GROWTH OF LÉVY TREES.

Thomas Duquesne* Matthias Winkel†

September 23, 2005

Abstract

We construct random locally compact real trees called Lévy trees that are the genealogical trees associated with continuous-state branching processes. More precisely, we define a growing family of discrete Galton-Watson trees with i.i.d. exponential branch lengths that is consistent under Bernoulli percolation on leaves; we define the Lévy tree as the limit of this growing family with respect to the Gromov-Hausdorff topology on metric spaces. This elementary approach notably includes supercritical trees and does not make use of the height process introduced by Le Gall and Le Jan to code the genealogy of (sub)critical continuous-state branching processes. We construct the mass measure of Lévy trees and we give a decomposition along the ancestral subtree of a Poisson sampling directed by the mass measure.

AMS 2000 subject classifications: 60J80.
Keywords: tree-valued Markov process, Galton-Watson branching process, genealogy, continuous-state branching process, percolation, Gromov-Hausdorff topology, continuum random tree, edge lengths, real tree.

1 Introduction

Continuous-state branching processes have been introduced by Jirina [20] and Lamperti [22, 21, 23]. They are the continuous analogues of the Galton-Watson Markov chains. Recall that the distribution of a continuous-state branching process is characterized by a real-valued function ψ defined on $[0, \infty)$ that is of the form

$$\psi(c) = \alpha c + \beta c^2 + \int_{(0,\infty)} (e^{-cx} - 1 + cx\mathbf{1}_{\{x<1\}})\Pi(dx), \tag{1}$$

where $\alpha \in \mathbb{R}$, $\beta \geq 0$, and Π is the Lvy measure which satisfies

$$\int_{(0,\infty)} (1 \wedge x^2)\Pi(dx) < \infty.$$

ψ is called the *branching mechanism* of the continuous-state branching process. More precisely, $Z = (Z_t, t \geq 0)$ is a continuous-state branching process with branching mechanism

*Université Paris 11, Mathématiques, 91405 Orsay Cedex, France; supported by NSF Grants DMS-0203066 and DMS-0405779

†University of Oxford, Department of Statistics, 1 South Parks Road, Oxford OX1 3TG, UK; email winkel@stats.ox.ac.uk; supported by Aon and the Institute of Actuaries, le département de mathématique de l'Université d'Orsay and NSF Grant DMS-0405779

ψ (a CSBP(ψ) for short) iff it is a $[0,\infty]$-valued Feller process whose transition kernel is characterized by

$$\mathbb{E}\left[\exp(-\lambda Z_{s+t})\,|Z_s\right] = \exp(-u(t,\lambda)Z_s)\ ,$$

where u is the unique non-negative solution of

$$\partial_t u(t,\lambda) = -\psi(u(t,\lambda)) \quad \text{and} \quad u(0,\lambda) = \lambda\ ,\ t,\lambda \geq 0.$$

This equation can be rewritten in the following integrated form

$$\int_{u(t,\lambda)}^{\lambda} \frac{dc}{\psi(c)} \ = \ t\ . \tag{2}$$

We shall mostly restrict our attention to the case where Z_t has a finite expectation which is equivalent to the fact that the right derivative of ψ at 0 is finite. We denote this right derivative by $m := \psi'(0+)$. We refer to the case $m \in [-\infty, 0)$ (resp. $m = 0$ and $m \in (0, +\infty)$) as to the *supercritical case* (resp. *critical case* and *subcritical case*).

We shall often assume that Z has a positive probability of extinction which is equivalent to the following analytical condition

$$\int^{\infty} \frac{dc}{\psi(c)} < \infty \tag{3}$$

(see [5]). In that case, we have

$$\mathbb{P}\left(\exists t \geq 0\ :\ Z_t = 0 |\, Z_0 = a\right) = \exp(-a\gamma),$$

where γ is the largest root of the equation $\psi(c) = 0$ (observe that $\gamma > 0$ only in the supercritical case: $m < 0$). If (3) is not satisfied then the underlying Lvy tree will fail to be separable.

The main goal of this paper is to construct an (a,ψ)-Lévy tree that can be interpreted as the genealogical tree of a population whose size evolves according a CSBP(ψ) Z with initial state $Z_0 = a$. We proceed by approximating the Lévy tree by Galton-Watson trees with exponential edge lengths. More precisely, recall that a Galton-Watson tree with exponential edge lengths is the genealogical tree of an ancestor and its descendants, where individuals have independent and identically exponentially distributed lifetimes with a rate $c \in (0,\infty)$, and produce offspring at the end of their lives independently according to an offspring distribution ξ on $\{0,2,3,\ldots\}$. Instead of one single tree, we rather consider a random number of independent Galton-Watson trees, the random number of ancestors being a Poisson random variable with parameter a. We call such a forest a *Galton-Watson forest* (a GW(ξ,c,a)-forest for short).

Let F be a GW(ξ,c,a)-forest. We perform a *Bernoulli leaf colouring* on F i.e. we attach independent Bernoulli marks with parameter p to all leaves and interpret a mark 0 as black, a mark 1 as red. An elementary calculation will show that the subforest F_b spanned by the black leaves and the root is a GW(ξ_b,c_b,a_b)-forest, where ξ_b, c_b, a_b are calculated explicitly in terms of ξ, c, a and p (see Lemma 4.1).

One of the aims of the paper is to construct a family $(\mathcal{F}_\lambda\,;\,\lambda \geq 0)$ of random trees such that for any $\lambda \geq 0$, $\mathcal{F}_\lambda$ is a GW($\xi_\lambda,c_\lambda,a_\lambda$)-forest and that is consistent under Bernoulli leaf colouring: namely, for any $0 \leq \mu \leq \lambda$, we want $\mathcal{F}_\mu$ to be the black subtree obtained from

$\mathcal{F}_\lambda$ by a Bernoulli leaf colouring with parameter $1-p=\mu/\lambda$. Theorem 4.2 asserts that the distribution of such a leaf colouring consistent family can be parametrized by (a,ψ), where $a\in(0,\infty)$ and ψ is the branching mechanism of a continuous-state branching process (CSBP(ψ)) that is of the form (1); more precisely we have

$$a_\lambda = a\psi^{-1}(\lambda), \quad c_\lambda = \psi'(\psi^{-1}(\lambda)), \quad \varphi_\lambda(r) = \sum_{k=0}^{\infty} \xi_\lambda(k) r^k = r + \frac{\psi((1-r)\psi^{-1}(\lambda))}{\psi^{-1}(\lambda)\psi'(\psi^{-1}(\lambda))}. \qquad (4)$$

This offspring distribution appears in [7] in the Brownian case and Theorem 3.2.1 [11] in the critical and the subcritical cases as the distribution of the ancestral tree corresponding to Poisson marks on $[0,\infty)$ via the coding of the Lévy tree by the height process. We refer to Remark 5.4 for a detailed discussion of the connection of our results and the work in [11, 12].

Conversely, Proposition 4.4 asserts that to each (a,ψ) there corresponds a growing family $(\mathcal{F}_\lambda;\lambda\geq 0)$ of GW-trees with edge lengths, consistent under Bernoulli leaf colouring as explained before and whose distribution is specified by (4). This process can be viewed as a forest-valued continuous-time Markov chain whose characteristics are specified by Remark 4.8.

The leaf-colouring consistent forest growth processes that we consider can be viewed in a more general framework of Markovian forest growth processes. Several schemes to construct such processes preserving Galton-Watson forests and allowing to pass to continuous limits are more or less explicit in the literature. They are often more easily described by their co-transition rules. *Firstly*, Neveu [28] and Salminen [33] erase branches in general (non-explosive) Galton-Watson trees with exponential edge lengths continuously from their tips. Le Jan [26], Abraham [1] and Pitman [31] reverse the procedure to grow stable/Brownian trees and forests from appropriate Galton-Watson trees/forests. *Secondly*, Aldous and Pitman [2] perform percolation on the edges in general Galton-Watson trees (without edge lengths) and retain the connected component containing the root, as a tree-valued Markov process as the percolation probability varies. They call the procedure pruning of a Galton-Watson tree. The viewpoint is to gradually reduce the tree by consistently increasing the percolation probability. Geiger and Kaufmann [17] discount the offspring distribution to reduce a given Galton-Watson tree in a size-biased way. This can be seen as a special case of multiplicity-dependent pruning at vertices. We will see here that it is also related to the *third* scheme of reduction by Bernoulli leaf colouring, that we study in this paper.

Let us denote by $(Z_t^\lambda)_{t\geq 0}$ the population size process associated with $\mathcal{F}_\lambda$. Assume that m is finite. Then, it is easy to show that for any $t\geq 0$, a.s.

$$\frac{1}{\psi^{-1}(\lambda)} Z_t^\lambda \underset{\lambda\to\infty}{\longrightarrow} Z_t,$$

where Z is a CSBP(ψ) such that $Z_0=a$. Under the additional assumptions (3) we prove in Theorem 5.1 an a.s. convergence for the entire genealogy: as in the paper by Evans, Pitman and Winter [14], we consider genealogical trees as tree-like metric spaces and more precisely as locally compact rooted real trees, whose precise definition is given in Section 3.1. We introduce the set $\mathbb{T}$ of root preserving isometry classes of such trees equipped with the pointed Gromov-Hausdorff metric δ (see (17) Section 3.2 for the definition); we prove in Proposition 3.4 that $(\mathbb{T},\delta)$ is a Polish space. This is a simple generalization of the compact case proved in [14]. Then, we see the growing process of trees $(\mathcal{F}_\lambda; \lambda \geq 0)$ as a collection of locally compact rooted real trees $(\mathcal{F}_\lambda, d_\lambda, \rho)$, $\lambda \geq 0$, such that for any $0 \leq \mu \leq \lambda$

$$\mathcal{F}_\mu \subset \mathcal{F}_\lambda \quad \text{and} \quad d_{\lambda | \mathcal{F}_\mu \times \mathcal{F}_\mu} = d_\mu$$

(here ρ stands for the common root of the trees). Then Theorem 5.1 asserts that a.s.

$$\delta\left(\mathcal{F}_\lambda, \mathcal{F}\right) \underset{\lambda \to \infty}{\longrightarrow} 0 ,$$

where $\mathcal{F}$ is the completion of $\bigcup \mathcal{F}_\lambda$. The limiting random tree is called the (ψ, a)-*Lévy forest*.

This result is related to the work of Pitman and Winkel [32] who perform Bernoulli leaf colouring in the special case of *binary* Galton-Watson forests. They show, that in this case, the forest growth process has independent “increments”, expressed by a composition rule. It can be consistently extended to increase to the Brownian forest. This passage to the limit is understood by convergence of coding height processes via a Donsker type theorem. In the critical or subcritical case $m \geq 0$, it is also clear (see Remark 5.4 for a detailed explanation), that the distribution of the root preserving isometry class of $\mathcal{F}$ is the same as the distribution induced by the corresponding forest coded by the height process introduced by Le Gall and Le Jan [24] (see also [11]). Let us mention that a framework of real trees and the Gromov-Hausdorff metric has been developed for probabilistic applications by Evans in [13], Evans, Pitman, Winter in [14] and Evans, Winter in [15].

In Section 5.3, we define the ψ-excursion measure Θ that can be seen as the “distribution” of a single ψ-Lévy tree. More precisely, Proposition 5.5 asserts that the isometry classes of the connected components of $\mathcal{F} \backslash \{\rho\}$ form a Poisson point process on $\mathbb{T}$ with intensity $a\,\Theta$.

Our definition of the Lévy forest also allows to construct the mass measure on $\mathcal{F}$ denoted by $\mathbf{m}$ in the following way: let us denote by $\mathbf{m}_\lambda$ the empirical distribution of the leaves of $\mathcal{F}_\lambda$; then Theorem 5.2 asserts that $\mathbf{m}_\lambda / \lambda$ a.s. converges to $\mathbf{m}$ for the vague topology of the Radon measures on $\mathcal{F}$. It also asserts that the topological support of the mass measure is $\mathcal{F}$ and that the isometry class of the tree spanned by the root ρ and the points of a point Poisson process on $\mathcal{F}$ with intensity $\lambda \mathbf{m}$ has the same distribution as the isometry class of $\mathcal{F}_\lambda$.

In the last section, Theorem 5.6 provides a decomposition of $\mathcal{F}$ along $\mathcal{F}_\lambda$. In the supercritical case $m < 0$, it is easy to see that if $\mathcal{F}$ is infinite, then the infinite subtree of $\mathcal{F}$ is simply the tree $\mathcal{F}_0$ and the latter decomposition provides a decomposition of the Lévy forest along its infinite component which is distributed as a GW$(\xi_0, \psi'(\gamma), a)$-forest. This generalizes a decomposition known for Galton-Watson trees (see [27]).

This paper is organized as follows: in Section 2, we set notation concerning discrete trees and we discuss the Bernoulli leaf colouring of discrete Galton-Watson trees. In Section 3 we introduce real trees (Subsection 3.1) and we define the Gromov-Hausdorff topology on the isometry classes of locally compact rooted real trees (Subsection 3.2); in Subsections 5.2 and 5.4 for technical reasons we shall need to embed locally compact trees in the Banach space $\ell_1(\mathbb{N})$; the way to do that is explained in Subsection 3.4. In Section 4 we define the growth process of the Lévy forest: we first discuss in Subsection 4.1 the Bernoulli leaf colouring of Galton-Watson trees with exponential edge lengths and in particular we prove Theorem 4.2 that specifies the distribution of Bernoulli leaf colouring consistent families of Galton-Watson trees; Subsection 4.2 is devoted to the construction of the growth process; we briefly discuss the infinitesimal dynamics of the growth process and at the end of this subsection we also give a special probabilistic construction of the increments of the growing process that shall be used in the proofs of the next sections. Section 5 is devoted to the study of the Lévy forest: in Subsection 5.1 we prove the convergence result Theorem 5.1; in Subsection 5.2 we prove Theorem 5.2 that concerns the mass measure; Subsection 5.3 is devoted to the definition of the excursion measure; In Subsection 5.4 we discuss the decomposition of the Lévy forest along the ancestral tree of the points of a Poisson sample with intensity the mass measure.

2 Discrete trees

2.1 Basic definitions and notations.

Let us set

$$\mathbb{U} = \bigcup_{n=0}^{\infty} \mathbb{N}^{*n}$$

where $\mathbb{N}^* = \{1, 2, \ldots\}$ and by convention $\mathbb{N}^{*0} = \{\emptyset\}$. The concatenation of words in $\mathbb{U}$ is denoted $w = vu = (v_1, \ldots, v_m, u_1, \ldots, u_n)$ for $v = (v_1, \ldots, v_m), u = (u_1, \ldots, u_n) \in \mathbb{U}$. Following Neveu [28] we represent an ordered rooted tree as a subset $t \subset \mathbb{U}$ satisfying

- $\emptyset \in t$; $\emptyset$ is called the *ancestor* of t.
- $j \in \mathbb{N}, vj \in t \Rightarrow v \in t$; v is called the *parent* of vj.
- for any $v \in t$, there exists an integer $k_v(t)$ such that $vj \in t$, $1 \leq j \leq k_v(t)$. $k_v(t)$ is the *number of children* of v.

