbb{R}$. Then

$$
\mathbb{P}\Big(\exists i \leq N(t) : (X_{i,t}(s), s \in [0, t]) \in A\Big) \leq \mathbb{P}(\mathrm{e}^{\sigma B_t}; (\sigma B_s, s \in [0, t]) \in A) \qquad (7.15)
$$

where $(B_s, s \geq 0)$ is a standard Brownian motion under $\mathbb{P}$. This is the main tool we use in proving the following proposition.

Let $J_\eta(t) := \{i \leq N(t) : |X_i(t) - m_t| < \eta\}$ where $m_t = \frac{3}{2}\log t + C_B$ by (7.3). For $t \geq 1$ and $x > 0$, we define the good event $A_t(x, \eta)$ by

$$
A_t(x, \eta) := E_1(x, \eta) \cap E_2(x, \eta) \cap E_3(x, \eta)
$$

where the events E_i are defined by

$$
\begin{aligned}
E_1(x, \eta) &:= \Big\{\forall i \in J_\eta(t),\ \min_{[0,t]} X_{i,t}(s) \geq -x,\ \min_{[\frac{t}{2}, t]} X_{i,t}(s) \geq m_t - x\Big\}, \\
E_2(x, \eta) &:= \Big\{\forall i \in J_\eta(t), \forall s \in [x,\ \frac{t}{2}], X_{i,t}(s) \geq s^{1/3}\Big\}, \\
E_3(x, \eta) &:= \Big\{\forall i \in J_\eta(t), \forall s \in [\frac{t}{2},\ t - x],\ X_{i,t}(s) - X_i(t) \in [(t-s)^{1/3}, (t-s)^{2/3}]\Big\}.
\end{aligned}
$$

Theorem 79 (Arguin, Bovier and Kistler [3]; see also [2])**.** *For any $\varepsilon > 0$ and $\eta > 0$, there exists $x > 0$ large enough such that $\mathbb{P}(A_t(x, \eta)) \geq 1 - \varepsilon$ for t large enough.*

[1] We recall the Laplace functional of a point Poisson process $\mathscr{P}$: $\mathbb{E}[\exp(-\int f\, \mathrm{d}\mathscr{P})] = \exp[-\int (1 - \mathrm{e}^{-f})\, \mathrm{d}\mu]$, where μ is the intensity measure.

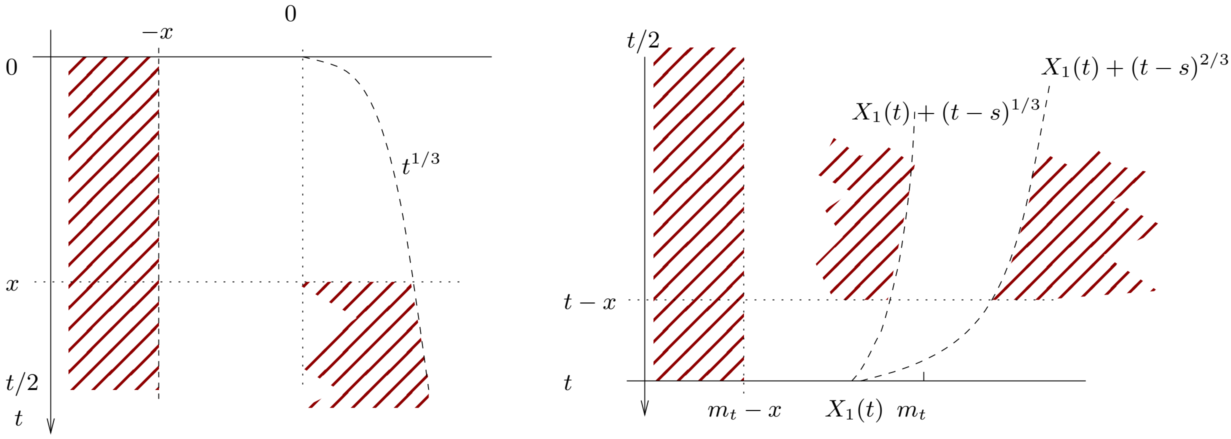


Figure 7.3: The events $E_1(x,\eta), E_2(x,\eta)$ and $E_3(x,\eta)$ together are the event that the paths of particles ending within distance η of m_t avoid all the dashed regions.

The proof of this result is quite involved. Essentially one starts by showing that for x large enough $\mathbb{P}(E_1(x,\eta)^{\complement}) < \epsilon$ (where ϵ, η are given). Then, conditionally on $E_1(x,\eta)$one shows that $\mathbb{P}(E_2(x,\eta)^{\complement}) < \epsilon$, and so on.

To give a taste of the technicalities involved here let us just focus on the proof of the bound of $\mathbb{P}(E_2(x,\eta)^{\complement})$.

Proof. We can restrict to the event $E_1(z,\eta)$ for z large enough. By the many-to-one principle, we get

$$\mathbb{P}(E_2(x,\eta)^{\complement}, E_1(z,\eta)) \le \mathrm{e}^{\eta + C_B} t^{3/2} \mathbb{P}(\widehat{B})$$

where $\mathbb{P}(\widehat{B})$ is defined by

$$\mathbb{P}(\widehat{B}) := \mathbb{P}\Big\{ \exists s \in [x,t/2] : \sigma B_s \le s^{1/3}, \sigma \underline{B}^{[0,t/2]} \ge -z, \sigma \underline{B}^{[t/2,t]} \ge m_t - z, \sigma B_t \le m_t + \eta \Big\}.$$

We will actually bound the probability

$$\begin{aligned} & \mathbb{P}(\widehat{B}, \mathrm{d}r) \qquad (7.16)\\ := \; & \mathbb{P}\Big\{ \exists s \in [x,t/2] : \sigma B_s \le s^{1/3}, \sigma \underline{B}^{[0,t/2]} \ge -z, \sigma \underline{B}^{[t/2,t]} \ge m_t - z, \sigma B_t \in m_t + \mathrm{d}r \Big\}. \end{aligned}$$

Applying the Markov property at time $t/2$ yields that

$$\begin{aligned} & \mathbb{P}(\widehat{B}, \mathrm{d}r) \\ = \; & \mathbb{E}\left[\mathbf{1}_{\{\exists s \in [x,t/2] : \sigma B_s \le s^{1/3}\}} \mathbf{1}_{\{\sigma \underline{B}^{[0,t/2]} \ge -z\}} \mathbb{P}_{B_{t/2}} \Big\{ \sigma \underline{B}^{[0,t/2]} \ge m_t - z, \sigma B_{t/2} \in m_t + \mathrm{d}r \Big\} \right] \\ \le \; & c(r+z) t^{-3/2} \mathbb{E}\left[\mathbf{1}_{\{\exists s \in [x,t/2] : \sigma B_s \le s^{1/3}\}} \mathbf{1}_{\{\sigma \underline{B}^{[0,t/2]} \ge -z\}} (\sigma B_{t/2} - m_t + z)_+ \right] \mathrm{d}r \\ \le \; & c(r+z) t^{-3/2} \mathbb{E}\left[\mathbf{1}_{\{\exists s \in [x,t/2] : \sigma B_s \le s^{1/3}\}} \mathbf{1}_{\{\sigma \underline{B}^{[0,t/2]} \ge -z\}} (\sigma B_{t/2} + z) \right] \mathrm{d}r \end{aligned}$$

where the second inequality comes from the joint law of a Brownian motion and its infimum), and we set $y_+ := \max(y, 0)$. We recognize the h-transform of the Bessel process. Therefore

$$\mathbb{P}(\widehat{B}, \mathrm{d}r) \leq cz(r+z)t^{-3/2}\mathbb{P}_z(\exists s \in [x, t/2] : \sigma R_s \leq z + s^{1/3})\,\mathrm{d}r \tag{7.17}$$

where as before $(R_s, s \geq 0)$ is a three-dimensional Bessel process. In particular, $\mathbb{P}(\widehat{B}) = \int_{-z}^{\eta} \mathbb{P}(\widehat{B}, \mathrm{d}r) \leq cz(z+\eta)^2 t^{-3/2}\mathbb{P}_z(\exists s \in [x, t/2] : \sigma R_s \leq z + s^{1/3})$. This yields that

$$\begin{aligned}\mathbb{P}(E_2(x,\eta)^{\complement}, E_1(z)) &\leq \mathrm{e}^{\eta + C_B} cz(z+\eta)^2 \mathbb{P}_z(\exists s \in [x, t/2] : \sigma R_s \leq z + s^{1/3}) \\ &\leq \mathrm{e}^{\eta + C_B} cz(z+\eta)^2 \mathbb{P}_z(\exists s \geq x : \sigma R_s \leq z + s^{1/3})\end{aligned}$$

and we deduce that $\mathbb{P}(E_2(x,\eta)^{\complement}, E_1(z)) \leq \varepsilon$ for x large enough.

□

7.6 The point process of the clan-leaders

We are now going to explain the structure of the limit point process as a decorated Poisson point process as follows. We show that by stopping particles when they first hit a certain position k and then considering only their leftmost descendants one recovers a Poisson point measure of intensity $\mathrm{e}^x\,\mathrm{d}x$ as $k \to \infty$. Then, we show that two particles near m_t have separated in a branching event that was either very recent or near the very beginning of the process and we finally combine those two steps to complete the proof of Theorem 75.

We employ a very classical approach: we stop the particles when they reach an increasing family of affine stopping lines and then consider their descendants independently. The same kind of argument with the same stopping lines appear in [21] and in [1].