We denote by $\mathbb{T}_{\mathrm{discr}}$ the space of all discrete ordered rooted trees. On each $t \in \mathbb{T}_{\mathrm{discr}}$, we have the *genealogical order* given by

$$v \preceq w \quad \Longleftrightarrow \quad vu = w \quad \text{for some } u \in \mathbb{U}.$$

Any tree $t \in \mathbb{T}_{\mathrm{discr}}$ is also totally ordered by the *lexicographical order* on $\mathbb{U}$ denoted by $\leq$. Note that if t is infinite, then $(\mathbb{T}_{\mathrm{discr}}, \leq)$ cannot in general be embedded in $(\mathbb{N}, \leq)$ in an order-preserving way.

Let $u \in t$. We say that u is *a leaf of* t iff $k_u(t) = 0$. We denote by $\mathrm{Lf}(t)$ the set of leaves of t. Note that $\mathrm{Lf}(t)$ is possibly empty. We define the shifted tree t at u by

$$\theta_u t = \{v \in \mathbb{U} \ : \ uv \in t\}\,.$$

Then $\theta_u t = \emptyset$ iff $u \in \mathrm{Lf}(t)$. Let $v \in t$. We denote by $[\![u, v]\!]$ the shortest path between u and v and by $u \wedge v$ the last common ancestor (or branching point) of u and v. We set $]\!]u, v]\!] := [\![u, v]\!] \backslash \{u\}$ and we define similarly $[\![u, v[\![$ and $]\!]u, v[\![$.

We endow $\mathbb{T}_{\mathrm{discr}}$ with the σ-algebra $\mathcal{G}_{\mathrm{discr}}$ generated by the countable family of subsets $\{t \in \mathbb{T}_{\mathrm{discr}} : \ u \in t\}$, $u \in \mathbb{U}$. Unless otherwise specified, the random variables that we consider in this paper are defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ which is assumed to be large enough to carry as many independent random variables as we require. Let ξ be a probability distribution on $\mathbb{N}$. We call *Galton-Watson tree* with offspring distribution ξ (a GW(ξ)-tree for short) any $\mathcal{G}_{\mathrm{discr}}$-measurable random variable τ whose distribution is characterized by the two following conditions:

(i) $\mathbb{P}(k_\emptyset(\tau) = i) = \xi(i)$, $i \geq 0$.

(ii) For every $i \geq 1$ such that $\xi(i) \neq 0$, the shifted trees $\theta_1(\tau), \ldots, \theta_i(\tau)$ under $\mathbb{P}(\cdot \mid k_\emptyset(\tau) = i)$ are independent copies of τ under $\mathbb{P}$.

We shall sometimes consider finite sequences of discrete trees $f = (t_1, \ldots, t_n)$. We call them *forests of discrete trees* and we denote their set by $\mathbb{F}_{\mathrm{discr}}$. The elements of the forest are ordered by putting first the vertices of the first tree, next the vertices of the second tree etc. The genealogical order on a forest is defined tree by tree. A Galton-Watson forest with n elements is just a sequence of n i.i.d. GW-discrete trees.

2.2 Bernoulli leaf colouring of Galton-Watson trees.

In this section we discuss Bernoulli colouring of the leaves of a GW-tree and we compute the distribution of the whole tree conditionally on the genealogy of the leaves remaining after the colouring. More precisely, let $p \in [0, 1]$ and let τ be a GW(ξ)-tree. We assume that τ has a.s. leaves which is obviously equivalent to the condition

$$\xi(0) > 0, \tag{5}$$

and it will be convenient to assume $\xi(1) = 0$.

We colour independently at random each leaf of τ in red with probability p and in black with probability $1 - p$. If there is at least one black leaf, we also colour in black the subtree spanned by the root and the black leaves, namely the ancestral tree of the black leaves; then, we colour in red the remaining vertices. If there is no black leaf, we colour all the tree in red.

Assume that τ is not completely red. Then, the black subtree is isomorphic to a random tree in $\mathbb{T}_{\mathrm{discr}}$ denoted by τ_{sub} and also called the *black subtree*. The *black tree* (which is distinct from the black subtree) is obtained as follows: define a graph with set of vertices V and set of edges E given by

$$V = \{\emptyset\} \cap \{u \in \tau_{\mathrm{sub}} \; : \; k_u(\tau_{\mathrm{sub}}) \neq 1\}$$

and

$$E = \{\{u, v\}\, ;\, u, v \in \tau_{\mathrm{sub}} \; : \; u \neq v \text{ and } V \cap]\!]u, v[\![= \varnothing\} \, .$$

Here $]\!]u, v[\![$ is the shortest path between u and v in τ_{sub}. Put on V the order inherited from τ_{sub}. Then (V, E), with the distinguished vertex $\emptyset$, is an ordered rooted tree isomorphic to a unique element τ_{b} in $\mathbb{T}_{\mathrm{discr}}$ that is taken as the definition of the *black tree* (see Figure 1).

The main goal of this section is to give the joint distribution of τ_{b}, τ_{sub} and τ in terms of ξ and p. More precisely, let us define the colour of each vertex $u \in \tau$ as the mark $c_u \in \{0, 1\}$: $c_u = 1$ if u is black and $c_u = 0$ if it is red. The two-colours tree is the $\{0, 1\}$-marked tree $\tilde{\tau} = (\tau; c_u, u \in \tau)$ distributed as follows:

- Conditionally on τ, the random variables are $\{c_u, u \in \mathrm{Lf}(\tau)\}$ i.i.d. Bernoulli random variables with expectation $1 - p$.
- For any $v \in \tau$, we set $c_v = 1$ if there is $u \in \mathrm{Lf}(\tau)$ such that $v \preceq u$ and $c_u = 1$; set $c_v = 0$ otherwise.

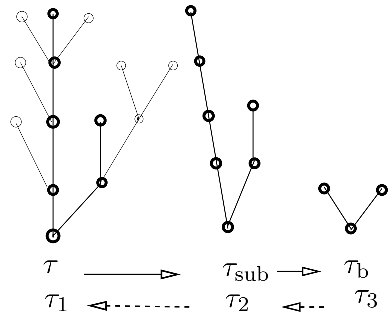

Figure 1: The black vertices are represented by the thick circles and the red ones by the thin circles. The dashed arrows represent the reconstruction procedure.

Let $u \in \tau$. We denote by $k_u^r(\tilde{\tau})$ the number of red children of u and by $k_u^b(\tilde{\tau})$ the number of black ones. Let $l \geq 0$ and $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_l) \in \{0,1\}^l$. Denote by l_r the number of 0 in ε and by l_b the number of 1. Let $f_1, \cdots, f_l$ be l nonnegative measurable functions on the set of two-coloured discrete trees equipped with the smallest σ-field making the marks measurable. Then it is easy to show that

$$\mathbb{E}\left[\prod_{i=1}^{l} f_i(\theta_i\tilde{\tau}) \; ; \; k_\emptyset(\tau) = l \; ; \; (c_1, \cdots, c_l) = \varepsilon\right] = \xi(l) g(p)^{l_r} (1-g(p))^{l_b} \prod_{i=1}^{l} \mathbb{E}\left[f_i(\tilde{\tau}) \,|c_\emptyset = \varepsilon_i\right], \quad (6)$$

where $g(p) = \mathbb{E}[p^{\#L(\tau)}]$. Here $\theta_j\tilde{\tau}$ stands for the marked tree shifted at the j-th children of the ancestor:

$$\theta_j\tilde{\tau} = (\theta_j\tau; c_{ju} \,,\, u \in \theta_j\tau).$$

Let us denote by φ the generating function of ξ:

$$\varphi(s) = \sum_{k \geq 0} \xi(k) s^k \,, \quad s \in [0,1].$$

By splitting τ at the root, we prove that g satisfies

$$g(s) = \varphi(g(s)) - \xi(0)(1-s) \,, \quad s \in [0,1]. \quad (7)$$

Formula (6) implies that $\tilde{\tau}$ is a two-types Galton-Watson tree whose branching mechanism is described as follows:

(a) The tree $\tilde{\tau}$ is completely red iff $c_\emptyset = 0$ which happens with probability $g(p)$. The tree conditioned to have no black vertices is a (completely red) GW(ξ_r)-tree where ξ_r is given by

$$\xi_r(l) = \begin{cases} \xi(l) g(p)^{l-1} & \text{if } l \geq 1 \,; \\ \xi(0) p / g(p) & \text{if } l = 0. \end{cases}$$

Then, the generating function φ_r of ξ_r is given by

$$\varphi_r(s) = 1 - \frac{\varphi(g(p)) - \varphi(g(p)s)}{g(p)} \,, \; s \in [0,1]. \quad (8)$$

(b) Conditionally on having at least one black leaf, the two-types offspring distribution is given by

$$\mathbb{P}\left((k^r_\emptyset(\tilde{\tau}), k^b_\emptyset(\tilde{\tau})) = (l_r, l_b) | c_\emptyset = 1\right) = \begin{cases} \xi(l) g(p)^{l_r} (1-g(p))^{l_b - 1} \dfrac{(l_b + l_r)!}{l_b! l_r!} & \text{if } l_b \geq 1, l_r \geq 0; \\ \xi(0) \dfrac{1-p}{1-g(p)} & \text{if } l_b = l_r = 0. \end{cases}$$

(c) Conditionally on $\{k^r_\emptyset(\tilde{\tau}) = l_r; k^b_\emptyset(\tilde{\tau}) = l_b\}$, $(c_1, \cdots, c_{l_r + l_b})$ is uniformly distributed among the $(l_b + l_r)!/l_r! l_b!$ possibilities.

(d) Conditionally on $\{(c_1, \cdots, c_{l_r + l_b}) = \varepsilon\}$, $\varepsilon \in \{0,1\}^l$, the marked trees $\theta_1 \tilde{\tau}, \cdots, \theta_l \tilde{\tau}$ are independent and $\theta_i \tilde{\tau}$ has the same distribution as $\tilde{\tau}$ under $\mathbb{P}(\,\cdot\,| c_\emptyset = \varepsilon_i)$.

Before giving the joint law of $\tau_{\rm b}$, $\tau_{\rm sub}$ and τ, we need to introduce some notation: we first define the "black" offspring distribution ξ_b by

$$\xi_b(l) = \begin{cases} \dfrac{\varphi^{(l)}(g(p))(1-g(p))^{l-1}}{1-\varphi'(g(p))} & \text{if } l \geq 2\,; \\ 0 & \text{if } l = 1; \\ \dfrac{\xi(0)(1-p)}{(1-g(p))(1-\varphi'(g(p)))} & \text{if } l = 0. \end{cases}$$

Then, its generating function is given by

$$\varphi_b(s) = s + \frac{\varphi(g(p) + s(1-g(p))) - g(p) - s(1-g(p))}{(1-g(p))(1-\varphi'(g(p)))}\,, \quad s \in [0,1]. \tag{9}$$

We also introduce for any $l \geq 1$ the following probability distribution on $\mathbb{N}$:

$$\nu_l(k) = \xi(l+k) \frac{(l+k)! g(p)^k}{k! \varphi^{(l)}(g(p))}\,, \quad k \geq 0. \tag{10}$$

The joint law of the black tree and the red forest is given by the following:

Reconstruction procedure for discrete GW-trees.

- **Step 1**: Let τ_1 be a GW(ξ_b)-tree. For any $u \in \tau_1$ distinct from the root, insert a line-tree with a random number N_u of edges at the end of the edge between u and its parent, and graft directly on the root a line-tree with a random number $N_\emptyset$ of edges. The N_u, $u \in \tau_1$ are distributed as follows: conditionally on τ_1, they are i.i.d. random variables with a geometric distribution given by

$$\mathbb{P}(N_u = k | u \in \tau_1) = (1-\varphi'(g(p)))\varphi'(g(p))^k\,, \quad k \geq 0. \tag{11}$$

  The resulting random element in $\mathbb{T}_{\rm discr}$ is denoted by τ_2 and has the same distribution as the black subtree.

- **Step 2**: Independently, on each vertex $u \in \tau_2$ such that $k_u(\tau_2) = l > 0$ graft a random number with distribution $\nu_l(\cdot)$ of red vertices. Insert these new red vertices uniformly at random among the l black ones. Then, graft independently on each newly added red vertex an independent GW(ξ_r)-tree. We obtain a two-colours tree denoted by $\widetilde{\tau}_3$.

We get the following identity:

$$(\tau_b, \tau_{\text{sub}}, \widetilde{\tau}) \quad \text{under} \quad \mathbb{P}(\,\cdot\,|c_\emptyset = 1) \overset{\text{(dist)}}{=} (\tau_1, \tau_2, \widetilde{\tau}_3)\,. \tag{12}$$

This identity is a consequence of an elementary computation based on (a), (b), (c) and (d), and it is left to the reader. Note that (12) implies in particular that the black tree τ_b is distributed as a GW(ξ_b)-tree.

Denote by N the number of red trees grafted on the black subtree of $\widetilde{\tau}$ if $\widetilde{\tau}$ is not completely red and set $N = 1$ if $\widetilde{\tau}$ is completely red. Denote by κ the generating function of N: $\kappa(s) := \mathbb{E}[s^N]$. By splitting $\widetilde{\tau}$ at the root and by an elementary computation based on (6), we show that $\kappa(s)$ satisfies the following equation

$$\varphi(\kappa(s)) - \kappa(s) = \varphi(sg(p)) - sg(p) - (\varphi(g(p)) - g(p))\,. \tag{13}$$

We shall use this identity in Section 5.1.

3 The space of locally compact rooted real trees.

3.1 Real trees.

Real trees form a class of loop-free length spaces, which turn out to be the class of limiting objects of many combinatorial and discrete trees, extending the class of trees with edge lengths. More precisely we say that a metric space (T, d, ρ) is *a rooted real tree* if it satisfies the following conditions:

- For all $s, t \in T$, there is a unique isometry $f_{s,t} : [0, d(s,t)] \to T$ such that $f_{s,t}(0) = s$ and $f_{s,t}(d(s,t)) = t$;
- If q is a continuous injective map from $[0, 1]$ into T, we have
$$q([0,1]) = f_{q(0),q(1)}([0, d(q(0), q(1))])$$
- $\rho \in T$ is a distinguished point, called the *root*.

Let us introduce some notation: we denote by $[\![s,t]\!]$ the trace of $f_{s,t}$: $[\![s,t]\!] := f_{s,t}([0, d(s,t)])$. We also denote by $]\!]s,t]\!]$, $[\![s,t[\![$ and $]\!]s,t[\![$ the respective images of $(0, d(s,t)]$, $[0, d(s,t))$ and $(0, d(s,t))$ by $f_{s,t}$. There is a nice characterization of real-trees that we use in the next subsection which is called the *four points condition*: let (X, d) be a complete path-connected metric space; then it is a real tree iff

$$d(s_1,s_2) + d(s_3,s_4) \le (d(s_1,s_3) + d(s_2,s_4)) \vee (d(s_3,s_2) + d(s_1,s_4)). \tag{14}$$

We refer to [8, 9, 10] for general results concerning real trees, [29, 30] for applications of real trees to group theory and to [13, 14, 15],[11] and also [19] for a probabilistic use of real trees.

In this paper we restrict our attention to *locally compact real trees*. By the Hopf-Rinow theorem (see for instance [18], Chapter 1) the closed balls are compact sets. For any $s \in T$ we denote by $\mathrm{n}(s,T)$ the *degree* of s, namely the (possibly infinite) number of connected components of $T \setminus \{s\}$. For convenience of notation, we often denote $\mathrm{n}(s,T)$ by $\mathrm{n}(s)$ when there is no risk of confusion. We denote by

$$\mathrm{Lf}(T) = \{s \in T \setminus \{\rho\} : \ \mathrm{n}(s,T) = 1\} \quad \text{and} \quad \mathrm{Br}(T) = \{s \in T \setminus \{\rho\} : \ \mathrm{n}(s,T) \geq 3\}$$

respectively the set of the *leaves* of T and the set of *branching points* of T. We also denote by $\mathrm{Sk}(T)$ the *internal skeleton* of T that is defined by $\mathrm{Sk}(T) = T \backslash \mathrm{Lf}(T)$. We can easily prove that for any sequence $(s_n, n \geq 1)$ dense in T, we have

$$\mathrm{Sk}(T) = \bigcup_{n \geq 1} [\![\rho, s_n[\![\, . \tag{15}$$

Then, the closure of $\mathrm{Sk}(T)$ is T. Note that the trace on $\mathrm{Sk}(T)$ of the Borel σ-field is generated by the "intervals" $[\![s, s']\!]$, $s, s' \in \mathrm{Sk}(T)$. Thus we can define a unique positive Borel measure $\ell_T(ds)$ on T such that

$$\ell_T(L(T)) = 0 \quad \text{and} \quad \ell_T([\![s, s']\!]) = d(s, s').$$

The measure ℓ_T is usually called the *length measure* of T. We next prove the following simple lemma.

Lemma 3.1 *The set of branching points of a locally compact real tree is at most countable.*

Proof: Let (T, d) be a locally compact real tree. Assume that $\mathrm{Br}(T)$ is uncountably infinite. Since $\mathrm{Br}(T) \subset \mathrm{Sk}(T)$, by (15) there is a positive integer n such that the set $[\![\rho, s_n]\!] \cap \mathrm{Br}(T)$ is uncountable. Thus we can find an injective map j from $\mathbb{R}$ into $[\![\rho, s_n]\!] \cap \mathrm{Br}(T)$. Then with any $x \in \mathbb{R}$, we can associate a connected component C_x of $T \setminus \{\mathrm{j}(x)\}$ such that $[\![\rho, s_n]\!] \cap C_x = \varnothing$ since $\mathrm{j}(x)$ is a branching point. A simple argument implies that $C_x \cap C_y = \varnothing$ for any $x \neq y$ and T cannot be separable, which contradicts the fact that it is locally compact. ■

In the present paper we define step by step a growing family of trees by recursively grafting independent random trees on nodes and branches of the tree of the previous step. Let us explain in the deterministic setting one step of this grafting procedure: let (T, d, ρ) be a locally compact real rooted tree, let (T_i, d_i, ρ_i), $i \in I$, be a family of locally compact real trees and let $(s_i, i \in I)$ be a collection of vertices of T. We specify T' as disjoint union

$$T' = T \coprod_{i \in I} T_i \setminus \{\rho_i\}$$

and we define a distance d' on $T' \times T'$ as follows: d' coincides with d on $T \times T$ and if $s \in T_i \backslash \{\rho_i\}$ and $s' \in T'$, then we set

$$d'(s, s') = \begin{cases} d_i(s, \rho_i) + d(s_i, s') & \text{if } s' \in T; \\ d_i(s, \rho_i) + d(s_i, s_j) + d_j(s', \rho_j) & \text{if } s' \in T_j \setminus \{\rho_j\}, \ i \neq j; \\ d_i(s, s') & \text{if } s' \in T_i \setminus \{\rho_i\}. \end{cases}$$

It is easy to prove that $(T', d', \rho' = \rho)$ is a real tree and we use the notation

$$(T, d', \rho') = T \circledast_{i \in I} (s_i, T_i)$$

to mean that (T', d', ρ') is obtained from (T, d, ρ) by this "grafting" procedure.

3.2 Gromov-Hausdorff convergence of pointed metric spaces.

The purpose of this section is to introduce a nice topology on the set $\mathbb{T}$ of *isometry classes* of locally compact rooted real trees: more precisely, we say that two pointed metric spaces (X_1, d_1, ρ_1) and (X_2, d_2, ρ_2) are equivalent iff there exists an isometry f from X_1 onto X_2 such that $f(\rho_1) = \rho_2$. Evans, Pitman and Winter [14] showed that the set $\mathbb{T}_{\rm cpct}$ of isometry classes of *compact* rooted real trees equipped with the Gromov-Hausdorff distance whose definition is recalled below, is a complete and separable metric space. Here we define a metric on $\mathbb{T}$ and prove a similar result in the the locally compact case. For sake of clarity, we actually prove this result for locally compact length spaces, the real tree case being a simple consequence of the four points conditions (14) that characterizes real trees.

Let us first recall the definition of the Gromov-Hausdorff distance of two pointed compact metric spaces (X_1, d_1, ρ_1) and (X_2, d_2, ρ_2): we set

$$\delta_{\rm cpct}(X_1, X_2) = \inf \{ d_{\rm Haus}(f_1(X_1), f_2(X_2)) \vee d(f_1(\rho_1), f_2(\rho_2)) \}$$

where the infimum is taken over all isometric embeddings $f_i : X_i \to E$, $i = 1, 2$ into a common metric space (E, d). Here $d_{\rm Haus}$ stands for the Hausdorff distance on the set of compact sets of E. Observe that $\delta_{\rm cpct}$ only depends on the isometry classes of the X_i's and we can show that it defines a metric on the set of isometry classes of all pointed compact metric spaces (see [18]).

There is a useful way to control $\delta_{\rm cpct}(X_1, X_2)$ via ε-isometries. Namely, we say that a (possibly not continuous) map $f : X_1 \to X_2$ is *a pointed ε-isometry* if

(i) $f(\rho_1) = \rho_2$

(ii) $\mathrm{dis}(f) := \sup\{ |d_1(x, y) - d_2(f(x), f(y))| \, ; \; x, y \in X_1 \} \; < \; \varepsilon$;

(iii) $f(X_1)$ is an ε-net of X_2.

The quantity $\mathrm{dis}(f)$ is called the *distortion* of f. The following lemma is a straightforward consequence of the non-pointed case stated in Corollary 7.3.28 in [6].

Lemma 3.2 *Let (X_1, d_1, ρ_1) and (X_2, d_2, ρ_2) be two pointed compact metric spaces. Then,*

(a) *If $\delta_{\rm cpct}(X_1, X_2) < \varepsilon$, then there exists a 4ε-isometry from X_1 to X_2.*

(b) *If there exists a ε-isometry from X_1 to X_2, then $\delta_{\rm cpct}(X_1, X_2) < 4\varepsilon$.*

Let us now recall from [6], Chapter 8, a way to extend the Gromov-Hausdorff convergence to non-compact metric spaces. Let (X, d) be a metric space. Let $r \geq 0$ and $\rho \in X$. We denote by $B_X(\rho, r)$ the closed ball centered at ρ with radius r. Let (X_n, d_n, ρ_n), $n \geq 1$, be a sequence of pointed metric spaces; we say that this sequence converges in the pointed Gromov-Hausdorff sense to the pointed metric space (X, d, ρ) if for any $r, \varepsilon > 0$ there exists $n_0 = n_0(r, \epsilon) \geq 1$ such that for every $n \geq n_0$, there is a map $f_n : B_{X_n}(\rho_n, r) \to X$ satisfying the following conditions:

(i') $f_n(\rho_n) = \rho$;

(ii') $\mathrm{dis}(f_n) < \varepsilon$;

(iii') The ε-neighbourhood of $f_n(B_{X_n}(\rho_n, r))$ contains $B_X(\rho, r-\varepsilon)$.

We use the following notation:

$$(X_n, d_n, \rho_n) \xrightarrow[n\to\infty]{\text{G-H}} (X, d, \rho). \tag{16}$$

Let us briefly recall from [6] useful properties of pointed Gromov-Hausdorff convergence. Assume that (16) holds. Then,

(a) (8.1.8 and 8.1.9 [6]) If the X_n's are locally compact length spaces and if X is complete, then X is a locally compact length space.

(b) (8.1.2 [6]) If the X_n's are compact and if X is compact, then

$$\delta_{\text{cpct}}(X_n, X) \underset{n\to\infty}{\longrightarrow} 0.$$

(c) (8.1.3 [6]) If X is a length space, then for any $r > 0$

$$\delta_{\text{cpct}}(B_{X_n}(\rho, r), B_X(\rho, r)) \underset{n\to\infty}{\longrightarrow} 0.$$

(d) (8.1.9 [6]) (*Pre-compactness*): Let $\mathcal{C}$ be a set of pointed metric spaces. Assume that for any $r, \varepsilon > 0$, there exists $N(r, \varepsilon)$ such that for every $(X, d, \rho) \in \mathcal{C}$ the closed ball $B_X(\rho, r)$ admits an ε-net with at most $N(r, \varepsilon)$ points. Then, any sequence of elements of $\mathcal{C}$ contains a converging subsequence in the pointed Gromov-Hausdorff sense.

For locally compact length spaces, the pointed Gromov-Hausdorff convergence is compatible with the following metric: let (X_1, d_1, ρ_1) and (X_2, d_2, ρ_2) be two pointed locally compact length spaces; under our assumptions $(B_{X_i}(\rho_i, r), d_i, \rho_i)$ is a pointed compact space so it makes sense to define

$$\delta(X_1, X_2) = \sum_{k\geq 1} 2^{-k} \delta_{\text{cpct}}(B_{X_1}(\rho_1, k), B_{X_2}(\rho_2, k)). \tag{17}$$

Clearly, δ only depends on the isometry classes of X_1 and X_2. Let us denote by $\mathbb{X}$ the set of isometry classes of pointed locally compact length spaces and by $\mathbb{X}_{\text{cpct}}$ the set of pointed compact length spaces.

Proposition 3.3 *Let (X_n, d_n, ρ_n), $n \geq 1$ and (X, d, ρ) be representatives of elements in $\mathbb{X}$. Then*

$$(X_n, d_n, \rho_n) \xrightarrow[n\to\infty]{\text{G-H}} (X, d, \rho) \quad \Longleftrightarrow \quad \lim_{n\to\infty} \delta(X_n, X) = 0.$$

Moreover $(\mathbb{X}, \delta)$ is complete and separable.

Proof: The fact that the δ-convergence implies the pointed Gromov-Hausdorff convergence is easy to deduce from properties (b) and Lemma 3.2. The converse is a consequence of (c).

Next, we prove that δ is a metric on $\mathbb{X}$. Since δ_{cpct} satisfies the triangle inequality, so does δ. Let (X_1, d_1, ρ_1) and (X_2, d_2, ρ_2) be two pointed locally compact length spaces such that $\delta(X_1, X_2) = 0$. Then, for every $k \geq 1$ there exists an isometry f_k from $B_{X_1}(\rho_1, k)$ onto $B_{X_2}(\rho_2, k)$ with $f_k(\rho_1) = \rho_2$. Let $(x_n, n \geq 1)$ be a dense sequence in X_1. By the Cantor diagonal procedure we can find an increasing sequence of indices $(k_i, i \geq 1)$ such that for any $n \geq 1$, $(f_{k_i}(x_n), i \geq 1)$ converges in X_2. Set $f(x_n) = \lim_{i\to\infty} f_{k_i}(x_n)$: it defines an isometric embedding of $(x_n, n \geq 1)$ into X_2 such that $f(\rho_1) = \rho_2$ which can be easily extended to an isometry f from X_1 into X_2.

It remains to prove that $f(X_1) = X_2$. By exchanging the roles of X_1 and X_2, we get an isometric embedding g from X_2 into X_1 such that $g(\rho_2) = \rho_1$. Then, for any $k \geq 1$, $f \circ g$ is an isometric map from the compact set $B_{X_2}(\rho_2, k)$ into itself. Thus, it is a bijective map and we get $f(B_{X_1}(\rho_1, k)) = B_{X_2}(\rho_2, k)$ for any $k \geq 1$ which easily proves that f is actually onto X_2.

It remains to prove that $\mathbb{X}$ equipped with the metric δ is complete and separable. Since the set of isometry classes of compact metric spaces equipped with δ_{cpct} is separable, so is $(\mathbb{X}, \delta)$ for $\mathbb{X}_{\mathrm{cpct}}$ is dense in $(\mathbb{X}, \delta)$ by definition of δ.

We have to show that $(\mathbb{X}, \delta)$ is complete. Let (X_n, d_n, ρ_n), $n \geq 1$, be a Cauchy sequence of representatives of elements of $\mathbb{X}$. To prove that this sequence converges, we only have to prove that it forms a δ-precompact set. Fix $r, \varepsilon > 0$; choose $k > r + 1$ and $n_0 \geq 1$ such that for any $n, m \geq n_0$, $\delta(X_n, X_m) < 2^{-k}\varepsilon/12$. It implies

$$\delta_{\mathrm{cpct}}(B_{X_n}(\rho_n, k), B_{X_{n_0}}(\rho_{n_0}, k)) < \varepsilon/12 , \quad n \geq n_0. \tag{18}$$

By Lemma 3.2, there exists a pointed $\varepsilon/3$-isometry

$$f_n : B_{X_{n_0}}(\rho_{n_0}, k) \longrightarrow B_{X_n}(\rho_n, k).$$

Let $\{x_1, \cdots, x_N\}$ be a $\varepsilon/3$-net of $B_{X_{n_0}}(\rho_{n_0}, k)$. Then, for any $n \geq n_0$, the set

$$\{f_n(x_1), \ \cdots f_n(x_N)\}$$

is an ε-net of $B_{X_n}(\rho_n, k)$ and thus, of $B_{X_n}(\rho_n, r)$. So we can find $N(r, \varepsilon) = N$ such that for any $n \geq 1$ the closed ball $B_{X_n}(\rho_n, r)$ admits an ε-net with at most $N(r, \varepsilon)$ points. The compactness criterion (d) completes the proof. ■

Recall that $\mathbb{T}$ denotes the set of isometry classes of locally compact rooted real trees. Since the four points condition is obviously a closed condition for δ, it implies that $\mathbb{T}$ is a closed subset of $\mathbb{X}$ and we deduce from Proposition 3.3 the following result.