Fix $k \geq 1$ and consider $\mathscr{H}_k$ the set of all particles which are the first in their line of descent to hit the spatial position k. (For the formalism of particle labelling, see Neveu [27].) Under the conditions we work with, we know that almost surely $\mathscr{H}_k$ is a finite set. The set $\mathscr{H}_k$ is again a *dissecting stopping line* at which we can apply the the strong Markov property (see e.g. [11]). We see that conditionally on $\mathscr{F}_{\mathscr{H}_k}$ — the sigma-algebra generated by the branching Brownian motion when the particles are stopped upon hitting the position k — the subtrees rooted at the points of $\mathscr{H}_k$ are independent copies of the branching Brownian motion started at position k and at the random time at which the particle considered has hit k. Define $H_k := \#\mathscr{H}_k$ and

$$Z_k := k\mathrm{e}^{-k} H_k.$$

Neveu ([27], equation (5.4)) shows that the limit Z of the derivative martingale in (7.4) can also be obtained as a limit of Z_k (it is the same martingale on a different stopping line)

$$Z = \lim_{k\to\infty} Z_k = \lim_{k\to\infty} k\mathrm{e}^{-k} H_k \tag{7.18}$$

almost surely. Let us further define $\mathscr{H}_{k,t}$ as the set of all particles which are the first in their line of descent to hit the spatial position k, and which do so before time t.

For each $u \in \mathscr{H}_{k,t}$, let us write $X_1^u(t)$ for the minimal position at time t of the particles which are descendants of u. If $u \in \mathscr{H}_k \backslash \mathscr{H}_{k,t}$ we define $X_1^u(t) = 0$. This allows us to define the point measure

$$\mathscr{P}^*_{k,t} := \sum_{u \in \mathscr{H}_k} \delta_{X_1^u(t) - m_t + \log(CZ_k)}.$$

We further define

$$\mathscr{P}^*_{k,\infty} := \sum_{u \in \mathscr{H}_k} \delta_{k + W(u) + \log(CZ_k)}$$

where, conditionally on $\mathscr{F}_{\mathscr{H}_k}$, the $W(u)$ are independent copies of the random variable W in (7.2).

Proposition 80. *The following convergences hold in distribution*

$$\lim_{t \to \infty} \mathscr{P}^*_{k,t} = \mathscr{P}^*_{k,\infty}$$

and

$$\lim_{k \to \infty} (\mathscr{P}^*_{k,\infty}, Z_k) = (\mathscr{P}, Z)$$

where $\mathscr{P}$ is as in Theorem 75, Z is as in (7.5), and $\mathscr{P}$ and Z are independent.

Proof. Fix $k \geq 1$. Recall that $\mathscr{H}_k$ is the set of particles absorbed at level k, and $H_k = \#\mathscr{H}_k$. Observe that for each $u \in \mathscr{H}_k$, $X_1^u(t)$ has the same distribution as $k + X_1(t - \xi_{k,u})$, where $\xi_{k,u}$ is the random time at which u reaches k. By (7.2) and the fact that $m_{t+c} - m_t \to 0$ for any c, we have, for all $k \geq 1$ and all $u \in \mathscr{H}_k$,

$$X_1^u(t) - m_t \stackrel{law}{\to} k + W, \qquad t \to \infty.$$

Hence, the finite point measure $\mathscr{P}_{k,t} := \sum_{u \in \mathscr{H}_k} \delta_{X_1^u(t) - m_t}$ converges in distribution as $t \to \infty$, to $\mathscr{P}_{k,\infty} := \sum_{u \in \mathscr{H}_k} \delta_{k+W(u)}$, where conditionally on $\mathscr{H}_k$, the $W(u)$ are independent copies of W. This proves the first part of Proposition 80.

The proof of the second part relies on some classical extreme value theory. We refer the reader to [29] for a thorough treatment of this subject. Let us state the result we will use. Suppose we are given a sequence $(X_i, i \in \mathbb{N})$ of i.i.d. random variables such that

$$\mathbb{P}(X_i \geq x) \sim Cx\mathrm{e}^{-x}, \text{ as } x \to \infty.$$

Call $M_n = \max_{i=1,\dots,n} X_i$ the record of the X_i. Then it is not hard to see that if we let $b_n = \log n + \log\log n$ we have as $n \to \infty$

$$\begin{aligned}
\mathbb{P}(M_n - b_n \leq y) &= (\mathbb{P}(X_i \leq y + b_n))^n \\
&= (1 - (1 + o(1))C(y + b_n)\mathrm{e}^{-(y+b_n)})^n \\
&\sim \exp\Big(- nC(y + b_n)\frac{1}{n \log n}\mathrm{e}^{-y}\Big) \\
&\sim \exp(-C\mathrm{e}^{-y})
\end{aligned}$$

and therefore

$$\mathbb{P}\left(M_n - b_n - \log C \le y\right) \sim \exp(-\mathrm{e}^{-y}).$$

By applying Corollary 4.19 in [29] we immediately see that the point measure

$$\zeta_n := \sum_{i=1}^{n} \delta_{X_i - b_n - \log C}$$

converges in distribution to a Poisson point measure on $\mathbb{R}$ with intensity $\mathrm{e}^{-x}\,\mathrm{d}x$.

This result applies immediately to the random variables $-W(u)$ (recalling from (7.6) that $\mathbb{P}(-W \ge x) \sim Cx\mathrm{e}^{-x}$, $x \to \infty$) and thus the point measure

$$\sum_{u \in \mathscr{H}_k} \delta_{W(u)+(\log H_k + \log\log H_k + \log C)}$$

converges (as $k \to \infty$) in distribution towards a Poisson point measure on $\mathbb{R}$ with intensity $\mathrm{e}^{x}\,\mathrm{d}x$ (it is e^{x} instead of e^{-x} because we are looking at the leftmost particles) independently of Z (this identity comes from (7.18)). By definition $H_k = k^{-1}\mathrm{e}^{k} Z_k$, thus

$$\begin{aligned}\log H_k &= k + \log Z_k - \log k\\ \log\log H_k &= \log k + \log(1 + o_k(1))\end{aligned}$$

where the term $o_k(1)$ tends to 0 almost surely when $k \to \infty$. Hence,

$$\log H_k + \log\log H_k = \log Z_k + k + o_k(1).$$

We conclude that for $u \in \mathscr{H}_k$

$$k + W(u) + \log(CZ) = W(u) + (\log H_k + \log\log H_k + \log C) + o_k(1).$$

Hence we conclude that

$$\mathscr{P}^*_{k,\infty} = \sum_{u \in \mathscr{H}_k} \delta_{k+W(u)+\log(CZ)}$$

also converges (as $k \to \infty$) towards a Poisson point measure on $\mathbb{R}$ with intensity $\mathrm{e}^{x}\,\mathrm{d}x$ independently of $Z = \lim_k Z_k$. This concludes the proof of Proposition 80. □

7.7 Genealogy near the extrema

The following result shows that near the rightmost particle at time t, particles have either a very recent common ancestor, or they have branched a very long time ago.

Recall that $J_\eta(t) := \{i \le N(t) : |X_i(t) - m_t| \le \eta\}$ is the set of indices which correspond to particles near m_t at time t and that $\tau_{i,j}(t)$ is the time at which the particles $X_i(t)$ and $X_j(t)$ have branched from one another.

Proposition 81. (Arguin, Bovier and Kistler [3]) *Fix* $\eta > 0$ *and any function* $\zeta : [0, \infty) \to [0, \infty)$ *which increases to infinity. Define the event*

$$\mathcal{B}_{\eta,k,t} := \{\exists i, j \in J_\eta(t) : \tau_{i,j}(t) \in [\zeta(k), t - \zeta(k)]\} .$$

One has

$$\lim_{k\to\infty} \lim_{t\to\infty} \mathbb{P}\left[\mathcal{B}_{\eta,k,t}\right] = 0. \tag{7.19}$$

We only sketch the beginning of a spine proof of this result here.

Proof. Fix $\eta > 0$ and $k \to \zeta(k)$ an increasing sequence going to infinity. We want to control the probability of

$$\mathcal{B}_{\eta,k,t} = \{\exists i, j \in J_\eta(t) : \tau_{i,j}(t) \in [\zeta(k), t - \zeta(k)]\}$$

the "bad" event that particles have branched at an intermediate time when $t \to \infty$ and then $k \to \infty$.

By choosing x large enough, we have for all $\zeta \geq 0$ and t large enough

$$\begin{aligned}
&\mathbb{P}(\exists i, j \in J_\eta(t) : \tau_{i,j}(t) \in [\zeta, t - \zeta]) \\
&\leq \mathbb{P}(A_t(x, \eta)^{\complement}) + \mathbb{P}(\exists i, j \in J_\eta(t) : \tau_{i,j}(t) \in [\zeta, t - \zeta], A_t(x, \eta)) \\
&\leq \varepsilon + \mathbb{E}\left[\mathbf{1}_{A_t(x,\eta)} \sum_{i \in J_\eta(t)} \mathbf{1}_{\{\exists j \in J_\eta(t) : \tau_{i,j}(t) \in [\zeta, t-\zeta]\}}\right].
\end{aligned}$$

Using the many-to-one principle, we have

$$\mathbb{E}\left[\mathbf{1}_{A_t(x,\eta)} \sum_{i \in J_\eta(t)} \mathbf{1}_{\{\exists j \in J_\eta(t) : \tau_{i,j}(t) \in [\zeta, t-\zeta]\}}\right] = \mathbb{E}_{\mathbb{Q}}\left[e^{\Xi(t)} \mathbf{1}_{A_t(x,\eta)} \mathbf{1}_{\{|\Xi(t) - m_t| \leq \eta, \exists j \in J_\eta(t) : \tau_{\Xi,j}(t) \in [\zeta, t-\zeta]\}}\right]$$

where $\tau_{\Xi,j}(t)$ is the time at which the particle $X_j(t)$ has branched off the spine Ξ. In particular, using the description of the process under $\mathbb{Q}$, we know that $\Xi(t)$ is σ times a standard Brownian motion, and that independent branching Brownian motions are born at rate 2 (at times $(\tau_i^{(\Xi_t)}(t), i \geq 1)$) from the spine Ξ. The event $\{\exists j \in J_\eta(t) : \tau_{\Xi,j}(t) \in [\zeta, t - \zeta]\}\}$ means that there is an instant $\tau_i^{(\Xi_t)}(t)$ between ζ and $t - \zeta$, such that the branching Brownian motion that separated from Ξ at that time has a descendant at time t in $[m_t - \eta, m_t + \eta]$. In particular, the minimum of this branching Brownian motion at time t is lower than $m_t + \eta$. Thus

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[e^{\Xi(t)} \mathbf{1}_{A_t(x,\eta)} \mathbf{1}_{\{|\Xi(t) - m_t| \leq \eta, \exists j \in J_\eta(t) : \tau_{\Xi,j}(t) \in [\zeta, t-\zeta]\}}\right] \\
&\qquad \leq \mathbb{E}_{\mathbb{Q}}\left[e^{\Xi(t)} \mathbf{1}_{A_t(x,\eta)} \mathbf{1}_{\{|\Xi(t) - m_t| \leq \eta\}} \sum_{\tau \in [\zeta, t-\zeta]} \mathbf{1}_{\{X_{1,t}^{\tau} \leq m_t + \eta\}}\right]
\end{aligned}$$

where $X_{1,t}^{\tau}$ is the leftmost particle at time t descended from the particle which branched off Ξ at time τ, and the sum goes over all times $\tau = \tau_i^{(\Xi_t)}(t) \in [\zeta, t - \zeta]$

at which a new particle is created. Recall that $G_v(x) = \mathbb{P}(X_1(v) \le x)$ so that by conditioning we obtain

$$\mathbb{E}_{\mathbb{Q}}\Big[e^{\Xi(t)}\mathbf{1}_{A_t(x,\eta)}\mathbf{1}_{\{|\Xi(t)-m_t|\le\eta,\exists j\in J_\eta(t):\tau_{\Xi,j}(t)\in[\zeta,t-\zeta]\}}\Big] \le \mathbb{E}_{\mathbb{Q}}\Big[e^{\Xi(t)}\mathbf{1}_{A_t(x,\eta)}\mathbf{1}_{\{|\Xi(t)-m_t|\le\eta\}}\sum_{\tau\in[\zeta,t-\zeta]}G_{t-\tau}(m_t+\eta-X_{\Xi_\tau}(\tau))\Big].$$

The rest of the proof works roughly as follows: the event $A_t(x,\eta)$ localizes the path Ξ as explained above, which allows us to lower bound the path Ξ by $s^{1/3}$ on $[\zeta, t/2]$ and by $(t-s)^{1/3}$ on $[t/2, t-\zeta]$. This in turn allows us to upper bound the last expression. □

7.8 The last piece

Recall that $\forall u \in \mathscr{H}_k, X_1^u(t)$ is the position at time t of the leftmost descendent of u (or 0 if $u \notin \mathscr{H}_{k,t}$), and let $X_{1,t}^u(s), s \le t$ be the position at time s of the ancestor of this leftmost descendent (or 0 if $u \notin \mathscr{H}_{k,t}$). For each t, ζ and $u \in \mathscr{H}_k$ define

$$\mathscr{Q}_{t,\zeta}^{(u)} = \delta_0 + \sum_{i:\tau_i^u > t-\zeta} \mathscr{N}_i^u$$

where the τ_i^u are the branching times along the path $s \mapsto X_{1,t}^u(s)$ enumerated backward from t and the $\mathscr{N}_i^u$ are the point measures of particles whose ancestor was born at τ_i^u (this measure has no mass if $u \notin \mathscr{H}_{k,t}$). Thus, $\mathscr{Q}_{t,\zeta}^{(u)}$ is the point measure of particles which have branched off the path $s \mapsto X_{1,t}^u(s)$ at a time which is posterior to $t-\zeta$, including the particle at $X_1^u(t)$.

In the same manner we define $\mathscr{Q}_\zeta$ as the point measure obtained from $\mathscr{Q}$ (in Theorem 77) by only keeping the particles that have branched off $s \mapsto Y(s)$ before ζ. More precisely, conditionally on the path Y, we let π be a Poisson point process on $[0,\infty)$ with intensity $2\big(1 - G_t(-Y(t))\big)\,dt = 2\big(1 - \mathbb{P}_{Y(t)}(X_1(t) < 0)\big)\,dt$. For each point $t \in \pi$ such that $t < \zeta$, start an independent branching Brownian motion $(\mathscr{N}_{Y(t)}^*(u), u \ge 0)$ at position $Y(t)$ conditioned to have $\min \mathscr{N}_{Y(t)}^*(t) > 0$. Then define $\mathscr{Q}_\zeta := \delta_0 + \sum_{t\in\pi, t<\zeta} \mathscr{N}_{Y(t)}^*(t)$.

Lemma 82. *For each fixed k and ζ, the following limit holds in distribution*

$$\lim_{t\to\infty} (\mathscr{P}_{k,t}^*, (\mathscr{Q}_{t,\zeta}^{(u)})_{u\in\mathscr{H}_k}) = (\mathscr{P}_{k,\infty}^*, (\mathscr{Q}_\zeta^{(u)})_{u\in\mathscr{H}_k})$$

where $(\mathscr{Q}_\zeta^{(u)})_{u\in\mathscr{H}_k}$ is a collection of independent copies of $\mathscr{Q}_\zeta$, independent of $\mathscr{P}_{k,\infty}^*$.

Proof. Conditionally on $\mathscr{H}_k$, the random variables $(X_{1,t}^u(\cdot), \mathscr{Q}_{t,\zeta}^{(u)})_{u\in\mathscr{H}_k}$ are independent by the branching property. By Theorem 77, for every $u \in \mathscr{H}_k$, the pair $(X_1^u(t) - m_t, \mathscr{Q}_{t,\zeta}^{(u)})$ converges in law to $(k + W(u), \mathscr{Q}_\zeta^{(u)})$ where $\mathscr{Q}_\zeta^{(u)}$ is a copy of $\mathscr{Q}_\zeta$ independent of $W(u)$.

To conclude, observe that $\sum_{u\in\mathscr{H}_k}\delta_{k+W(u)} = \mathscr{P}^*_{k,\infty} - \log(CZ_k)$ by Proposition 80. Since for each $u\in\mathscr{H}_k$ the point measure $\mathscr{Q}^{(u)}_\zeta$ is independent of $W(u)$ and of all $W(v)$ for $v\in\mathscr{H}_k$ and $v\neq u$, it follows that $\mathscr{Q}^{(u)}_\zeta$ is independent of $\mathscr{P}^*_{k,\infty}$. We conclude that

$$\lim_{t\to\infty}(\mathscr{P}^*_{k,t},(\mathscr{Q}^{(u)}_{t,\zeta})_{u\in\mathscr{H}_k}) = (\mathscr{P}^*_{k,\infty},(\mathscr{Q}^{(u)}_{\zeta})_{u\in\mathscr{H}_k})$$

in distribution where the two components of the limit are independent. □

7.9 Putting the pieces back together: Proof of Theorem 75

Let $\bar{\mathscr{N}}^{(k)}(t)$ be the extremal point measure seen from the position $m_t - \log(CZ_k)$

$$\bar{\mathscr{N}}^{(k)}(t) := \mathscr{N}(t) - m_t + \log(CZ_k).$$

Let $\zeta : [0,\infty)\to[0,\infty)$ be any function increasing to infinity. Observe that on $\mathcal{B}^{\complement}_{\eta,k,t}$ (an event of probability tending to one when $t\to\infty$ and then $k\to\infty$ by Proposition 81) we have

$$\bar{\mathscr{N}}^{(k)}(t)|_{[-\eta,\eta]} = \sum_{u\in\mathscr{H}_k}\Big(\mathscr{Q}^{(u)}_{t,\zeta(k)} + X^u_{1,t} - m_t + \log(CZ_k)\Big)\,|_{[-\eta,\eta)]}.$$

Now by Lemma 82 we know that in distribution

$$\lim_{t\to\infty}\sum_{u\in\mathscr{H}_k}\Big(\mathscr{Q}^{(u)}_{t,\zeta(k)} + X^u_{1,t} - m_t + \log(CZ_k)\Big) = \sum_{x\in\mathscr{P}^*_{k,\infty}}(x+\mathscr{Q}^{(x)}_{\zeta(k)})$$

where the $\mathscr{Q}^{(x)}_{\zeta(k)}$ are independent copies of $\mathscr{Q}_{\zeta(k)}$, and independent of H_k. Moreover, we know that $\lim_{t\to\infty} Z(t) = Z$ almost surely.

By the second limit in Proposition 80, we have that $(\sum_{x\in\mathscr{P}^*_{k,\infty}}(x+\mathscr{Q}^{(x)}_{\zeta(k)}), Z_k)$ converges as $k\to\infty$ to $(\mathscr{L}, Z)$ in distribution, $\mathscr{L}$ being independent of Z. In particular, $(\mathscr{L}, Z)$ is also the limit in distribution of $(\sum_{x\in\mathscr{P}^*_{k,\infty}}(x+\mathscr{Q}^{(x)}_{\zeta(k)}), Z)$. We conclude that in distribution

$$\lim_{k\to\infty}\lim_{t\to\infty}(\bar{\mathscr{N}}^{(k)}(t)|_{[-\eta,\eta)]}, Z(t)) = (\mathscr{L}|_{[-\eta,\eta)]}, Z).$$

Hence, $\lim_{k\to\infty}\lim_{t\to\infty}(\bar{\mathscr{N}}^{(k)}(t), Z(t)) = (\mathscr{L}, Z)$ in distribution. Since $\bar{\mathscr{N}}(t)$ is obtained from $\bar{\mathscr{N}}^{(k)}(t)$ by the shift $\log(CZ) - \log(CZ_k)$, which goes to 0 by (7.18), we have in distribution $\lim_{t\to\infty}(\bar{\mathscr{N}}(t), Z(t)) = (\mathscr{L}, Z)$ which yields the content of Theorem 75.