Proposition 3.4 $(\mathbb{T}, \delta)$ *is a complete and separable metric space.*

Following the proof of Lemma 2.7 in [14], we prove the following lemma that we shall use in the next section.

Lemma 3.5 *Let* $((T_n, d_n, \rho))_{n\geq 1}$ *be a Cauchy sequence of representatives of elements of* $(\mathbb{T}, \delta)$ *such that* $T_n \subset T_{n+1}$ *and* $d_{n+1}|_{T_n\times T_n} = d_n$, $n \geq 1$. *Set for any* $a, b \in T_n$, $n \geq 1$:

$$d(a, b) = d_n(a, b), \quad a, b \in T_n, n \geq 1.$$

This defines a metric on $T_\infty := \bigcup_{n\geq 1} T_n$. *Furthermore, all metric completions of* (T_∞, d, ρ) *are isometric and form the limit in* $(\mathbb{T}, \delta)$ *of the sequence* $((T_n, d_n, \rho))_{n\geq 1}$.

3.3 Galton-Watson real trees with exponential edge lengths.

Let us consider a discrete tree with positive marks. Namely let $t \in \mathbb{T}_{\mathrm{discr}}$ and let $m = (m_u; u \in t)$ be a collection of marks in $[0, \infty]$. We assume that if $m_u = \infty$ then u has no child: $k_u(t) = 0$. Such a pair (t, m) is called a *marked tree* and the set of marked trees is denoted by $\mathbb{T}_{\mathrm{mark}}$. We denote by $\mathcal{G}_{\mathrm{mark}}$ the σ-algebra generated by the events $\{(t, m) : u \in t \, , m_u > a\}$, $u \in \mathbb{U}$ and $a \in \mathbb{R}$. Thinking of the marks as distances between the nodes of t, we can associate with (t, m) a real tree denoted by $T(t, m) = (T, d, \rho)$ as follows: set $\rho = (\emptyset, 0)$ and

$$T = \{\rho\} \bigcup_{u \in t \, : \, m_u < \infty} \{(u, s) \, , s \in (0, m_u]\} \bigcup_{u \in t \, : \, m_u = \infty} \{(u, s) \, , s \in (0, \infty)\}$$

and we define d as follows : let $\sigma = (u, s) \in T \setminus \{\rho\}$, then we set

$$d(\rho, \sigma) = s + \sum_{v \in [\![\emptyset, u [\![} m_v$$

where we recall notation $[\![\emptyset, u [\![= [\![\emptyset, u]\!] \setminus \{u\}$. Let $\sigma' = (u', s') \in T \setminus \{\rho\}$. We define

$$d(\sigma, \sigma') = \begin{cases} d(\rho, \sigma) + d(\rho, \sigma') - 2 \displaystyle\sum_{v \in [\![\emptyset, u \wedge u']\!]} m_v & \text{if} \quad u \wedge u' \notin \{u, u'\} \\ |d(\rho, \sigma) - d(\rho, \sigma')| & \text{otherwise.} \end{cases}$$

It is easy to check that $T(t, m) = (T, d, \rho)$ is a real tree. Instead of a single tree, consider now a *marked forest* (f, m) that is a finite sequence $(f, m) = ((t_i, m_i); 1 \leq i \leq n)$ of marked trees; the set of marked forests is denoted by $\mathbb{F}_{\mathrm{mark}}$. With a marked forest (f, m) we associate the real tree $T(f, m)$ defined by

$$T(t, m) = \{\rho\} \circledast_{1 \leq i \leq n} (\rho, T(t_i, m(i)))$$

which obtained by pasting at ρ the trees $T(t_i, m(i))$. We also denote by $\overline{T}(f, m)$ the equivalence class of $T(f, m)$ up to root preserving isometries. Note that $T(f, m)$ may fail to be locally compact. For instance if $T(f, m)$ is locally compact if for any infinite line of descent: $u_0 \preceq \ldots \preceq u_n \preceq \ldots$, we have

$$\sum_{n \geq 0} m_{u_n} = +\infty. \tag{19}$$

If (19) is satisfied then the real tree $T(f, m)$ that is obtained from f and m, is called a *a discrete tree with edge lengths*, namely a rooted real tree (T, d, ρ) such that

$$\forall r > 0 \, , \quad \# B_T(\rho, r) \cap \mathrm{Br}(T) \; < \; \infty \quad \text{and} \quad n(\sigma, T) < \infty \, , \quad \sigma \in T. \tag{20}$$

Conversely, with each discrete tree with edge lengths (T, d, ρ) we can associate a discrete forest $f \in \mathbb{F}_{discr}$ and a set of marks $m = (m_u, u \in f)$ such that $(T, d, \rho) = T(f, m)$. One way to proceed is the following: we call an *edge* of T the connected components of $T \setminus (\mathrm{Br}(T) \cup \{\rho\})$; each edge is isometric to an interval of the real line (that possibly has one infinite end); by convention, the *left end* of an edge is the closest end to the root; observe that T is the closure of the union of its edges by (20); fix an order on each group of edges sharing the same left end and then label the edges of T by words written with integer in the following recursive way:

- Each of the $\mathrm{n}(\rho, T)$ edges of T having the root ρ as a left end are labelled by the empty word $\varnothing$.
- Take a finite edge whose right end is denoted by $y \in T$. Assume that this edge is labelled by $u \in \mathbb{U}$ and consider the edges whose left end is y: the j-th edge with respect to the fixed order is then labelled by the word uj.

In this way we construct a discrete forest f. Consider the edge labelled by the word $u \in \mathbb{U}$. There are two cases: if the edge is infinite, then set $m_u = \infty$; if the edge is finite, then set $m_u = d(\rho, y) - d(\rho, x)$, where x and y stand for its resp. left and right ends. We clearly have $T(f, m) = (T, d, \rho)$. Note that such a marked forest (f, m) is by no way unique. However, it is uniquely determined if we assume first that f is *proper* that is $k_u(f) \neq 1$, $u \in f$, and then if we specify some order on the edges of T sharing the same left end.

Let ξ be an offspring distribution and let c be a positive real number. Let τ be a GW(ξ)-tree and conditionally on τ, let $m = (m_u, u \in \tau)$ be i.i.d. exponentially distributed random variables with parameter c. The random real tree $T(\tau, m) = (\mathcal{T}, d, \rho)$ is called a *Galton-Watson real tree* with parameters (ξ, c) (a GW(ξ, c)-real tree for short). Define for any $t \geq 0$, $Z_t(\mathcal{T}) = \#\{v \in \mathcal{T} : d(0, v) = t\}$. Then, we can show that $(Z_t(\mathcal{T}), t \geq 0)$ is a continuous-time Markov branching process. Moreover, if we denote by φ the generating function of ξ, then

$$\mathbb{E}\left[\exp(-\theta Z_t(\mathcal{T}))\right] = \exp(-v(t, \theta)),$$

where $v(t, \theta)$ is the unique non-negative solution of the integral equation

$$\int_{e^{-\theta}}^{e^{-v(t,\theta)}} \frac{dr}{\varphi(r) - r} = ct \tag{21}$$

(see Chapter III, Section 3, p. 106 [4]). $\mathcal{T}$ is a discrete tree with edge lengths (namely, $\mathcal{T}$ satisfies (19)) iff $Z_t(\mathcal{T})$ is a.s. finite for all $t \geq 0$, which is equivalent to the following analytical condition

$$\int^{1-} \frac{dr}{|\varphi(r) - r|} = \infty. \tag{22}$$

Unless otherwise specified, we assume that *all the GW-real trees that we consider in this paper satisfy* (22).

Define the height of $\mathcal{T}$ by $h(\mathcal{T}) := \sup\{d(\rho, \sigma)\ ,\ \sigma \in \mathcal{T}\} \in [0, \infty]$. Then observe that $\mathbb{P}(h(\mathcal{T}) \leq t) = \exp(-v(t))$, where for any $t \geq 0$ we set $v(t) = \lim_{\theta \to \infty} v(t, \theta)$. It satisfies

$$\int_0^{e^{-v(t)}} \frac{dr}{\varphi(r) - r} = ct. \tag{23}$$

We end this subsection by precisely defining the class of random discrete trees that we shall consider: more specifically, let $(\tau_i; i \geq 1)$ be an i.i.d. sequence of GW(ξ)-trees and conditionally on the τ_i's, let $(m_u(i), u \in \tau_i, i \geq 1)$ be independent exponentially distributed random variables with parameter c. Fix a positive real number $a > 0$ and denote by N a Poisson random variable with expectation a that is assumed to be independent of the $m(i)$'s and of the τ_i's. Set $(f, m) = (\tau_i, m(i); 1 \leq i \leq N)$. The real tree $T(f, m) = (\mathcal{F}, d, \rho)$ is called a *Galton-Watson real forest* with parameters (ξ, c, a) (a GW(ξ, c, a)-real forest for short).

3.4 Isometrical embeddings of real trees in $\ell_1(\mathbb{N})$.

For technical reasons we shall sometimes need to consider specific representatives of real trees rather that isometry classes. Following Aldous's idea (see [3]), we may choose to embed locally compact rooted trees in the vector space $\ell_1(\mathbb{N})$ of the summable real-valued sequences equipped with the $||\cdot||_1$-norm. Namely,

$$\ell_1(\mathbb{N}) = \left\{ x = (x_n)_{n\geq 0} \in \mathbb{R}^{\mathbb{N}} \; : \quad ||x||_1 := \sum_{n\geq 0} |x_n| < \infty \right\} .$$

We introduce the space $\mathbb{T}_{\ell_1}$ of the subsets $T \subset \ell_1(\mathbb{N})$ such that $(T, ||\cdot||_1, 0)$ is a locally compact rooted real tree. Let us denote by d_{Haus} the Hausdorff distance on compact subsets of $\ell_1(\mathbb{N})$. Then, for any $T, T' \in \mathbb{T}_{\ell_1}$, define

$$\mathbf{d}(T, T') = \sum_{k\geq 0} 2^{-k} d_{\text{Haus}}\left(B_T(0,k), B_{T'}(0,k)\right) .$$

Note that

$$\delta(T, T') \leq \mathbf{d}(T, T'). \tag{24}$$

Proposition 3.6 $(\mathbb{T}_{\ell_1}, \mathbf{d})$ *is a Polish space.*

Proof: It is easily proved that $(\mathbb{T}_{\ell_1}, \mathbf{d})$ is a separable metric space. Let us prove it is complete. Let $(T_n, n \geq 0)$ be a Cauchy sequence of elements of $(\mathbb{T}_{\ell_1}, \mathbf{d})$. Then, for any $k \geq 1$, $B_{T_n}(0,k), n \geq 0$ is a d_{Haus}-Cauchy sequence of closed subsets of $\ell_1(\mathbb{N})$. Thus, by a well-known property of Hausdorff distances, for any $k \geq 0$ there exists a closed set $C_k \subset \ell_1(\mathbb{N})$ such that

$$\lim_{n\to\infty} d_{\text{Haus}}\left(B_{T_n}(0,k), C_k\right) = 0 ,$$

which implies $\lim_{n\to\infty} \delta\left(B_{T_n}(0,k), C_k\right)$ by (24). By Theorem 3.4, $(C_k, ||\cdot||_1, 0)$ has to be a rooted compact real tree. Moreover, for any $k' \geq k$ we have $C_k \subset C_{k'}$ and Property (c) in Section 3.2 implies that

$$B_{C_{k'}}(0,k) = C_k .$$

Now set $T = \bigcup_{k\geq 0} C_k$. The previous observations easily implies that $(T, ||\cdot||_1, 0)$ is a locally compact rooted real tree and that $\lim_{n\to\infty} \mathbf{d}\left(T_n, T\right) = 0$, which completes the proof. ■

Let us now briefly explain how to isometrically embed a discrete tree with edge lengths (T, d, ρ) in $\ell_1(\mathbb{N})$. Recall from Section 3.3 that we can find a discrete forest $f \in \mathbb{F}_{discr}$ and marks $m = (m_u, u \in f)$ such that $\overline{T}(f, m) = (T, d, \rho)$. Recall also the definition of an edge of T and recall that to each vertex $u \in f$ corresponds an edge in T. We now order the vertices of f as follows: order the roots of f and put them first; then order the vertices at height 1 and put them after the roots of f; order the vertices at height 2 and put them next ... etc. Recall from the previous section the definition of an edge of T. For any $k \geq 0$, denote by $I(k)$ the edge of T corresponding to the k-th vertex of f visited with respect to the linear order above defined and denote by x_k the left end of $I(k)$. Clearly x_k belongs to the closure of the set $\{\rho\} \bigcup_{j<k} I(j)$. Then, let us introduce for any $k \geq 0$ the sequence $e_k \in \ell_1(\mathbb{N})$ given by $e_k(n) = 0$ if $n \neq k$ and $e_k(k) = 1$. Let $P = (n_k, k \geq 0)$ be an $\mathbb{N}$-valued increasing sequence; we recursively define the map f_P from T to $\ell_1(\mathbb{N})$ as follows.

- $f_P(\rho) = 0$;
- For any $k \geq 1$, and any $\sigma \in I(k)$,

$$f_P(\sigma) = f_P(x_k) + d(x_k, \sigma) e_{n_k} \ .$$

It is easy to check that f_P is an isometry. Thus (T, d, ρ) and $(f_P(T), ||\cdot||_1, 0)$ are equivalent. Now we prove the following proposition.

Proposition 3.7 *Every element of* $\mathbb{T}$ *has a representative in* $\mathbb{T}_{\ell_{\mathbf{1}}}$.

Proof: We have to prove that any locally compact rooted real tree (T, d, ρ) can be embedded isometrically in $\ell_{\mathbf{1}}(\mathbb{N})$. It is possible to find a non-decreasing sequence of subsets K_n, $n \geq 0$, with no limit points and such that K_n is a 2^{-n}-net of T. We set

$$T_n = \bigcup_{\sigma \in K_n} [\![\rho, \sigma]\!] \quad \text{and} \quad T_\infty = \bigcup_{n \geq 0} T_n \ .$$

Clearly, the T_n's are discrete trees with edge lengths and the closure of T_∞ is T. We recursively define a map f from T_∞ to $\ell_{\mathbf{1}}(\mathbb{N})$ in the following way:

- Let P_0 and $P_{n,i}$, $n \geq 0, i \geq 0$ be disjoint subsets of $\mathbb{N}$. We consider f_{P_0}, the isometrical embedding of T_0 into $\ell_{\mathbf{1}}(\mathbb{N})$ as defined above and we require that f coincides with f_{P_0} on T_0.
- Assume that f is defined on T_n; Denote by $T^o_{n,i}$, $i \in I_n$, the connected components of $T_{n+1} \backslash T_n$. Denote by $\rho_{n,i}$ the closest point to the root of the closure $T_{n,i}$ of $T^o_{n,i}$. Then, $\rho_{n,i} \in T_n$ and the $(T_{n,i}, d, \rho_{n,i})$ are rooted discrete trees with edge lengths. We assume for convenience of notations that the sets of indices I_n are subsets of $\mathbb{N}$. Then, for any $\sigma \in T_{n,i}$, we set
$$f(\sigma) = f(\rho_{n,i}) + f_{P_{n,i}}(\sigma) \ ,$$
where $f_{P_{n,i}}$ stands for the above defined isometrical embedding of $T_{n,i}$ in $\ell_{\mathbf{1}}(\mathbb{N})$.

Thus, f is an isometrical embedding of T_∞ into $\ell_{\mathbf{1}}(\mathbb{N})$, which has a unique extension to the closure T of T_∞. This completes the proof of the proposition. ■

4 The growth process.

4.1 Bernoulli colouring of the leaves and extensibility of GW-real trees.

In this section we discuss the Bernoulli colouring of the leaves of GW-real trees and forests. In particular, we introduce the class of Lévy GW-real trees that is, roughly speaking, the class of GW-real trees consistent under Bernoulli colouring. More precisely, let T be a discrete tree with edge lengths, that is a rooted real tree satisfying (20). Let $p \in [0, 1]$. Then, colour independently each leaf of T in black with probability $1 - p$ and in red with probability p. Denote by A the set of the black leaves. If A is non-empty, then colour in black the following subtree:

$$T_{black} = \bigcup_{\sigma \in A} [\![\rho, \sigma]\!];$$

Then, colour in red the remaining part $T\backslash T_{black}$ of the tree. If A is empty, then colour in red the whole tree T and set $T_{black} = \{\rho\}$. As in the discrete case such a colouring is called a *p-Bernoulli leaf colouring of T*.

Remark 4.1 Observe that if T has leaves and if it is not reduced to a point, then the black subtree is reduced to the root iff T is completely red.

Let ξ be an offspring distribution on $\mathbb{N}$ such that $\xi(0) > 0$. Let us assume that ξ is proper, namely $\xi(1) = 0$. Fix two positive real numbers $a, c > 0$ and denote by $\mathcal{T}$ (resp. $\mathcal{F}$) a GW(ξ, c)-real tree (resp. a GW(ξ, c, a)-real forest). Let $p \in (0,1)$. Denote by $\mathcal{T}_{black}$ (resp. $\mathcal{F}_{black}$) the black subtree of $\mathcal{T}$ (resp. $\mathcal{F}$) resulting from a p-Bernoulli leaf colouring (here the extra random variables used for the Bernoulli colourings are chosen independent of $\mathcal{T}$ and of $\mathcal{F}$). As in the reconstruction procedure discussed in Section 2.2, we first compute the distribution of $\mathcal{T}$ (resp. $\mathcal{F}$) conditionally on $\mathcal{T}_{black}$ (resp. $\mathcal{F}_{black}$). To that end, recall the notation $g(p), \xi_b, \xi_r, \nu_l$ from Section 2.2. Let $\mathcal{T}'$ (resp. $\mathcal{F}'$) be a GW$(\xi_b, (1-\varphi'(g(p)))c)$-real tree (resp. a GW$(\xi_b, (1-\varphi'(g(p)))c, (1-g(p))a)$-real forest). Let $\mathcal{P} = \{\sigma_i; i \in I\}$ be a Poisson point process on $\mathcal{T}'$ (resp. on $\mathcal{F}'$) with intensity $\varphi'(g(p))c\,\ell_{\mathcal{T}'}$ (resp. $\varphi'(g(p))c\,\ell_{\mathcal{F}'}$).

Reconstruction procedure on GW-real trees.

- For $\mathcal{T}'$: on each vertex $\sigma \in \mathcal{P} \cup \mathrm{Br}(\mathcal{T}')$ graft independently a random number N_σ of independent GW(ξ_r, c)-real trees; conditionally on $\mathcal{P} \cup \mathrm{Br}(\mathcal{T}')$ the N_σ's are independent and the conditional distribution of N_σ is ν_l where $l = \mathrm{n}(\sigma, \mathcal{T}') - 1$. Denote by $\mathcal{T}''$ the resulting tree.

- For $\mathcal{F}'$: do the same thing as for $\mathcal{T}'$ and graft on the root N_ρ additional independent GW(ξ_r, c)-real trees, where N_ρ stands for an independent Poisson random variable with parameter $ag(p)$. Denote by $\mathcal{F}''$ the resulting tree.

Lemma 4.1 *Assume that (22) holds. Then,*

$$\left(\overline{\mathcal{T}}_{black}\,,\,\overline{\mathcal{T}}\right) \;\text{ under }\; \mathbb{P}(\,\cdot\,|\mathcal{T}_{black} \neq \{\rho\}) \quad \stackrel{(d)}{=} \quad \left(\overline{\mathcal{T}'}, \overline{\mathcal{T}''}\right)$$

and

$$\left(\overline{\mathcal{F}}_{black}\,,\,\overline{\mathcal{F}}\right) \quad \stackrel{(d)}{=} \quad \left(\overline{\mathcal{F}'}, \overline{\mathcal{F}''}\right).$$

Proof: Recall from Section 3.3 the definition of an edge of a discrete tree with edge lengths. Let us assume that $\mathcal{T} = T(\tau, m)$ where τ is a GW(ξ)-tree and where $m = (m_u, u \in \tau)$ is a collection of independent exponential random variables with parameter c. Similarly we can write

$$\overline{\mathcal{T}}_{black} = \overline{T}(\tau_{black}, m_{black}) \quad \text{and} \quad \overline{\mathcal{T}'} = \overline{T}(\tau', m').$$

Since the leaves of $\mathcal{T}$ are exactly the leaves of τ, τ_{black} is obtained from τ by a p-Bernoulli leaf colouring. Thus, by the result of Section 2.2, conditionally on $\{\mathcal{T}_{black} \neq \{\rho\}\}$, τ_{black} is distributed as τ', namely as a GW(ξ_b)-real tree. Moreover, the marks m' are independent exponential random variables with parameter $c(1-\varphi'(g(p)))$. We need the following elementary claim whose proof is left to the reader.

- *Claim.* Let M be an exponential random variable with parameter $\alpha > 0$; consider an independent Poisson process on $[0, \infty)$ with intensity $\beta > 0$, which splits the interval $[0, M]$ into N subintervals with lengths $L_1, \ldots, L_N$; then, N is a geometric random variable with parameter $\beta/(\alpha + \beta)$:

$$\mathbb{P}(N = k+1) = \frac{\alpha}{\beta + \alpha} \left(\frac{\beta}{\beta + \alpha} \right)^k .$$

Moreover, conditionally on N the L_i's are independent exponentially distributed random variables with parameter $\alpha + \beta$.

Now consider one edge $I \subset \mathcal{T}'$ that corresponds to a vertex $u \in \tau'$ as explained in Section 3.3. Condition on τ' and use the claim with

$$M = m_u \; , \; \alpha = c(1 - \varphi'(g(p))) \quad \text{and} \quad \beta = c\varphi'(g(p))$$

in order to show that the Poisson point process $\mathcal{P}$ splits I into N subintervals whose lengths are independent exponential variables with parameter c; Moreover N has a geometric distribution with parameter $\varphi'(g(p))$. Now observe that adding the points of $\mathcal{P}$ in $\mathcal{T}'$ corresponds to adding the line-trees to τ' as in Step 1 of the reconstruction procedure for discrete trees in Section 2.2. Then, note that we next graft on $\mathcal{T}'$ independent red GW(ξ_r, c)-real trees according to Step 2 of the reconstruction procedure for discrete trees. Thus, we can write $\overline{\mathcal{T}''} = \overline{T}(\tau'', m'')$ where τ'' is obtained by Steps 1 and 2 of the reconstruction procedure for discrete trees in Section 2.2 and where m'' is a collection of independent exponential variables with parameter c. This proves the first identity of the lemma. The second one is a simple consequence of the first one and its proof is left to the reader. ■

We now discuss the converse problem to determine the possible offspring distributions that appear as "black" distributions; more precisely, we say that a proper offspring distribution ξ_b is *p-extensible* if we can find a proper offspring distribution ξ such that ξ_b is the "black" distribution associated with a p-Bernoulli leaf colouring of a GW(ξ)-tree.

Theorem 4.2 *Let ξ_b be a proper offspring distribution on $\mathbb{N}$. Then, the two following assertions are equivalent*

(I) *ξ_b is p-extensible for all sufficiently large $p \in (0, 1)$.*

(II) *There exists ψ that is the branching mechanism of a CSBP (thus of the form (1)) such that*

$$\varphi_b(r) = r + \psi(1 - r) \; , \quad r \in [0, 1] \, ,$$

where φ_b stands for the generating function of ξ_b.

Proof: Let us first prove that (I) implies (II). With any $p \in (0, 1)$ we can associate the p-extension ξ of ξ_b (ξ depends on p but we skip it for convenience of notation). Recall (9) and set $v_p = g(p)/(1 - g(p))$ where g is defined by (7). Observe that (9) implies that φ_b is C^∞ on $(-v_p, 1)$ and continuous on $[-v_p, 1]$. Moreover

$$\forall v \in (-v_p, 1) \; , \; \forall n \geq 2 \; : \quad \varphi_b^{(n)}(v) \geq 0 \, . \tag{25}$$

We first prove the following equation

$$p = 1 - \frac{\varphi_b(0)}{v_p + \varphi_b(-v_p)}. \tag{26}$$

To that end, first note that

$$\varphi_b(0) = \xi_b(0) = (1+v_p)(1-\varphi_b'(-v_p))(\varphi(g(p)) - g(p)) \tag{27}$$

Then, observe that

$$\varphi(0) = \xi(0) = \frac{v_p + \varphi_b(-v_p)}{(1+v_p)(1-\varphi_b'(-v_p))}. \tag{28}$$

Deduce from (7) that

$$1 - p = \frac{\varphi(g(p)) - g(p)}{\xi(0)}$$

and use (27)and (28) to prove (26).

Let us now define $v_{max} \in (0, \infty]$ by

$$v_{max} = \sup\{v \geq 0 \ : \ \varphi_b^{(n)}(u) \geq 0 \ , \ u \in (-v, 1) \ , \ n \geq 2\}.$$

Suppose that $v_{max} < \infty$. First observe that by (26) we can find an increasing sequence $p_n \in (0,1) \to 1$ such that

$$\lim_{n\to\infty} \varphi_b(-v_{p_n}) = +\infty.$$

Since φ_b is convex on $(-v_{max}, 1]$, it implies that

$$\lim_{v \to v_{max}} \varphi_b(-v) = +\infty \quad \text{and} \quad \lim_{v\to v_{max}} \varphi_b'(-v) = -\infty.$$

But the second limit is impossible for φ_b' is a convex non-decreasing function on $(-v_{max}, 1)$. Thus, we must have $v_{max} = \infty$ and (25) implies that

$$\forall v \in (-\infty, 1) \ , \ \forall n \geq 2 \ : \quad \varphi_b^{(n)}(v) \geq 0 \, .$$

Set $\psi(u) = \varphi_b(1-u) - 1 + u$, $u \in [0, +\infty)$. The previous observation implies that ψ has the following properties

(a) $\psi(0) = 0$ and $\psi'(1) = 1$;

(b) ψ'' is completely monotone on $[0, +\infty)$.

Bernstein's Theorem and a standard integration argument adapted from the proof of Theorem 2, Chapter XIII.7 in [16] imply that ψ is of the form (1).

The fact that (II) implies (I) is an easy consequence of the following computation (which is left to reader). If $\varphi_b(r) = r + \psi(1-r)$, $r \in [0,1]$ and if $p \in (0,1)$, then the offspring distribution ξ, whose generating function φ is given by

$$\varphi(r) = r + \frac{\psi\left((1-r)\psi^{-1}\left(\psi(1)/1-p\right)\right)}{\psi^{-1}\left(\psi(1)/1-p\right)\psi'\left(\psi^{-1}\left(\psi(1)/1-p\right)\right)} \tag{29}$$

is a p-extension of ξ_b. ■

Remark 4.2 Observe that Theorem 4.2 is true for offspring distributions that do not satisfy (22). If φ_b is of the form given by Theorem 4.2 (II), then it is easy to prove that if even

$$\varphi_b'(1) = \sum_{k\geq 0} k\xi_b(k) < \infty ,$$

then, $\psi'(0+)$ is finite.

The main objects that we discuss in this paper are families of GW-real forests that are consistent under Bernoulli leaf colouring. More precisely, let $(\mathcal{F}_\lambda; \lambda \in [0,\infty))$ be a collection of random locally compact rooted trees such that for any $\lambda \geq 0$, $\mathcal{F}_\lambda$ is a GW$(\xi_\lambda, c_\lambda, a_\lambda)$-real forest, such that for any $\lambda > 0$, ξ_λ is a proper offspring distribution satisfying $\xi_\lambda(0) > 0$ and such that c_λ and a_λ are non-negative real numbers. We say that $(\mathcal{F}_\lambda; \lambda \in [0,\infty))$ is *Bernoulli leaf colouring consistent* if for any $0 \leq \mu \leq \lambda$, $\mathcal{F}_\mu \subset \mathcal{F}_\lambda$ and $\mathcal{F}_\mu$ is obtained from $\mathcal{F}_\lambda$ as the "black" tree resulting from a p-Bernoulli leaf colouring with $1-p = \mu/\lambda$. According to Lemma 4.1 it implies that ξ_λ is the $(1-\mu/\lambda)$-extension of ξ_μ. Therefore, ξ_μ is p-extensible for any sufficiently large p and ξ_μ has to be of the form given by Theorem 4.2 (II). Accordingly, up to a linear time change of the family $(\mathcal{F}_\lambda; \lambda \in [0,\infty))$, there is a unique function ψ satisfying (1) such that for any $\lambda \geq 0$:

$$\xi_\lambda(k) = \frac{\psi^{-1}(\lambda)^{k-1}\left|\psi^{(k)}(\psi^{-1}(\lambda))\right|}{k!\psi'(\psi^{-1}(\lambda))} \quad \text{if} \quad k \neq 1 \tag{30}$$

and $\xi_\lambda(1) = 0$. The generating function φ_λ of ξ_λ is then given by

$$\varphi_\lambda(s) = s + \frac{\psi\left((1-s)\psi^{-1}(\lambda)\right)}{\psi^{-1}(\lambda)\psi'(\psi^{-1}(\lambda))}. \tag{31}$$

Remark 4.3 Recall that γ is the largest root of ψ. Thus $\gamma > 0$ iff $m = \psi'(0+) < 0$. Observe that if $\gamma > 0$, then

$$\varphi_0(s) = s + \frac{\psi\left((1-s)\gamma\right)}{\gamma\psi'(\gamma)}$$

and $\xi_0(0) = 0$. A GW(ξ_0)-tree is infinite with no leaf.