7.10 The approach and description of Arguin et al.

In [4], Arguin, Bovier and Kistler using the path localization argument obtained in [3] are able to show that if one only considers particles that have branched off from one another far enough into the past (the point process of clan leaders), then it converges to a Poisson point process with exponential intensity ([4], Theorem 2). Their proof relies on the convergence of Laplace functionals (for which a first Lalley-Sellke type representation is given) and not from the classical results about records of iid variables as here.

In [5] a complete description of the extremal point process of the branching Brownian motion is given. There, they show that $\bar{\mathscr{N}}(t)$ (actually in [5] the point process $\mathscr{N}$ is centered by m_t instead of $m_t - \log(CZ)$) converges in distribution to a limiting point process which is necessarily an exponential Poisson point process whose atoms are “decorated" with iid point measures. They give a complete description of this decoration point measure as follows. Let $\mathcal{D}(t) = \sum_{i=1}^{\infty} \delta_{X_i(t)-X_1(t)}$ which is a random point measure on $\mathbb{R}_+$. Conditionally on the event $X_1(t) < 0$ it converges in distribution to a limit $\mathcal{D}$.

One of the key argument in [5] is to identify the limit extremal point process of the branching Brownian motion with the limit of an auxiliary point process. This auxiliary point process is constructed as follows. Let $(\eta_i, i \in \mathbb{N})$ be the atoms of a Poisson point process on $\mathbb{R}_+$ with intensity

$$a(xe^{bx})\,\mathrm{d}x$$

for some constants a and b. For each i, they start form η_i an independent branching Brownian motion (with the same λ, σ, ϱ parameters as the original one) and call $\Pi(t)$ the point process of the position of all the particles of all the branching Brownian motions at time t. Theorem 2.5 in [5] shows that $\lim_{t\to\infty} \Pi(t) = \lim_{t\to\infty} \bar{\mathscr{N}}(t)$. This solves what Lalley and Sellke [22] call the conjecture on the *standing wave of particles*. The proof is based on the analysis of Bramson [7] for the solution of the F-KPP equation with various initial conditions and the subsequent work of Lalley and Sellke [22] and Chauvin and Rouault [10] which allows them to show convergences of Laplace type functionals of the extremal point process.

Chapter 8

Branching Brownian motion with absorption

In this chapter we are going to start our analysis of *branching Brownian motion with absorption*. In its simplest instance, the definition of the model is straightforward. We start a usual branching Brownian motion with dirt $\mu \in \mathbb{R}$ from a single particle at position $x > 0$ and we kill particles when they hit the origin.

Such a model is clearly coupled with the usual branching Brownian motion as follows. Let $(X(t), t \geq 0)$ be a usual $\mathbb{P}_x$-branching Brownian motion with drift μ started from x and $\mathcal{N}(t)$ be the set of particles alive at time t for this full BBM. Define

$$\mathcal{N}_{\text{abs}}(t) = \{u \in \mathcal{N}(t) : \inf_{s \leq t} X_u(s) > 0\}.$$

Then $(\{X_u(t), u \in \mathcal{N}_{\text{abs}}(t)\}, t \geq 0)$ is the branching Brownian motion with absorption. We have just kept all the particles whose path has not touched 0.

Since the branching Brownian motion with or without absorption are thus defined on the same probability space, we use the same notation $\mathbb{P}_x$ for their law, or sometimes $\mathbb{P}_{\mu,x}$ when we wish to emphasize the dependence on the drift parameter.

8.1 Survival and Kesten's result

The first question we may ask about this process is wether it survives or not. Clearly this will depend on the value of μ. Recall that under $\mathbb{P}_{0,0}$ when there is no drift, we have shown that

$$\frac{M(t)}{t} \to \sqrt{2}, \qquad \text{almost surely.}$$

Therefore one can guess the following result which is first proved by kesten in 78 [**?**]:

Theorem 83. *Let $\zeta := \inf\{t \geq 0 : \mathcal{N}_{abs}(t) = \emptyset\}$ be the extinction time of the BBM with absorption. Then, $\mathbb{P}(\zeta < \infty) = 1$ if and only if $\mu \leq -\sqrt{2}$.*

Proof. Observe that this is not a simple consequence of the almost sure limit of $M(t)/t$ which only implies that if $\mu \leq -\sqrt{2}$ then $\mathbb{P}(\zeta < \infty) = 1$ (in fact for the equality case we need to use that $M(t) - \sqrt{2}t \to -\infty$).

So we only need to show that for $\mu > -\sqrt{2}$ there is a positive probability that the process survives forever. Observe that this equivalent to asking that in a BBM with zero drift started form 0, there exists a particle $u \in \bar{\mathcal{U}}$ such that

$$X_u(t) \geq -\mu t - x, \forall t \geq 0$$

[insert figure]

This can be done most efficiently through the use of the change of probability $W_\lambda(t)$. Indeed, choose $\lambda \in (-\sqrt{2}, \mu)$ so that $\mathbb{Q}_\lambda \sim \mathbb{P}$. Under $\mathbb{Q}_\lambda$ we know that Ξ is a BM with drift $-\lambda$ and therefore

$$\mathbb{Q}_\lambda\big(\forall t \geq 0 : \Xi(t) > -\lambda t - x\big) > 0$$

and thus

$$\mathbb{P}\big(\exists u \in \bar{\mathcal{U}} \text{ s.t. } \forall t \geq 0 : X_u(t) > -\lambda t - x\big) > 0.$$

□

8.2 Refinement By Feng-Zeitouni and Jaffuel

8.3 The number of absorbed particles: regime A (Maillard's result)

8.4 The number of absorbed particle: Regime C (the distribution of the all time minimum)

8.5 The number of absorbed particles: Regime B (convergence to the traveling wave

Chapter 9

Populations under selection and Brunet-Derrida's conjectures

In this chapter, I will try to describe a series of conjectures by Brunet and Derrida concerning branching random walks -or branching Brownian motion- with *sélection*. I will then présent briefly some of the results we have obtained concerning these conjectures in [**?**] and [**?**] as well as the beautiful results of Maillard [**?**].

9.1 Brunet-Derrida conjectures

9.1.1 BRW and BBM with selection

In [**?**, **?**, **?**] Brunet, Derrida, Mueler and Munier study various models of population under selection and use this to obtain results concerning the *noisy FKPP equation*. To fix things, let us consider the following two models.

Model 1: the BRW with selection We have a population of size N with asexual reproduction. Each individual $i \leq N$ is completely characterized by a number $x_i \in \mathbb{R}$ which measures its salective advantage and that we interpret as his *fitness*. So at a given time the population is simple a collection of N points on the real line. Time is discrete and at each generation the whole population is entirely renewed according to the following two-steps mechanism

> **Reproduction-mutation :** each individual has k offsprings (so that after this step there is momentarily kN particles) and the relative positions of offsprings with respect to their parents are given by i.i.d. copies of a certain displacement law ρ.
>
> **Selection :** Just keep the N rightmost particles among the kN you have just created.

A good example to keep in mind is the case $k = 2$ and ρ the uniform distribution on $[0, 1]$ (typically want second moments and no lattice).

Model 2: the BBM with selection We still have a cloud of N particles that leaves on $\mathbb{R}$. The evolution of the population is still essentially following the same steps

> **Reproduction-mutation :** each particle moves in $\mathbb{R}$ according to an independent Brownian motion and branches at rate 1 into two new particles which then start to follow the same behavior and so on. If we stop here, this is just the standard branching Brownian motion.
>
> **Selection :** At each branching event, kill the leftmost particle to keep the population size constant.

If the leftmost particle tries to branch, the event is simply ignored.

9.1.2 The speed conjecture

So what can we say about the branching random walk with selection? Brunet and Derrida makes three sticking predictions: The first one concerns the speed of the system. Let us note $X_1(t) \leq X_2(t) \leq \ldots \leq X_N(t)$ for the positions of the particles at time t.

It is not hard to convince oneself (by subadditivity for instance) that

$$\lim_{t\to\infty} \frac{X_1(t)}{t} = \lim_{t\to\infty} \frac{X_N(t)}{t} = v_N$$

for a certain velocity v_N which depends on N. In fact it is clear that $X_N(t) - X_1(t)$ is an ergodic process that stays of order one, since the time at which you discover a particle that is going to be the common ancestor of the whole population at a future time will give you some renewal times.

Conjecture 84 (Brunet, Derrida, Mueller, Munier). *$v_N \to v_\infty < \infty$ when $N \to \infty$ and (cf. equation (76) in [?])*

$$v_\infty - v_n = \frac{c}{2(\log N)^2} - c\frac{3\log\log N}{(\log N)^3} + \ldots \tag{9.1}$$

where c is a constant which is explicitly determined in terms of the displacement law. In case of model 2 this constant is [to compute]

The first order term of this correction has been proved in the context of model 1 (the BRW with selection) by Bérard and Gouéré [?]. In their work they consider model 1 with $k = 2$ (dyadic branching) and a displacement law ρ which has the following properties. Define

$$\Lambda(t) := \log \int e^{tx}\rho(dx.)$$

We must have

$$\sigma := \sup\{t \geq 0 \Lambda(-t) < +\infty\} > 0\ ;\ \zeta := \sup\{t \geq 0 \Lambda(t) < +\infty\} > 0$$

and

$$\exists t^* \in]0, \zeta[\text{ such that } t^*\Lambda'(t^*) - \Lambda(t^*) = \log 2.$$

Theorem 85 (Bérard and Gouéré). *Under the above hypothesis, $v_N \to v_N(\rho) \to v_\infty(\rho) = \Lambda'(t^*)$ when $N \to \infty$ and*

$$v_\infty(\rho) - v_N(\rho) \sim \chi(\rho)(\log N)^{-2}$$

where $\chi(\rho) := \frac{\pi^2}{2} t^* \Lambda''(t^*)$.