By definition, the black distribution associated with ξ_λ via a $(1-\mu/\lambda)$-Bernoulli leaf colouring is ξ_μ. It is also easy to compute the function g that solve (7). Namely,

$$g(s) := 1 - \frac{\psi^{-1}\left((1-s)\lambda\right)}{\psi^{-1}(\lambda)} . \tag{32}$$

Thus, the probability for a GW(ξ_λ)-tree to be completely red is

$$g(p) = 1 - \frac{\psi^{-1}(\mu)}{\psi^{-1}(\lambda)}. \tag{33}$$

The red distribution associated with ξ_λ via a $(1-\mu/\lambda)$-Bernoulli leaf colouring is denoted by $\xi_{\mu,\lambda} := \xi_r$ and is given by

$$\xi_{\mu,\lambda}(k) = \frac{\left|\psi^{(k)}(\psi^{-1}(\lambda))\right|}{\psi'(\psi^{-1}(\lambda))} \frac{\left(\psi^{-1}(\lambda)-\psi^{-1}(\mu)\right)^{k-1}}{k!} , \quad k \geq 2, \tag{34}$$

$$\xi_{\mu,\lambda}(1) = 0 \quad \text{and} \quad \xi_{\mu,\lambda}(0) = \frac{\lambda-\mu}{(\psi^{-1}(\lambda)-\psi^{-1}(\mu))\,\psi'(\psi^{-1}(\lambda))}.$$

The generating function of $\xi_{\mu,\lambda}$ is denoted by $\varphi_{\mu,\lambda}$ and is given by

$$\varphi_{\mu,\lambda}(s) = s + \frac{\psi\left(\psi^{-1}(\lambda) - s(\psi^{-1}(\lambda)-\psi^{-1}(\mu))\right) - \mu}{(\psi^{-1}(\lambda)-\psi^{-1}(\mu))\psi'(\psi^{-1}(\lambda))}. \tag{35}$$

Remark 4.4 Observe that $\varphi'_{\mu,\lambda}(1) = 1 - \psi'(\psi^{-1}(\mu))/\psi'(\psi^{-1}(\lambda)) < 1$. Thus, for $\mu < \lambda$, $\xi_{\mu,\lambda}$ is a subcritical offspring distribution and therefore, any GW($\xi_{\mu,\lambda}$)-real tree is a.s. finite.

For any $l \geq 1$ we denote by $\nu_l^{\mu,\lambda}$ the distribution given by (10) with $\varphi = \varphi_\lambda$, $\xi = \xi_\lambda$ and $g(p)$ as in (33). For any $l \geq 2$, $\nu_l^{\mu,\lambda}$ is given by

$$\nu_l^{\mu,\lambda}(k) = \frac{\left|\psi^{(l+k)}(\psi^{-1}(\lambda))\right|}{\left|\psi^{(l)}(\psi^{-1}(\mu))\right|} \frac{\left(\psi^{-1}(\lambda)-\psi^{-1}(\mu)\right)^k}{k!}\,, \quad k \geq 0 \tag{36}$$

and for $l = 1$

$$\nu_1^{\mu,\lambda}(k) = \frac{\left|\psi^{(1+k)}(\psi^{-1}(\lambda))\right|}{\psi'(\psi^{-1}(\lambda))-\psi'(\psi^{-1}(\mu))} \frac{\left(\psi^{-1}(\lambda)-\psi^{-1}(\mu)\right)^k}{k!}\,, \quad k \geq 1, \tag{37}$$

with $\nu_1^{\mu,\lambda}(0) = 0$. Now observe that the parameter $\varphi'(g(p))$ of the geometric distribution in (11) is given by

$$\varphi'(g(p)) = 1 - \frac{\psi'(\psi^{-1}(\mu))}{\psi'(\psi^{-1}(\lambda))}\,. \tag{38}$$

Then, according to Lemma 4.1 we have

$$a_\mu/a_\lambda = \psi^{-1}(\mu)/\psi^{-1}(\lambda) \quad \text{and} \quad c_\mu/c_\lambda = \psi'(\psi^{-1}(\mu))/\psi'(\psi^{-1}(\lambda)).$$

We choose the following normalization:

$$a_\lambda = a\,\psi^{-1}(\lambda) \quad \text{and} \quad c_\lambda = \psi'(\psi^{-1}(\lambda)), \tag{39}$$

where $a > 0$. Such a Bernoulli leaf colouring consistent family $(\mathcal{F}_\lambda; \lambda \in [0,\infty))$ whose distribution is specified by (30) and (39) is called an (a,ψ)-*Lévy growth process*.

Remark 4.5 For any $\mu \geq 0$, we set

$$\psi_\mu(x) = \psi(x + \psi^{-1}(\mu)) - \mu.$$

Then, $\psi'_\mu(0+) = \psi'(\psi^{-1}(\mu))$. If $\mu > 0$, then $\psi'_\mu(0+)$ is finite. It is also easy to check that

$$\psi_\mu^{-1}(x) = \psi^{-1}(x+\mu) - \psi^{-1}(\mu);$$

Thus for any $\mu \leq \lambda$

$$\left(\psi_{\lambda-\mu}\right)_\mu = \psi_\lambda.$$

Note that ξ_λ and $\xi_{\mu,\lambda}$ actually depend on ψ: $\xi_{\lambda;\psi} = \xi_\lambda$, $\xi_{\mu,\lambda;\psi} = \xi_{\mu,\lambda}$. Then, it is easy to check that for any $\mu_0 \leq \mu \leq \lambda$:

$$\xi_{\mu,\lambda} = \xi_{\lambda-\mu;\psi_\mu} \quad \text{and} \quad \xi_{\mu,\lambda;\psi_{\mu_0}} = \xi_{\mu+\mu_0,\lambda+\mu_0}. \tag{40}$$

Notation 4.1 Fix $0 \le \mu \le \lambda$ and $a > 0$. We shall use the following notation. We denote by

- $\Delta_\lambda(d\overline{T})$ the distribution on $\mathbb{T}$ of the isometry class of a $\mathrm{GW}(\xi_\lambda, \psi'(\psi^{-1}(\lambda)))$-real tree,
- $\Delta^a_\lambda(d\overline{T})$ the distribution on $\mathbb{T}$ of the isometry class of a $\mathrm{GW}(\xi_\lambda, \psi'(\psi^{-1}(\lambda)), a\psi^{-1}(\lambda))$-real forest,
- $\Delta_{\mu,\lambda}(d\overline{T})$ the distribution on $\mathbb{T}$ of the isometry class of a $\mathrm{GW}(\xi_{\mu,\lambda}, \psi'(\psi^{-1}(\lambda)))$-real tree,
- $\Delta^a_{\mu,\lambda}(d\overline{T})$ the distribution on $\mathbb{T}$ of the isometry class of a $\mathrm{GW}(\xi_{\mu,\lambda}, \psi'(\psi^{-1}(\lambda)),$ $a(\psi^{-1}(\lambda) - \psi^{-1}(\mu))$-real forest.

According to the previous remark, we get

$$\Delta^a_{\mu,\lambda} = \Delta^a_{\lambda-\mu;\psi_\mu} \quad \text{and} \quad \Delta_{\mu,\lambda} = \Delta_{\lambda-\mu;\psi_\mu} \tag{41}$$

with an obvious notation. Observe also that $\Delta^0_\lambda = \delta_{\overline{\{\rho\}}}$ that is the Dirac mass at the isometry class $\overline{\{\rho\}}$ of the point tree. Thus $\Delta^0_\lambda \neq \Delta_\lambda$.

4.2 Construction of the growth process.

In this section we discuss how to grow a tree in order to obtain Bernoulli colouring consistent families of GW-real trees and related tree-valued processes. The definition given in this subsection is slightly more general for we want to start the growth process at any discrete real tree with edge lengths. Let (T, d, ρ) be such a tree. Fix $0 \le \mu \le \lambda$ and $a > 0$. Let ψ be of the form (1) such that $\psi'(0+)$ is finite. We first define a random tree denoted by $Q^a_{\mu,\lambda}(T)$ via the following grafting procedure:

- **The grafting procedure on** T: Let $\mathcal{P}$ be a Poisson point process on T with intensity

  $$\big(\psi'(\psi^{-1}(\lambda)) - \psi'(\psi^{-1}(\mu))\big)\ \ell_T\ .$$

  Graft a random number N_σ of independent $\mathrm{GW}(\xi_{\mu,\lambda}, \psi'(\psi^{-1}(\lambda)))$-real trees on each vertex $\sigma \in \mathcal{P} \cup \mathrm{Br}(T)$; here N_σ has distribution $\nu^{\mu,\lambda}_l$, where $l = \mathrm{n}(\sigma, T) - 1$. The resulting tree is denoted by $Q_{\mu,\lambda}(T)$. Then, graft on ρ a random number N_ρ of independent GW-real trees with the same distribution, where N_ρ is a Poisson random variable with parameter $a(\psi^{-1}(\lambda) - \psi^{-1}(\mu))$. Denote by $Q^a_{\mu,\lambda}(T)$ the resulting tree.

Remark 4.6 Observe that $Q^0_{\mu,\lambda}(T) = Q_{\mu,\lambda}(T)$ and note that if T reduces to its root ρ then $Q_{\mu,\lambda}(T) = \{\rho\}$.

Consider a $\mathrm{GW}(\xi_\lambda, \psi'(\psi^{-1}(\lambda)))$-real tree (resp. a $\mathrm{GW}(\xi_\lambda, \psi'(\psi^{-1}(\lambda)), a\psi^{-1}(\lambda))$-real forest) denoted by $\mathcal{T}(\lambda)$ (resp. by $\mathcal{F}(\lambda)$). Denote by $\mathcal{T}_\mu(\lambda)$ (resp. by $\mathcal{F}_\mu(\lambda)$) the black subtree obtained by a $(1-\mu/\lambda)$-Bernoulli leaf colouring of $\mathcal{T}(\lambda)$ (resp. of $\mathcal{F}(\lambda)$). Let $\mathcal{T}'$ (resp. $\mathcal{F}'$) be a $\mathrm{GW}(\xi_\mu, \psi'(\psi^{-1}(\mu)))$-real tree (resp. a $\mathrm{GW}(\xi_\mu, \psi'(\psi^{-1}(\mu)), a\psi^{-1}(\mu))$-real forest). The grafting procedure corresponds to the reconstruction procedure explained at the beginning of the previous section and Lemma 4.1 implies that

$$\big(\overline{\mathcal{T}}_\mu(\lambda)\ ,\ \overline{\mathcal{T}}(\lambda)\big) \ \text{under}\ \mathbb{P}\left(\ \cdot\ |\mathcal{T}_\mu(\lambda) \neq \{\rho\}\right) \quad \overset{(d)}{=} \quad \big(\overline{\mathcal{T}'}, \overline{Q}_{\mu,\lambda}(\mathcal{T}')\big) \tag{42}$$

and

$$\left(\overline{\mathcal{F}}_\mu(\lambda)\ ,\ \overline{\mathcal{F}}(\lambda)\right) \quad \stackrel{(d)}{=} \quad \left(\overline{\mathcal{F}'}, \overline{Q}^a_{\mu,\lambda}(\mathcal{F}')\right). \tag{43}$$

Here $\overline{Q}_{\mu,\lambda}(\mathcal{T}')$ and $\overline{Q}^a_{\mu,\lambda}(\mathcal{F}')$ stand for the isometry classes of resp. $Q_{\mu,\lambda}(\mathcal{T}')$ and $Q^a_{\mu,\lambda}(\mathcal{F}')$ (the extra random variables used to define the grafting procedures on $\mathcal{T}'$ and $\mathcal{F}'$ are chosen independent of these trees).

The grafting procedure enjoys a Markov-like property in the following sense: fix $a \geq 0$ and let $0 \leq \lambda_1 < \lambda_2 < \lambda_3$. Set $\mathcal{F} = Q^a_{\lambda_1,\lambda_3}(T)$. Let U_σ , $\sigma \in \mathrm{Lf}(\mathcal{F})\backslash\mathrm{Lf}(T)$ be $[0,1]$-uniform independent random variables conditionally on $\mathcal{F}$. We define

$$\mathcal{F}_b = T \bigcup \left\{ [\![\rho, \sigma]\!]\ ;\ \sigma \in \mathrm{Lf}(\mathcal{F})\backslash\mathrm{Lf}(T)\ :\ U_\sigma \leq \frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1} \right\} .$$

$\mathcal{F}_b$ is thus the black tree resulting from a $(1-(\lambda_2-\lambda_1)/(\lambda_3-\lambda_1))$-Bernoulli colouring of the leaves of $\mathcal{F}$ that are not in T.

Proposition 4.3 *For any discrete tree with edge lengths T, any $a \geq 0$ and any $0 \leq \lambda_1 < \lambda_2 < \lambda_3$, we have*

$$\left(\overline{Q}^a_{\lambda_1,\lambda_2}(T)\ ,\ \overline{Q}^a_{\lambda_2,\lambda_3}\left(Q^a_{\lambda_1,\lambda_2}(T)\right)\right) \quad \stackrel{(d)}{=} \quad \left(\overline{\mathcal{F}}_b\ ,\ \overline{\mathcal{F}}\right)$$

(here the extra random variables used to define $Q^a_{\lambda_2,\lambda_3}$ are chosen independent of $Q^a_{\lambda_1,\lambda_2}(T)$).

Proof: By Remark 4.5, we only have to prove

$$\left(\overline{Q}^a_{0,\mu}(T)\ ,\ \overline{Q}^a_{\mu,\lambda}\left(Q^a_{0,\mu}(T)\right)\right) \quad \stackrel{(d)}{=} \quad \left(\overline{\mathcal{F}}_b\ ,\ \overline{\mathcal{F}}\right) \tag{44}$$

by replacing ψ by ψ_{λ_1} and by taking $\mu = \lambda_2 - \lambda_1$ and $\lambda = \lambda_3 - \lambda_1$ in (44).

Let us denote by $\mathcal{P}$ the Poisson point process on T involved in the grafting procedure defining $\mathcal{F}$. For any $\sigma \in \mathcal{P} \cup \mathrm{Br}(T) \cup \{\rho\}$, we denote by $\mathcal{T}^i_\sigma$, $1 \leq i \leq N_\sigma$, the trees grafted on σ. Denote by $\mathcal{T}^i_b(\sigma)$ the tree $\mathcal{F}_b \cap \mathcal{T}^i_\sigma$ and set

$$J_r(\sigma) = \left\{ i \in \{1, \ldots, N_\sigma\}\ :\ \mathcal{T}^i_b(\sigma) = \{\sigma\} \right\} \quad \text{and} \quad J_b(\sigma) = \{1, \ldots, N_\sigma\} \backslash J_r(\sigma).$$

Then, observe that performing a $(1-\mu/\lambda)$-Bernoulli leaf colouring on $\mathrm{Lf}(\mathcal{F})\backslash\mathrm{Lf}(T)$ is the same as performing independent $(1-\mu/\lambda)$-Bernoulli leaf colourings on the $\mathcal{T}^i_\sigma$'s. Accordingly, conditionally on $J_r(\sigma)$ and on $J_b(\sigma)$ the pairs of trees $(\mathcal{T}^i_\sigma, \mathcal{T}^i_b(\sigma))$, $i \in J_b(\sigma)$, and the trees $\mathcal{T}^i_\sigma$, $i \in J_r(\sigma)$, are independent; moreover, by (42), conditionally on $J_r(\sigma)$ and on $J_b(\sigma)$ the isometry classes of $(\mathcal{T}^i_\sigma, \mathcal{T}^i_b(\sigma))$, $i \in J_b(\sigma)$, are independent copies of $(\overline{Q}_{\mu,\lambda}(\mathcal{T}), \overline{\mathcal{T}})$ where $\mathcal{T}$ is a $\mathrm{GW}(\xi_\mu, \psi'(\psi^{-1}(\mu)))$-real tree. To simplify notation we assume that for any $\sigma \in \mathcal{P} \cup \mathrm{Br}(T) \cup \{\rho\}$ and any $i \in J_b(\sigma)$

$$Q_{\mu,\lambda}(\mathcal{T}^i_b(\sigma)) = \mathcal{T}^i_\sigma. \tag{45}$$

Now recall that conditionally on N_σ, the events that $\mathcal{T}^i_\sigma$ is completely red, $i \in \{1, \ldots, N_\sigma\}$, are independent events and have probability $1 - \psi^{-1}(\mu)/\psi^{-1}(\lambda)$. Since N_σ has distribution $\nu^{0,\lambda}_l$, with $l = \mathrm{n}(\sigma, T) - 1$, we get

$$\mathbb{P}\left(\# J_b(\sigma) = k_b ; \# J_r(\sigma) = k_r | \mathcal{P}\right) =$$

$$\frac{(k_b+k_r)!}{k_b!k_r!}\left(1-\frac{\psi^{-1}(\mu)}{\psi^{-1}(\lambda)}\right)^{k_r}\left(\frac{\psi^{-1}(\mu)}{\psi^{-1}(\lambda)}\right)^{k_b}\nu_l^{0,\lambda}(k_b+k_r). \quad (46)$$

Now set $\mathcal{P}_1=\{\sigma\in\mathcal{P}\ :\ \#J_b(\sigma)\geq 1\}$ and $\mathcal{P}_2=\mathcal{P}\backslash\mathcal{P}_1$. It is easy to deduce from the latter observations that $\mathcal{P}_1$ and $\mathcal{P}_2$ are independent Poisson point processes with respective intensities

$$\psi^{-1}(\mu)\,\ell_T \quad \text{and} \quad (\psi^{-1}(\lambda)-\psi^{-1}(\mu))\,\ell_T\ .$$

A long but straightforward computation based on (46) (which is left to the reader) implies that conditionally on $\mathcal{P}_1$ and $\mathcal{P}_2$ the following assertions are true:

(a) If $\sigma\in\mathcal{P}_1$, then $\#J_b(\sigma)$ has distribution $\nu_1^{0,\mu}$ and conditionally on $\#J_b(\sigma)=j$, $\#J_r(\sigma)$ has distribution $\nu_{j+1}^{\mu,\lambda}$;

(b) If $\sigma\in\mathcal{P}_2$, then $\#J_r(\sigma)$ has distribution $\nu_1^{\mu,\lambda}$;

(c) If $\sigma\in\mathrm{Br}(T)$, then $\#J_b(\sigma)$ has distribution $\nu_l^{0,\mu}$, where $l=\mathrm{n}(\sigma,T)-1$. Moreover, conditionally on $\#J_b(\sigma)=j$, $\#J_r(\sigma)$ has distribution $\nu_{j+l}^{\mu,\lambda}$;

(d) $\#J_b(\rho)$ and $\#J_r(\rho)$ are independent Poisson random variables with respective parameters $\psi^{-1}(\mu)a$ and $(\psi^{-1}(\lambda)-\psi^{-1}(\mu))a$.

Next, observe that

$$\mathcal{F}_b=T\circledast_{\substack{\sigma\in\mathcal{P}_1\cup\mathrm{Br}(T)\cup\{\rho\}\\ i\in J_b(\sigma)}}\left(\sigma,\mathcal{T}_b^i(\sigma)\right). \quad (47)$$

According to the distribution of $\mathcal{P}_1$ and of $J_b(\sigma)$, $\sigma\in\mathcal{P}_1\cup\mathrm{Br}(T)\cup\{\rho\}$, (47) implies that $\mathcal{F}_b$ is obtained from T by the grafting procedure corresponding to the "grafting operator" $Q^a_{0,\mu}$ and, more precisely, that $\overline{\mathcal{F}_b}$ has the same distribution as the isometry class of $Q^a_{0,\mu}(T)$. To simplify notation we assume that

$$\mathcal{F}_b=Q^a_{0,\mu}(T). \quad (48)$$

We now graft trees on $\mathcal{F}_b$ according to the "grafting operator" $Q^a_{\mu,\lambda}$. Observe that this procedure can be split in the three following steps:

(i) Graft trees according the "grafting operator" $Q_{\mu,\lambda}$ independently on each $\mathcal{T}_b^i(\sigma)$, $i\in J_b(\sigma)$, $\sigma\in\mathcal{P}_1\cup\mathrm{Br}(T)\cup\{\rho\}$. Note that by (45) the resulting trees have the same distribution as the $\mathcal{T}_\sigma^i$'s .

(ii) Choose additional grafting points on T according to a Poisson point process with the same distribution as $\mathcal{P}_2$. We denote this set of points by $\mathcal{P}_2'$.

(iii) Graft a random number of independent $\mathrm{GW}(\xi_{\mu,\lambda},\psi'(\psi^{-1}(\lambda)))$-real trees at each $\sigma\in\mathcal{P}_1\cup\mathcal{P}_2'\cup\mathrm{Br}(T)\cap\{\rho\}$, the random number of trees grafted on σ having distribution $\nu_l^{\mu,\lambda}$, with

$$l=\mathrm{n}(\sigma,\mathcal{F}_b)-1=\mathrm{n}(\sigma,T)-1+\#J_b(\sigma).$$

If $\#J_b(\sigma)=j$, then by the grafting procedure, $\#J_r(\sigma)$ is distributed $\nu_{l+j}^{\mu,\lambda}$, and the resulting trees have the same distributions as $\mathcal{T}_\sigma^i$, $i\in J_b(\sigma)$, $\sigma\in\mathcal{P}_1\cup\mathcal{P}_2\cup\mathrm{Br}(T)\cup\{\rho\}$.

This implies that the isometry class of $Q^a_{\mu,\lambda}(\mathcal{F}_b)$ has the same distribution as $\overline{\mathcal{F}}$ and it completes the proof of (44) by (47). ■

Fix $\lambda > 0$, set $\mathcal{F}(\lambda) = Q^a_{0,\lambda}(T)$, and for all $\mu \in [0,\lambda]$ set

$$\mathcal{F}_\mu(\lambda) = T \bigcup \{[\![\rho,\sigma]\!] \; ; \; \sigma \in \mathrm{Lf}(\mathcal{F}(\lambda))\backslash \mathrm{Lf}(T) \; : \; U_\sigma \leq \mu/\lambda\} \,,$$

where the U_σ's are i.i.d. $[0,1]$-uniform variables conditionally on $\mathcal{F}(\lambda)$. The following proposition discusses how to construct an (a,ψ)-*growth process* starting from a discrete tree with edge lengths (T,d,ρ).

Proposition 4.4 *Assume that $m = \psi'(0+)$ is finite. Then, there exists a family of random rooted locally compact real trees $(\mathcal{F}_\lambda, d_\lambda, \rho)$, $\lambda \in [0,\infty)$ such that a.s.*

(i) *For any $0 \leq \mu \leq \lambda$*

$$\mathcal{F}_\mu \subset \mathcal{F}_\lambda \quad \text{and} \quad d_\mu = d_{\lambda \,|\mathcal{F}_\mu \times \mathcal{F}_\mu} \,.$$

(ii) *The map $\lambda \longrightarrow \overline{\mathcal{F}}_\lambda$ is cadlag in $(\mathbb{T},\delta)$ and*

$$\left(\overline{\mathcal{F}}_\mu \; , \; 0 \leq \mu \leq \lambda\right) \overset{(d)}{=} \left(\overline{\mathcal{F}}_\mu(\lambda) \; , \; 0 \leq \mu \leq \lambda\right) \; .$$

Proof: Let $(\lambda_n; n \geq 0)$ be an increasing sequence that goes to ∞ and such that $\lambda_0 = 0$. Set $\mathcal{F}_0 = T$ and define the sequence $(\mathcal{F}_{\lambda_n}; n \geq 1)$ by $\mathcal{F}_{\lambda_{n+1}} = Q^a_{\lambda_n,\lambda_{n+1}}(\mathcal{F}_{\lambda_n}), n \geq 0$, where the extra random variables used in the grafting procedure at step n are chosen to be independent of $\mathcal{F}_{\lambda_n}$. Associate a random variable V_σ with any $\sigma \in \cup \mathrm{Lf}(\mathcal{F}_{\lambda_n})\backslash \mathrm{Lf}(T)$ such that conditionally on the sequence $(\mathcal{F}_{\lambda_n}; n \geq 1)$, the V_σ's are i.i.d. uniformly distributed in $[0,1]$. Then, for any $\lambda \in [\lambda_n, \lambda_{n+1})$ we define the growth process as follows:

$$\mathcal{F}_\lambda = \mathcal{F}_{\lambda_n} \cup \bigcup \left\{ [\![\rho,\sigma]\!] \; , \; \sigma \in \mathrm{Lf}(\mathcal{F}_{\lambda_{n+1}})\backslash \mathrm{Lf}(\mathcal{F}_{\lambda_n}) \; : \; V_\sigma \leq \frac{\lambda - \lambda_n}{\lambda_{n+1} - \lambda_n} \right\}$$

and

$$d_\lambda = d_{\lambda_{n+1} \,|\mathcal{F}_\lambda \times \mathcal{F}_\lambda} \,,$$

Thus, point (i) clearly holds and it implies that $\lambda \longrightarrow \overline{\mathcal{F}}_\lambda$ is cadlag in $(\mathbb{T},\delta)$. Fix $n \geq 0$ and take $\lambda = \lambda_{n+1}$. Then use Proposition 4.3 successively with $\mu = \lambda_0, \ldots, \lambda_n$ to prove that the joint distribution of $\overline{\mathcal{F}}(\lambda_{n+1})$ and

$$\left(\overline{\mathcal{F}}_{\lambda_k}(\lambda_{n+1}) \, ; \; U_\sigma \, , \; \sigma \in \mathrm{Lf}(\mathcal{F}_{\lambda_{k+1}}(\lambda_{n+1}))\backslash \mathrm{Lf}(\mathcal{F}_{\lambda_k}(\lambda_{n+1})) \, , \; 0 \leq k \leq n\right)$$

is the same as the joint distribution of $\overline{\mathcal{F}}_{\lambda_{n+1}}$ and

$$\left(\left(\overline{\mathcal{F}}_{\lambda_k} \, ; \; \frac{\lambda_k + (\lambda_{k+1} - \lambda_k)V_\sigma}{\lambda_{n+1}} \, , \; \sigma \in \mathrm{Lf}(\mathcal{F}_{\lambda_{k+1}})\backslash \mathrm{Lf}(\mathcal{F}_{\lambda_k})\right) \; , \; 0 \leq k \leq n\right) .$$

Thus, for any $n \geq 0$:

$$\left(\overline{\mathcal{F}}_\mu \; , \; 0 \leq \mu \leq \lambda_{n+1}\right) \overset{(d)}{=} \left(\overline{\mathcal{F}}_\mu(\lambda) \; , \; 0 \leq \mu \leq \lambda_{n+1}\right) \; ,$$

which implies the second part of the proposition by an easy argument. ■

Remark 4.7 Following the construction given in the proof of Proposition 3.7, we can embed the growth process in $\ell^1(\mathbb{N})$ and we obtain a non-decreasing cadlag process in $(\mathbb{T}_{\ell_1}, \mathbf{d})$.

Remark 4.8 Observe that the distribution of $\overline{Q}^a_{\mu,\lambda}(T)$ only depends on the isometry class of (T, d, ρ) so it makes sense to denote by $\mathbf{P}_{\mu,\lambda}(\overline{T}, d\overline{T})$ the distribution on $\mathbb{T}$ of $\overline{Q}^a_{\mu,\lambda}(T)$. Proposition 4.3 and (43) imply that the isometry classes $(\overline{\mathcal{F}}_\lambda; \lambda \geq 0)$ of a (a,ψ)-Lévy growth process as defined in the end of Section 4.1 is a $\mathbb{T}$-valued inhomogeneous Markov process with transition kernel $\mathbf{P}_{\mu,\lambda}(\overline{T}, d\overline{T})$ (in the Brownian case $\psi(\lambda) = \lambda^2/2$, Pitman and Winkel in [32] proved that this process has independent growth increments expressed by a composition rule). Observe, however, that $\overline{Q}^a_{\mu,\lambda}(T)$ is only defined for discrete trees with edge lengths.