It should be possible to extend this result to the case of a random number of children per branching and to the case of model 2. One of the main ingredient of the proof is the result of Gantert, Hu and Shi [?] about the survival probability of the BRW killed under a certain line.

[cite Durrett Remenik]

9.1.3 The genealogy conjecture

Consider model 2 and suppose that in the selection phase, instead of picking the N rightmost individuals we just sample uniformly N particles out of the kN produced. In this case we are back in the context of a *neutral* model (no selective advantage). The probability that two individuals chosen at random at generation $n+1$ have the same ancestor in generation n is roughly of the order $1/N$ so we know that we are in the universality class where on a timescale of N the population genealogy will converge when $N \to \infty$ to an object called the *Kingman coalescent*.

Can we say anything similar in the case of populations under selection? This is of course an important question in population genetics and in ecology.

Conjecture 86 (Brunet, Derrida, Mueller, Munier). *In model 1 or 2, on a timescale of order $(\log N)^3$, the genealogy of the population converges to a Bolthausen-Sznitman coalescent.*

To understand this statement one has to know what a Bolthausen-Sznitman coalescent is. We make a small stop to explain heuristically what this object is.

Description of Λ-coalescents Suppose you have a countable population of individuals $i = 1, 2, \dots$ for which you can follow backward their ancestral lineages. At the time of the most recent common ancestor of two individuals, i and j say, their lineages coalesce. Thus if you look at the k fist individuals $\{1, 2, \dots, k\}$ their ancestral lineages trace a tree with k leaves (the genealogical tree). The coalescent is a Markov process which describes how the lineages merge when one goes backward in time.

- In Kingman coalescent, each pair of lineages merge at rate 1.
- In a Λ-coalescent, there is a point process (t_i, p_i) in $\mathbb{R}_+ \times [0,1]$ with intensity $\mathrm{d}t \otimes p^{-2}\Lambda(\,\mathrm{d}p)$ where Λ is a finite measure on $[0,1]$. At the time t_i of an atom you select by independent coin-flipping a proportion p of the active lineages and merge them in a single one.
- the Bolthausen-Sznitman coalescent is the Λ-coalescent one obtain when $\Lambda(\,\mathrm{d}p) = \mathrm{d}p\mathbf{1}_{\{p\in[0,1]\}}$.

Thus, the action of the selection changes qualitatively the model. The precise form of the conjecture of Brunet Derrida is the following: if T_p denotes the time of the MRCA (most recent common ancestor) of p individuals, the authors predict that the statistics $\langle T_p\rangle/\langle T_2\rangle$ where $\langle\cdot\rangle$ is the physicists notation for the expectation converge to the values which are the same as those obtained for the Bolthausen-Sznitman coalescent. The sequence $\langle T_p\rangle/\langle T_2\rangle$ characterizes (up to a multiplicative factor) the distribution Λ.

9.2 Universality

An important idea in the work of Brunet and Derrdida is that those predictions (about the speed correction and about the genealogy) should be very robust to the precise details of the model. In this section we present two models which are strongly conjectured to be in the same universality class (in some sense).

9.2.1 Directed polymers

Dans [?, ?] puis [?], il est conjecturé que le coalescent de Bolthausen Sznitman est également la généalogie de certains modèles de polymères dirigés. L'état du système est de nouveau une population de N points sur la ligne réelle qui évoluent par générations discrètes (on note $x_i(n)$ la position de l'individu i à la génération n). Pour chaque i on tire uniformément avec remise deux individus j_i et j_i' dans la génération n et on pose

$$x_i(n+1) = \max\{x_{j_i} + \alpha_i; x_{j_i'} + \alpha_i'\}$$

où les α_i, α_i' sont des variables iid, et plus précisément Brunet et Derrida prennent des variables de Bernoulli de paramètre p. Dans [?], les auteurs montrent à l'aide de simulations numériques que la correction de la vitesse est encore en $k/(\log N)^2$.

Dans [?], un modèle un peu différent est exploré où

$$x_i(n+1) = \max_{j=1,\dots,N}\{x_j(n) + S_{i,j}(n)\}$$

où les $S_{i,j}(n)$ sont iid de loi ρ. Après avoir montré comment ce modèle est relié à l'équation FKPP bruitée (et est donc dans la même classe d'universalité que les marches aléatoires branchantes avec sélection) un cas particulier est étudié en détail (on prend la distribution de Gumbel pour ρ). Malheureusement ce cas est spécial en ce sens que la vitesse du nuage de particules diverge avec N et ne permet pas de retrouver la correction en $(\log N)^{-2}$.

Finallement, dans [?] le modèle suivant de polymère dirigé en champs moyen et en environement aléatoire est proposé. Chaque génération consiste en N sites qui sont chacun connectés à M site choisis uniformément dans la génération précédente, $2 \leq M \leq N$. Chaque lien (AB) entre deux sites$A$ et B porte une énergie $e_{(A,B)}$. Un polymère dirigé est simplement un chemin dans la structure de graphe ainsi défini qui avance (sans saut) dans les générations. L'énergie E d'un polymère est la somme des énergies des arrêtes qu'il traverse. Le modèle est spécifié par la donnée de M et la loi des $e_{(AB)}$. On prendra toujours des variables $e_{(A,B)}$ iid, des simulations numériques sont faites dans [?] pour le cas de variables uniformes sur $[0, 1]$.

On fixe à présent un site racine dans la génération 0 et pour chaque site A d'une génération $n > 0$ on définit E_A comme l'énergie minimale du polymère sur tous les chemins dirigés possible qui relient la racine à A. Clairement, les chemins d'énergie minimum pour un ensemble de $k \leq N$ sites qui appartiennent à la même génération forment un arbre à k feuilles (au moins quand n est assez grand).

Les simulations effectuées dans [?] suggèrent que, comme pour les marches branchantes avec sélection, cet arbre, sur une échelle de temps d'ordre $(\log N)^3$, converge vers l'arbre de Bolthausen Sznitman.

Les auteurs proposent également la conjecture fascinante suivante. Les modèles de polymères dirigés présentent une transition de phase autour d'une température critique. Les auteurs s'attendent donc à ce que sous cette température critique l'arbre de Bolthausen Sznitman continue à décrire la structure des polymères, tandis qu'au dessus, la situation correspondrait aux marches aléatoires coalescentes et au coalescent de Kingman.

9.2.2 Le lien avec l'équation FKPP bruitée

Ce paragraphe est une tentative de résumer (avec un très faible degré de précision) certains des liens qui relient l'étude de ces marches branchantes avec sélection à l'équation de FKPP bruitée.

En 1937, Fisher [?] d'une part et Kolmogorov, Petrovski et Piskunov [?] d'autre part ont introduit l'équation aux dérivées partielles suivante qui décrit la propagation d'un front dans un milieu instable (dans le cas de Fisher il s'agissait de décrire la propagation d'un gêne mutant dans une population)

$$\partial_t u = \partial^2_{xx} u + u(1 - u)$$

(je ne donne pas la version la plus générale de cette équation, en particulier on peut avoir un terme de non-linéarité différent de $u(1 - u)$). Cette équation a des solutions de type "onde voyageuse", et si la condition initiale décroît assez rapidement, la vitesse minimum (notée v_∞) est sélectionnée.

L'équation de FKPP bruitée est

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - u^2 + \sqrt{\frac{u(1-u)}{N}}\, W(x,t), \tag{9.2}$$

où $W(x,t)$ est un bruit blanc espace-temps. Cette équation est en dualité fonctionnelle avec le modèle de mouvement brownien branchant (à taux 1) et coalescent (les coalescences se produisant le long du temps local d'intersection avec taux $1/N$ (voir [?, ?]). Pour cette équation stochastique, avec de bonnes conditions initiales, on a des solutions (aléatoires) telles que

$$r(t) = \sup\{x \in \mathbb{R} : u(t,x) > 0\} < \infty$$

et vu de $r(t)$ le processus a une unique loi stationnaire non-dégénérée. En d'autre termes, un front se forme et on s'attend à ce qu'il ait une vitesse $v_N = \lim_{t\to\infty} r(t)/t$. Ces résultats ont été prouvés dans [?].

Il est remarquable que la conjecture de Brunet et Derrida concernant les corrections (9.1) pour l'équation (9.2) ait été récemment (partiellement) prouvée par Mueller, mytnik et Quastel [?]. Leur résultat principal est que la vitesse v_N de (9.2) est donnée par

$$2 - \pi^2(\log N)^{-2}$$

avec une erreur d'ordre $\log\log N/(\log N)^3$.

Quel est le lien avec les systèmes de particules du type marches aléatoires branchantes avec sélection ? Je résume de façon très schématique à présent la discussion que l'on peut trouver dans [?]. On peut décrire l'état $(X_1(n), \dots, X_N(n))$ du système au temps n par un front, c'est à dire une fonction

$$h_n(x) = \frac{1}{N}\#\{i = 1, \dots, N \text{ t.q. } X_i(N) \geq x\}.$$

Clairement, $h_n(-\infty) = 1$ et $h_n(\infty) = 0$. Soit $Nh^*_{n+1}(x)$ le nombre d'enfants produits par la génération n à droite de x avant l'étape de sélection. On a alors

$$h_{n+1}(x) = \min[1, h^*_{n+1}(x)].$$

On peut calculer aisément la moyenne $\mu(h_n, x)$ et la variance $\sigma(h_n, x)$ de $h^*_{n+1}(x)$ conditionnellement aux positions à la génération n en terme de la distribution de déplacement ρ. En particulier $\sigma(h_n, x)$ est d'ordre $1/\sqrt{N}$. On a alors

$$h_{n+1}(x) = \min[1, 2\int dy \rho(y) h_g(x-y) + \eta_n(x)\sqrt{\sigma(h_n, x)}]$$

où $\eta_n(x,1)$ est un bruit centré de variance 1 et le premier terme correspond au branchement binaire et au déplacement par ρ. Brunet et Derrida expliquent que cette équation est “similaire” à l'équation FKPP (9.2). La convolution par 2ρ joue le même rôle que l'opérateur de diffusion et étale le front, le fait de prendre le min avec

1 est un mécanisme de saturation comme le terme $-u^2$, et l'amplitude du terme de bruit est la même comme je viens de l'expliquer.

L'objet de ces modèles microscopiques est de comprendre le second ordre de correction de $v_\infty - v_N$ dans (9.1). En effet, le premier terme en $(\log N)^{-2}$ peut s'obtenir par des méthodes purement déterministes. Brunet et Derrida introduisent une équation déterministe avec cutoff

$$\partial_t u = \partial^2_{xx} u + u(1-u)a(u)$$

où $a(u) = 0$ dés que $u \leq 1/N$ et vaut 1 sinon. Ils montrent alors que la correction entre cette nouvelle équation et v_∞ est précisément ce terme $c/(2(\log N)^2)$ vu plus haut.

En revanche, le terme suivant en $\log\log N/(\log N)^3$ est lui dû aux fluctuations aléatoires de la position du front. Dans le langage du système de particules, cette correction (positive) à la vitesse correspond à des événements rares au cours desquels une particule arrive à s'avancer très loin devant ses poursuivants et peut alors "oublier" la sélection. Cette particule génère de nombreux descendant et fait avancer le front plus loin qu'il ne serait allé sinon, créant ainsi une fluctuation. Comme on le verra, cette explication heuristique qui est présentée dans [?] et qui est bien détaillée dans les notes de cours [?] est très proche de notre analyse de la généalogie du mouvement Brownien branchant avec sélection.

9.3 Résultats principaux

9.3.1 Mouvement Brownien branchant avec absorption

L'une des difficultés qui se pose lorsque l'on veut analyser rigoureusement le modèle à taille de population constante est que l'on perd l'indépendance entre les particules. Pour retrouver cette propriété on peut s'intéresser à un modèle voisin dans lequel la taille de la population n'est plus fixée (mais dans lequel la vitesse de sélection est imposée de façon exogène).

Dans ce modèle, on retrouve les deux ingrédients vus plus haut : mutation-reproduction et sélection.

> **Reproduction-sélection** : les individus se déplacent et se reproduisent selon le mécanisme d'un mouvement Brownien branchant. Plus précisément, chaque particule évolue indépendamment des autres en se déplaçant selon un mouvement Brownien standard et branche à taux 1 en donnant naissance à deux nouvelles particules identiques situées au lieu du branchement.
>
> **Sélection** : toutes les particules dont la position au temps t est inférieure ou égale à μt où $\mu \in \mathbb{R}$ sont immédiatement tuées et sortent du système.

Par un simple changement de repère, ce modèle de "mur qui avance" et de Brownien branchant sans dérive est équivalent au modèle dans lequel le mur est fixe en 0 (les particules sont immédiatement tuées en 0) et les particules Browniennes ont une dérive $-\mu$ (i.e. sont poussées vers 0). C'est ce modèle précisément qui va nous intéresser et que nous désignons sous le nom de mouvement Brownien branchant avec absorption en 0 (on note parfois kBBM(μ) pour killed Brownian motion with drift ($-\mu$)). (voir les figures 9.2 et 9.3 qui sont reproduites de la thèse de Damien Simon avec son aimable autorisation).

Ce modèle, dont l'étude remonte au moins à Kesten [?] a fait l'objet de travaux récents de Derrida et Simon [?, ?] dans lesquels une conjecture (étayée par une analyse perturbative de l'équation aux dérivées partielle de FKPP qui peut sans doute être rendue rigoureuse) concernant la probabilité de survie est formulée.

9.3.2 Survie

Dans la suite on notera $\mathbb{P}_x$ et $\mathbb{E}_x$ pour désigner la loi (et l'espérance sous cette loi) du kBBM qui a pour état initial une particule unique en position $x > 0$. On ne précise pas μ qui est en général fixé. Dans [?], Kesten montre un premier résultat simple sur la survie du processus, qui dans notre cadre peut s'énoncer ainsi :

Theorem 87 (Kesten, [?]). *Si $\mu \geq \sqrt{2}$ le processus s'éteint presque sûrement en temps fini (cas critique et sous-critique).*
Si $\mu < \sqrt{2}$ le processus survit avec probabilité positive et dans ce cas le nombre de particules en vie croît exponentiellement (cas sur-critique). Dans ce cas on note $Q_\mu(x)$ la probabilité sous $\mathbb{P}_x$ que le processus survive.

Plus récemment, quelques résultats supplémentaires ont été obtenus :

- Harris, Harris, et Kyprianou [?] montrent (cf. Théorème 13) que, pour $\mu < \sqrt{2}$, la fonction $x \mapsto Q_\mu(x)$ satisfait l'équation de Kolmogorov

$$\frac{1}{2}Q''_\mu(x) - \mu Q'_\mu(x) = Q_\mu(x)(1 - Q_\mu(x))$$

  avec conditions aux bords $\lim_{x\to 0} Q_\mu(x) = 0$ et $\lim_{x\to\infty} Q_\mu(x) = 1$. Ils montrent également (cf. Théorème 1) que pour chaque $\mu < \sqrt{2}$ fixé il y a une constante K telle que

$$\lim_{x\to\infty} e^{(\sqrt{\mu^2+2}-\mu)x}(1 - Q_\mu(x)) = K.$$

- Dans le cas sous-critique ($\mu > \sqrt{2}$) Harris et Harris [?] utilisent des techniques de martingales et de décomposition en épine dorsale pour calculer les asymptotiques de $\mathbb{P}_x(\zeta > t)$ quand t devient grand (et où ζ est l'instant d'extinction du processus).

- Dans [?] Simon et Derrida obtiennent des estimées pour la probabilité de survie $Q_\mu(x)$. Ils conjecturent qu'il existe K tel que quand $L - x \gg 1$,

$$Q_\mu(x) = KLe^{\sqrt{2}(x-L)}\left(\sin\left(\frac{\pi x}{L}\right) + O\left(\frac{1}{L^2}\right)\right) + O(e^{2\sqrt{2}(x-L)})$$

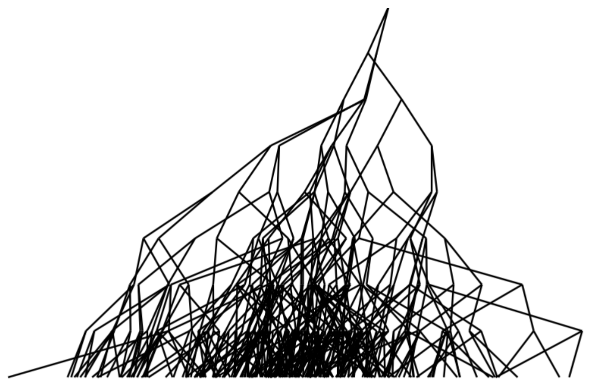


Figure 9.1: Mouvement Brownien branchant (ou marche branchante) sans sélection

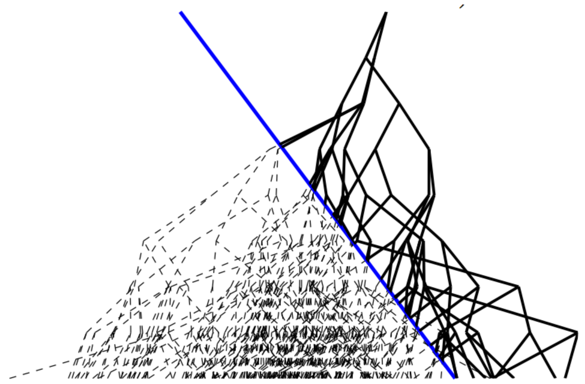


Figure 9.2: Mouvement Brownien branchant (ou marche branchante) avec sélection par absorption

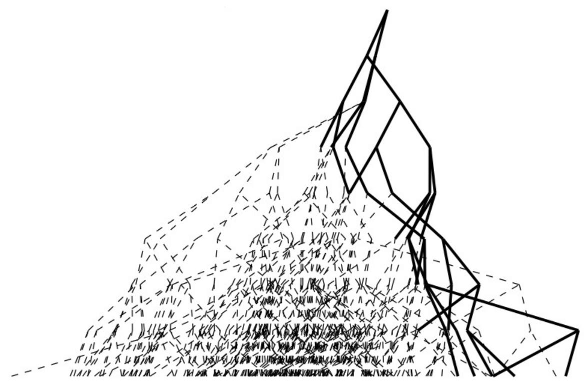


Figure 9.3: Mouvement Brownien branchant (ou marche branchante) avec sélection endogène (taille de population fixée à 6).

(équation (B.16)) et qu'il existe une autre constante c telle que quand $x > L$ ou $L - x$ est $O(1)$,

$$Q_\mu(x) = 1 - \theta(x - L + c) + O\left(\frac{1}{L^2}\right),$$

où θ résout l'équation différentielle (9.3) ci-dessous (équation (B.17)). Ces estimées sont obtenues à l'aide de méthodes entièrement analytiques qui ne sont cependant pas encore un argument tout à fait rigoureux.

On pose $\epsilon = \sqrt{2} - \mu > 0$, et on définit L par la relation $1 - \mu^2/2 - \pi^2/2L^2 = 0$, c'est-à-dire $L = \pi/\sqrt{2-\mu^2} = \pi/\sqrt{\epsilon}$. Le théorème suivant que nous obtenons dans [?] valide les prédictions de Derrida et Simon.

Theorem 88 (B.,Berestycki,Schweinsberg [?]). *Soit $\alpha \in \mathbb{R}$ fixé. Alors il existe une fonction $\theta : \mathbb{R} \to \mathbb{R}$ telle que*

$$\lim_{\epsilon \to 0} Q_\mu(L + \alpha) = \theta(\alpha),$$

où $\theta : \mathbb{R} \to (0,1)$ satisfait

$$\frac{1}{2}\theta'' - \sqrt{2}\theta' = \theta(1 - \theta). \tag{9.3}$$

Il existe une constante C telle que si $L - x \gg 1$, alors

$$Q_\mu(x) \sim CLe^{-\mu(L-x)} \sin\left(\frac{\pi x}{L}\right),$$

où $\sim$ désigne l'équivalence asymptotique quand $\epsilon \to 0$.

La preuve de ce résultat (dont je donne les grandes lignes plus loin) utilise les résultats que nous avons obtenus dans [?] et que je présente maintenant.

9.3.3 Généalogie et CSBP de Neveu

La présentation de ces résultats est légèrement modifiée par rapport à [?] ou un paramètrage différent est utilisé. Dans toute cette partie, pour chaque ϵ donné on travail sous la loi $\mathbb{P}_x$ où l'on fixe $x = L$. On se fixe $n \geq 1$ et l'on définit le processus de la partition ancestrale $(\Pi^\epsilon_s, 0 \leq s \leq t)$ à valeur dans $\mathcal{P}_n$ (les partitions de $\{1, \ldots, n\}$) en tirant uniformément n individus dans la populations au temps $t\epsilon^{-3/2}$ et en posant que i et j sont dans le même bloc de Π^ϵ_s si et seulement si les individus i et j ont le même ancêtre au temps $(t - s)\epsilon^{-3/2}$.

Theorem 89 (B.,Berestycki,Schweinsberg [?]).

$$(\Pi^\epsilon_s, 0 \leq s \leq t) \underset{\epsilon \searrow 0}{\overset{fdd}{\to}} (\Pi_{2^{3/2}\pi^3 s}, 0 \leq s \leq t)$$

où Π est le coalescent de Bolthausen-Sznitman restreint à $\{1, \ldots, n\}$.

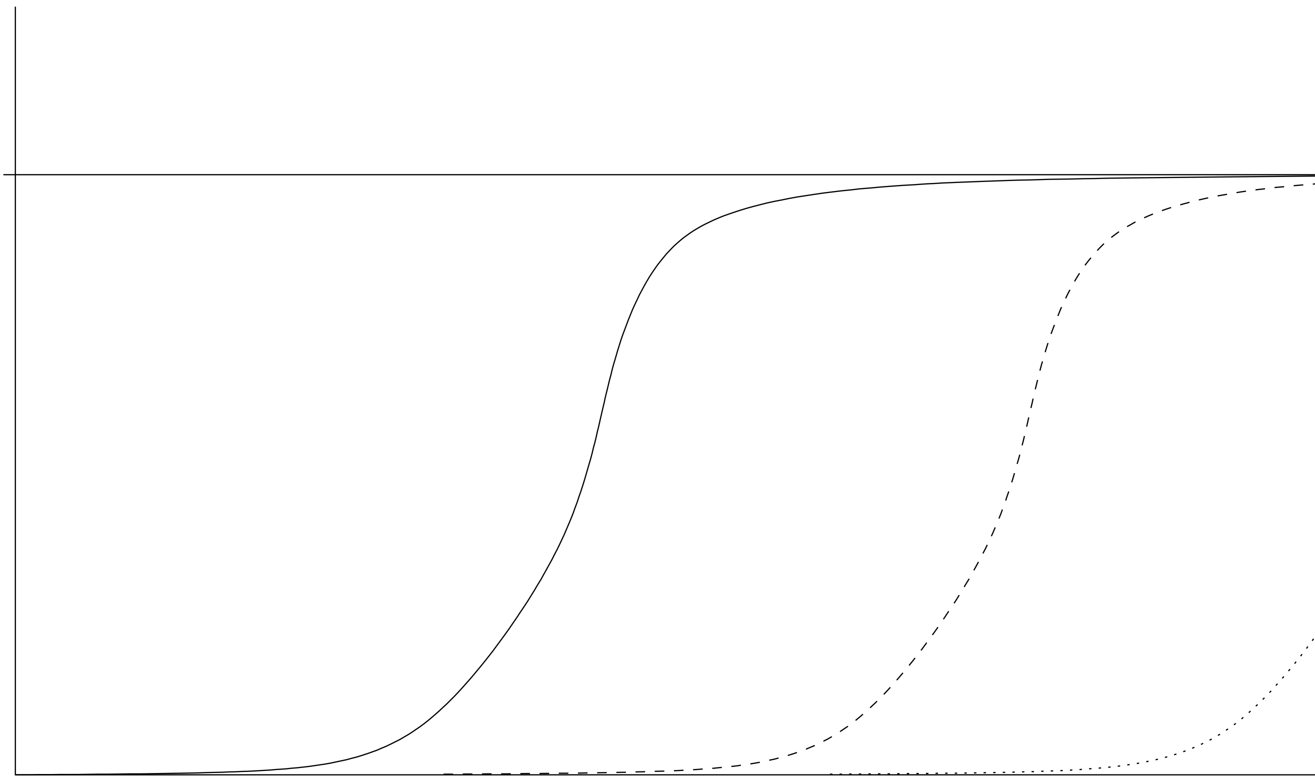


Figure 9.4: L'allure de la probabilité de survie $x \mapsto Q_\mu(x)$ pour $\mu - \sqrt{2} - \epsilon_{1,2,3}$ avec $\epsilon_1 > \epsilon_2 > \epsilon_3$. Un front se forme et s'éloigne de 0 (position L).

En outre, si l'on appelle M_t^ϵ le nombre de particules en vie au temps t on a

Theorem 90 (B.,Berestycki,Schweinsberg [?])**.**

$$\left(\frac{1}{\epsilon^{3/2}e^{\pi\sqrt{2/\epsilon}}}M^\epsilon_{t\epsilon^{-3/2}}, t\geq 0\right) \underset{\epsilon\searrow 0}{\overset{fdd}{\rightarrow}} (Z_t, t\geq 0)$$

où Z est un CSBP de mécanisme de branchement $\psi(u) = au + 2\pi^2 u\log u$, pour une certaine constante $a\in\mathbb{R}$, issu d'une population initiale Z_0 aléatoire de loi connue.

Le Gall et Bertoin avaient déjà observé dans [?] que la généalogie du CSBP de Neveu (de mécanisme de branchement $\psi(u) = u\log u$) était donné par le coalescent de Bolthausen-Sznitman. Ainsi, si l'on peut montrer que la population évolue selon une version du CSBP de Neveu il n'est pas surprenant que sa généalogie soit donnée par le coalescent de Bolthausen-Sznitman.

La renormalisation que l'on utilise dans le résultat ci-dessus montre que si l'on veut avoir de l'ordre de N particules en vie il faut prendre $\epsilon^{3/2}e^{\pi\sqrt{2/\epsilon}} \asymp N$ soit $\epsilon\sim 2\pi^2/(\log N+3\log\log N)^2$. On peut ainsi décider de paramètrer entièrement le modèle par N en prenant $\mu_N = \sqrt{2-\frac{2\pi^2}{(\log N+3\log\log N)^2}}$ et $L = (\log N + 3\log\log N)/\sqrt{2}$.

L'un des résultats clés dans [?] consiste à montrer que l'on peut mesurer la taille de la population en attribuant des poids aux individus tout en conservant la convergence du Théorème 90 vers le CSBP de Neveu (avec une renormalisation différente). Plus précisément, si l'on définit

$$Z_N(t) := \sum_{i=1}^{M_N(t)} e^{\mu X_i(t)}\sin(\pi X_i(t)/L)\mathbf{1}_{\{X_i(t)\leq L\}}, \qquad t\geq 0,$$

on a alors le résultat suivant

Theorem 91 (B.,Berestycki,Schweinsberg [?])**.** *Sous $\mathbb{P}_x$ on a*

$$\left(\frac{Z_N(t)}{N(\log N)^2}, t\geq 0\right) \underset{\epsilon\searrow 0}{\overset{fdd}{\rightarrow}} (Z_t, t\geq 0)$$

où Z est un CSBP de mécanisme de branchement $\psi(u) = au + 2\pi^2 u\log u$ pour une certaine constante $a\in\mathbb{R}$ issu d'une population initiale Z_0 aléatoire de loi connue.

Dans [?] nous ne travaillons pas sous $\mathbb{P}_x$, nous formulons le Théorème ci-dessus pour une suite de configuration initiales $X_1(0),\dots,X_{M_N(0)}(0)$ telles que $Z_N(0)/N(\log N)^2$ converge en distribution vers une certaine variable W non-dégénérée et tels que $Y_N(0) = o(N(\log N)^3)$ avec

$$Y_N(t) := \sum_{i=1}^{M_N(t)} e^{\mu X_i(t)}, \qquad t\geq 0.$$

Dans ce cas W est bien sûr la loi de la population initiale Z_0. Pour prouver les Théorèmes 90 et 91 tels que je les énonce ici, il faut donc montrer que si l'on démarre avec une particule en L, alors en un temps très court (négligeable devant $(\log N)^3$) le système se trouve dans une configuration $X_1, \ldots, X_N$ (où N est aléatoire) telle que pour ces position $Z_N/N(\log N)^2$ converge en distribution et $Y_N = o(N(\log N)^3)$. La loi limite de $Z_N/N(\log N)^2$ donne la loi initiale de Z_0 qui est mentionnée dans les résultats ci-dessus.

9.3.4 Quelques idées de la preuve des Théorèmes 89, 90 et 91

Notre approche dans [?] consiste à traiter séparément les particules qui atteignent une distance d'ordre L du mur en 0. Ces particules sont assez loin pour ne plus "sentir" le mur et ainsi peuvent rapidement produire une grande descendance, ce qui conduit à un saut positif de la taille de la population et à des coalescences multiples de lignées ancestrales lorsque l'on remonte le temps.

Cette approche est à comparer à la description phénoménologique du modèle à population constante que donnent Derrida et ses coauteurs (dans [?, ?] ou dans les notes du mini-cours que Eric Brunet a donné à Marseille en 2007 [?]). Ils montrent comment la correction de deuxième ordre à la vitesse (le terme en $\log\log N/(\log N)^3$) est due à des événements rares durant lesquels une particule se retrouve en avance du front et produit une grande descendance avant d'être rattrapée, créant ainsi une fluctuation dans la position du front.

Pour commencer, on peut considérer un mouvement Brownien branchant dans lequel les particules sont tuées lorsqu'elles atteignent 0 ou L. Si la configuration initiale consiste en une seule particule en position x, alors, pour t assez grand, la densité de particules attendue autour du point y est approximativement $p_t(x, y)\, dy$, où

$$p_t(x, y) = \frac{2}{L} e^{(1-\mu^2/2-\pi^2/2L^2)t} \cdot e^{\mu x} \sin\left(\frac{\pi x}{L}\right) \cdot e^{-\mu y} \sin\left(\frac{\pi y}{L}\right). \tag{9.4}$$

On peut tirer plusieurs informations sur le comportement du mouvement Brownien branchant dans une bande de cette formule. Le paramètre t n'apparaît que dans le premier facteur exponentiel. La taille de la population devrait donc rester stable dés que $1 - \mu^2/2 - \pi^2/2L^2 = 0$ (ce qui correspond en effet à notre choix). Une seconde observation est que la densité est proportionnelle à $e^{\mu x} \sin(\pi x/L)$, c'est-à-dire, en sommant sur toutes les particules au temps t, à $Z_N(t)$. On voit donc que $Z_N(t)$ est un bon prédicteur du nombre de particules que l'on trouvera dans un ensemble à un certain temps futur. Ainsi, $Z_N(t)$ est la bonne façon de mesurer la "taille" de la population. Enfin, la densité est également proportionnelle à $e^{-\mu y} \sin(\pi y/L)$. En conséquence, dés que t est assez grand (d'ordre $(\log N)^2$) le système a "oublié" sa configuration initiale et les particules sont dans une configuration stable dont la densité de particules à y est proportionnelle à $e^{-\mu y} \sin(\pi y/L)$.

L'étape suivante consiste à observer qu'une particule qui atteint L conduit à un "saut" dans la taille de la population. En effet, si l'on démarre avec N particules tirée selon cette densité d'équilibre entre 0 et L, alors $Z_N(0)$ vaut approximativement

$$N \int_0^L e^{\mu y} \sin\left(\frac{\pi y}{L}\right) \cdot C L e^{-\mu y} \sin\left(\frac{\pi y}{L}\right) dy,$$

qui est d'ordre NL^2. D'autre part, si on part avec une particule en L, ses descendants n'iront typiquement pas plus loin qu'une constante à droite de L. Pour estimer la contribution typique d'une particule en L au temps t on utilise (9.4) avec L à la place de x et $L+\alpha$ au lieu de L, avec $\alpha > 0$ une constante. La valeur de $Z_N(t)$ doit être du même ordre que

$$\int_0^L e^{\mu y} \sin\left(\frac{\pi y}{L}\right) \cdot \frac{2}{L+\alpha} e^{\mu L} \sin\left(\frac{\pi L}{L+\alpha}\right) e^{-\mu y} \sin\left(\frac{\pi y}{L+\alpha}\right) dy,$$

qui est d'ordre $L^{-1}e^{\mu L}$. En utilisant l'expression de L en fonction de N on obtient des ordres de grandeur identiques. Une particule à L produit donc une augmentation substantielle de la taille de la population.

Pour traiter les particules qui n'atteignent pas L, nous commençons par étudier un mouvement Brownien branchant tué en 0 et en L_A

$$L_A = \frac{1}{\sqrt{2}}\big(\log N + 3 \log\log N - A\big), \tag{9.5}$$

À l'aide de (9.4), nous obtenons des estimées pour les premiers et deuxièmes moments de différentes quantités ce qui nous permet ensuite de calculer le premier et deuxième moment de $Z_N(t)$ conditionnellement à l'état du processus $\theta(\log N)^3$ unités de temps avant. La borne du second moment est suffisante pour établir une loi des grand nombres quand A est grand qui permet de contrôler la distance entre $Z_N(t)$ et sa moyenne. La troncation à L_A est nécessaire car sinon les moments seraient dominés par des événements rares où une particule arrive très loin à droite et produit une grande descendance qui survit.

L'étape suivante consiste à compter le nombre de particules qui atteignent L_A. On peut montrer que si l'on part de N particules dans leur configuration stable, alors il faut attendre environs $(\log N)^3$ unités de temps pour voir une particule qui atteint L_A. On peut se convaincre qu'il s'agit de la bonne échelle de temps avec l'argument heuristique suivant. Si $\beta > 0$ est une constante, alors le nombre de particules entre $L-\beta$ et L au temps t est d'ordre

$$N \int_{L-\beta}^L C L e^{-\mu y} \sin\left(\frac{\pi y}{L}\right) dy,$$

c'est-à-dire d'ordre $1/(\log N)^3$. Chacune de ces particules a une probabilité positive d'atteindre L entre t et $t+1$ mais on peut montrer que les particules qui sont à une distance de L plus grande qu'une constante ne touche L avant $t+1$ qu'avec une faible probabilité. On a donc $O(1/(\log N)^3)$ particules qui atteignent L par unité de temps.

Lorsqu'une particule atteint L_A, elle pourrait avoir branché juste avant de toucher L_A permettant ainsi à un grand nombre de particules d'atteindre elles aussi L_A. On doit donc contrôler le second moment de ce nombre. Á cette fin on montre que conditionnellement à ce qu'au moins une particule atteigne L_A, l'espérance du nombre de particules qui atteignent L_A dans un intervalle de temps de durée $\theta(\log N)^3$, est bornée par une constante.

La clé pour analyser la contribution des particules qui atteignent L_A se trouve dans un résultat de Neveu [27]. Si une particule démarre à L_A et y est une grande constante, alors le nombre de descendants qui atteignent $L_A - y$ est $y^{-1}e^{\sqrt{2}y}W$, où W est une variable aléatoire. À l'aide de théorèmes Taubériens nous démontrons que pour x grand l'on a $P(W > x) \sim B/x$. C'est ce résultat qui conceptuellement explique pourquoi la généalogie est décrite par un coalescent de Bolthausen-Sznitman. La contribution d'une particule en L_A sera approximativement proportionnelle aux nombre de ses descendants qui atteignent $L_A - y$ pour y assez grand. La probabilité qu'une particule en L_A produise un saut de taille au moins x est ainsi proportionnelle à $1/x$ ce qui entraîne que la mesure de Lévy du CSBP limite aura une densité proportionnelle à x^{-2} comme dans le cas du CSBP de Neveu, conduisant ainsi au résultat de dualité avec le coalescent de Bolthausen-Sznitman. Techniquement, la preuve du Théorème 89 se fait en utilisant les flots de ponts introduits par Bertoin et Le Gall dans [**?**].

9.3.5 Un bref aperçut de la preuve du Théorème 88

L'essentiel du travail pour montrer le Théorème 88 consiste à montrer que pour α fixé, la quantité $Q_\mu(L + \alpha)$ converge vers une limite appelée $\theta(\alpha)$ quand $\epsilon \to 0$. Le Théorème 88 montre que si l'on démarre avec une particule en L, le processus $M_N(t)$ du nombre de particules, une fois renormalisé, converge vers une CSBP de Neveu $(Z_t, t \geq 0)$ issu d'une population initiale de taille W où W est la variable limite de Neveu que l'on vient de voir. Les trajectoires du CSBP de Neveu issu d'une population initiale x sont de deux types : soient elles croissent sur-exponentiellement, soit elles convergent vers 0 (mais n'atteignent pas 0 en temps fini). Si le mécanisme de branchement est $\psi(u) = au + bu \log u$ La probabilité de ce second événement est e^{-bx}. Tout le travail consiste donc à montrer que la probabilité d'extinction du processus $(M_N(t), t \geq 0)$ converge bien vers $\mathbb{E}(e^{-bW})$ la probabilité que le processus limite $(Z_t, t \geq 0)$ converge vers 0. Si on démarre avec une particule en $L + \alpha$ à présent, les mêmes arguments montrent que la probabilité d'extinction converge vers $\mathbb{E}(e^{-be^{\sqrt{2}\alpha}W})$ et l'on sait déjà que cette fonction de α résout l'équation KPP (9.3).

La seconde partie du théorème qui concerne le comportement de $Q_\mu(x)$ pour x fixé se fait selon les grandes lignes suivantes. Partant d'une particule à x on compte le nombre R de particules qui touchent $L - \alpha$ si on tue les particules à cet instant. On peut monter que

$$\mathbb{E}_x[R] = \frac{e^{\sqrt{2}\alpha}}{\pi\sqrt{2}\alpha} e^{\mu x} \sin\left(\frac{\pi x}{L-\alpha}\right) \frac{1}{N(\log N)^2}(1 + C_{N,\alpha})$$

où pour chaque $\alpha > 0, C_{N,\alpha} \to 0$ quand $N \to \infty$. Une borne sur le second moment de R permet alors de conclure que

$$Q_\mu(x) = \frac{e^{\sqrt{2}\alpha}}{\pi\sqrt{2}\alpha} \cdot e^{\mu x} \sin\left(\frac{\pi x}{L}\right) \cdot \frac{1}{N(\log N)^2} \cdot Q_\mu(L-\alpha)(1 + C_{N,\alpha} + o(\alpha^{-1})),$$

où $o(\alpha^{-1})$ est un terme qui tends vers 0 quand $\alpha \to \infty$ et est uniforme en N. Il suffit maintenant d'appliquer la première partie du théorème pour conclure.

Chapter 10

Branching Brownian motion with absorption: The critical case

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