More specifically, it is clear from the construction that the growth process $(\mathcal{F}_\lambda)_{\lambda\geq 0}$ is a pure jump process obtained by adding single branches. More precisely, we get the following jump-chain with holding times construction of the process of $(\mathcal{F}_\lambda)_{\lambda\geq 0}$ started at a *compact* discrete tree with edge lengths (T, d, ρ). The equivalence classes of $(\mathcal{F}_\lambda)_{\lambda\geq 0}$ have the same distribution as the equivalence classes of the non-decreasing family of real trees $(\widetilde{\mathcal{F}}_\lambda)_{\lambda\geq 0}$ that has a discrete set of jump times $(\Lambda_n)_{n\geq 1}$ at which branches of lengths $(L_n)_{n\geq 1}$ are added, at locations $(\Sigma_n)_{n\geq 1}$ and such that the process $(\Lambda_n, \Sigma_n, L_n, \widetilde{\mathcal{F}}_{\Lambda_n})_{n\geq 0}$ is a Markov chain with transition kernel

$$\begin{aligned}\mathbb{P}\Big(\Lambda_{n+1} \in d\lambda\,;\, \Sigma_{n+1} \in d\sigma\,;\, L_{n+1} = dy\,;\, \widetilde{\mathcal{F}}_{\Lambda_{n+1}} \in dT' \,\Big|\, \Lambda_n = \mu\,;\, \widetilde{\mathcal{F}}_{\Lambda_n} = T\Big) \\ = \psi'(\psi^{-1}(\lambda)) \exp\left(-\psi'(\psi^{-1}(\lambda))\, y - \int_\mu^\lambda ds < M_{s,T} >\right) \\ \times \quad d\lambda\; M_{\lambda,T}(d\sigma)\; dy\; \delta_{\{T * (\sigma, [\![0,y]\!])\}}(dT')\end{aligned}$$

where

$$\psi'(\psi^{-1}(\lambda))\; M_{\mu,T}(d\sigma) = \psi''(\psi^{-1}(\mu))\ell_T(d\sigma) \quad + \sum_{v\in Br(T)\setminus\{\rho\}} \frac{|\psi^{(\mathrm{n}(\sigma,T))}(\psi^{-1}(\mu))|}{|\psi^{(\mathrm{n}(\sigma,T)-1)}(\psi^{-1}(\mu))|}\delta_v(d\sigma) + a\delta_\rho(d\sigma)$$

and $< M_{s,T} >$ stands for the total mass of $M_{s,T}$. Since the result is not important in the sequel, we skip the proof that is an easy consequence of the grafting procedure.

For technical purposes, we end the subsection by providing an alternative definition of the grafting procedure that is less direct but that is used in the proofs of the results of the next section. For convenience of notation we set

$$q = \psi'(\psi^{-1}(\lambda)) \quad \text{and} \quad c = \psi^{-1}(\lambda) - \psi^{-1}(\mu). \tag{49}$$

Then, for any non-negative integer $l \geq 2$, we define the distribution $\eta_{\mu,l}(dx)$ on $[0,\infty)$ by

$$\eta_{\mu,l}(dx) = \frac{2\beta\, \mathbf{1}_{\{l=2\}}}{|\psi^{(2)}(\psi^{-1}(\mu))|}\delta_0(dx) + \frac{x^l e^{-x\psi^{-1}(\mu)}}{|\psi^{(l)}(\psi^{-1}(\mu))|}\Pi(dx).$$

It is easy to check that $\eta_{\mu,l}(dx)$ is a probability measure. Let

$$\mathcal{P}_1 = \{(\sigma_i^{(1)}, x_i)\;,\; i \in I^{(1)}\} \quad \text{and} \quad \mathcal{P}_2 = \{(\sigma_i^{(2)}, y_i)\;,\; i \in I^{(2)}\}$$

be two independent Poisson point processes on $\mathbb{T} \times [0, \infty)$ with respective intensities

$$\ell_T(d\sigma) \otimes xe^{-\psi^{-1}(\mu)x}\Pi(dx) \quad \text{and} \quad 2\beta\, \ell_T(d\sigma) \otimes dy.$$

We shall use the following notation: define for any $k \in \{1, 2\}$,

$$S_\mu^{(k)} = \left\{ \sigma_i^{(k)} \ ,\ i \in I^{(k)} \right\}$$

and set $S_\mu = S_\mu^{(1)} \cup S_\mu^{(2)} \cup \mathrm{Br}(T) \cup \{\rho\}$ and

$$S'_\mu = S_\mu^{(1)} \cup \mathrm{Br}(T) \cup \{\rho\} \,.$$

We then introduce the collection of random variables $\mathcal{A}_\mu = \{a_\sigma(\mu),\ \sigma \in S'_\mu\}$ that are distributed as follows:

- $a_{\sigma_i^{(1)}}(\mu) = x_i \ ,\ i \in I^{(1)}$;
- $(a_\sigma(\mu),\ \sigma \in \mathrm{Br}(T) \cup \{\rho\})$ is a set of independent real-valued random variables independent of $\mathcal{P}_1$ and $\mathcal{P}_2$. Moreover, $a_\rho(\mu) = a$ and for any $\sigma \in \mathrm{Br}(T) \setminus \{\rho\}$, $a_\sigma(\mu)$ is distributed according to $\eta_{\mu,l}(dx)$ where $l = \mathrm{n}(\sigma, T) - 1$.

We next define a collection of random trees $\{(F_\sigma(\lambda), d_{\sigma,\lambda}, \rho_{\sigma,\lambda})\ ,\ \sigma \in S_\mu\}$ independent conditionally on $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{A}_\mu$, and whose conditional distribution is given as follows.

- If $\sigma \in S'_\mu$, then $(F_\sigma(\lambda), d_{\sigma,\lambda}, \rho_{\sigma,\lambda})$ is distributed as a $\mathrm{GW}(\xi_{\mu,\lambda}, q, c\, a_\sigma)$-real forest *with the convention that* $F_\sigma(\lambda) = \{\rho_{\sigma,\lambda}\}$ *if* $a_\sigma = 0$.
- If $\sigma = \sigma_j^{(2)}$, $j \in I^{(2)}$, then $(F_\sigma(\lambda), d_{\sigma,\lambda}, \rho_{\sigma,\lambda})$ is a single $\mathrm{GW}(\xi_{\mu,\lambda}, q)$-real tree if $y_j \le \psi^{-1}(\lambda) - \psi^{-1}(\mu)$ and it is simply the point tree $\{\rho_{\sigma,\lambda}\}$ otherwise.

We set

$$S_{\mu,\lambda} = \{\sigma \in S_\mu \ :\ F_\sigma(\lambda) \neq \{\rho_{\sigma,\lambda}\}\}$$

and

$$T' = T \circledast_{\sigma \in S_{\mu,\lambda} \setminus \{\rho_{\sigma,\lambda}\}} (\sigma, F_\sigma(\lambda)) \quad \text{and} \quad F' = T' * (\rho, F_\rho(\lambda)).$$

The following lemma implies that

$$\overline{T}' \overset{(d)}{=} \overline{Q}_{\mu,\lambda}(T) \quad \text{and} \quad \overline{F}' \overset{(d)}{=} \overline{Q}^a_{\mu,\lambda}(T) \,. \tag{50}$$

Lemma 4.5 *Assume that $\psi'(0+)$ is finite.*

(I) *Let E be a connected component of $T \backslash (\mathrm{Br}(T) \cup \{\rho\})$ (an edge of T). Then, $E \cap S_{\mu,\lambda}$ is a Poisson point process with intensity*

$$\left(\psi'(\psi^{-1}(\lambda)) - \psi'(\psi^{-1}(\mu))\right) \mathbf{1}_E(\sigma) \ell_T(d\sigma).$$

(II) *Conditionally on $S_{\mu,\lambda}$, the random real forests $(F_\sigma(\lambda), d_{\sigma,\lambda}, \rho_{\sigma,\lambda})$, $\sigma \in S_{\mu,\lambda}$, are independent. Moreover, for any $\sigma \in S_{\mu,\lambda}\backslash\{\rho\}$, the forest $F_\sigma(\lambda)$ consists of a random number $N_\sigma(\lambda)$ of independent GW($\xi_{\mu,\lambda}, \psi'(\psi^{-1}(\lambda))$)-real rooted trees, whose conditional distribution is given by*

$$\mathbb{P}\left(N_\sigma(\lambda) = k \mid S_{\mu,\lambda}\right) = \nu_l^{\mu,\lambda}(k) \ , \ k \geq 1,$$

where $l = \mathrm{n}(\sigma, T) - 1$.

Proof: Set $S_\mu^{1,2} = S_\mu^1 \cup S_\mu^2$ and let M be the measure on $\mathbb{T}$ given by

$$M(d\overline{\mathcal{T}}) = 2\beta(\psi^{-1}(\lambda) - \psi^{-1}(\mu))\Delta_{\mu,\lambda}(d\overline{\mathcal{T}}) + \int_{(0,\infty)} \Pi(dx)\, x e^{-\psi^{-1}(\mu)x} \Delta^x_{\mu,\lambda}(d\overline{\mathcal{T}}).$$

An easy computation implies that

$$\begin{aligned} M(\overline{\mathcal{T}} \neq \{\rho\}) &= 2\beta(\psi^{-1}(\lambda) - \psi^{-1}(\mu)) + \int_{(0,\infty)} \Pi(dx) x e^{-\psi^{-1}(\mu)x}\left(1 - e^{-(\psi^{-1}(\lambda) - \psi^{-1}(\mu))x}\right) \\ &= \psi'(\psi^{-1}(\lambda)) - \psi'(\psi^{-1}(\mu)). \end{aligned}$$

If we set $\widetilde{M} = M(\,\cdot \mid \overline{\mathcal{T}} \neq \{\rho\})$, then standard results on Poisson point processes imply that

$$\left\{(\sigma, F_\sigma(\lambda)) \ , \ \sigma \in S_\mu^{1,2} \ : \ F_\sigma(\lambda) \neq \{\rho\}\right\}$$

is a Poisson point process with intensity

$$\left(\psi'(\psi^{-1}(\lambda)) - \psi'(\psi^{-1}(\mu))\right) \ell_T(d\sigma) \otimes \widetilde{M}(d\overline{\mathcal{T}}).$$

This implies the first point of the lemma.

Now, observe that if $\overline{\mathcal{T}}$ has distribution $\widetilde{M}$, then $\overline{\mathcal{T}}$ is obtained by pasting at the root N independent copies of GW(ξ_r, q)-real rooted trees, where $\xi_r = \xi_{\mu,\lambda}$, q is given by (49), and the distribution of N is given first by $\widetilde{M}(N = 0) = 0$ and for any $k \geq 1$ by

$$\begin{aligned} \left(\psi'(\psi^{-1}(\lambda)) - \psi'(\psi^{-1}(\mu))\right) \widetilde{M}(N = k) &= 2\beta\, c\, \mathbf{1}_{\{k=1\}} \ + \\ &\quad \int_{(0,\infty)} \Pi(dx) x e^{-\psi^{-1}(\mu)x} e^{-cx} (cx)^k / k! \\ &= (-1)^{k+1} \psi^{(k+1)}(\psi^{-1}(\lambda))\, c^k / k! \ . \end{aligned}$$

Accordingly, $\widetilde{M}(N = k) = \nu_1^{\mu,\lambda}(k)$, $k \geq 0$, which implies the second part of the lemma in the $\sigma \in S_\mu^{1,2} \cap S_{\mu,\lambda}$ case.

It remains to consider $\sigma \in \mathrm{Br}(T) \cup \{\rho\}$: in that case the forest $F_\sigma(\lambda)$ is composed of N_σ independent random GW(ξ_r, q)-real rooted trees, where N_σ is a mixture of Poisson random variables whose distribution is given for any $k \geq 0$ by :

$$\begin{aligned} \mathbb{P}\left(N_\sigma = k \mid S_{\mu,\lambda}\right) &= \mathbb{E}\left[e^{-ca_\sigma} \frac{(ca_\sigma)^k}{k!}\right] \\ &= \frac{2\beta \mathbf{1}_{\{l=2\}}}{|\psi^{(2)}(\psi^{-1}(\mu))|} + \frac{1}{|\psi^{(l)}(\psi^{-1}(\mu))|} \int_{(0,\infty)} \Pi(dx) x^{k+l} c^k e^{-x(c+\psi^{-1}(\mu))} / k! \\ &= \nu_l^{\mu,\lambda}(k) \end{aligned}$$

(here again $l = \mathrm{n}(\sigma, T) - 1$). This completes the proof of the lemma. ■

Remark 4.9 Deduce from the definition of the $F_\sigma(\lambda)$'s that the sets of random variables

$$\mathcal{P}_1(\lambda) = \{(\sigma, \overline{F}_\sigma(\lambda)) \; , \; \sigma \in S^1_\mu \cap S_{\mu,\lambda}\} \; , \; \mathcal{P}_2(\lambda) = \{(\sigma, \overline{F}_\sigma(\lambda)) \; , \; \sigma \in S^2_\mu \cap S_{\mu,\lambda}\}$$

and $\mathcal{P}_3(\lambda) = \{(\sigma, \overline{F}_\sigma(\lambda)) \; , \; \sigma \in \mathrm{Br}(T) \cup \{\rho\}\}$ are independent. Their distributions are given as follows:

(i) $\mathcal{P}_1(\lambda)$ is a Poisson point process on $T \times \mathbb{T}$ with intensity measure

$$\ell_T(d\sigma) \otimes \int_{(0,\infty)} \Pi(dr) r e^{-r\psi^{-1}(\mu)} \Delta^r_{\mu,\lambda}(d\overline{\mathcal{T}} \cap \{\overline{\mathcal{T}} \neq \overline{\{\rho\}}\})$$

(Recall that $\overline{\{\rho\}}$ stands for the isometry class of the point tree).

(ii) $\mathcal{P}_2(\lambda)$ is a Poisson point process on $T \times \mathbb{T}$ with intensity measure

$$2\beta \ell_T(d\sigma) \otimes (\psi^{-1}(\lambda) - \psi^{-1}(\mu)) \, \Delta_{\mu,\lambda}(d\overline{\mathcal{T}})$$

(iii) For every $\sigma \in \mathrm{Br}(T)$, $\overline{F}_\sigma(\lambda)$ has distribution

$$\int_{[0,\infty)} \eta_{\mu,l}(dr) \, \Delta^r_{\mu,\lambda}(d\overline{\mathcal{T}}) \, ,$$

where $l = \mathrm{n}(\sigma, F_\sigma(\lambda)) - 1$ and $\overline{F}_\rho(\lambda)$ has distribution $\Delta^a_{\lambda,\mu}(d\overline{\mathcal{T}})$.

Remark 4.10 Fix $r > 0$. Denote the total number of trees added on $B_T(\rho, r)$ by $N_{\mu,\lambda}(r)$. Then, note that

$$N_{\mu,\lambda}(r) = \sum_{\sigma \in S_{\mu,\lambda} \cap B_T(\rho,r)} (\mathrm{n}(\rho_{\sigma,\lambda}, F_\sigma(\lambda)) - 1).$$

Then, conditionally on $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{A}_\mu$, $N_{\mu,\lambda}(r)$ is distributed as a Poisson random variable with parameter $cA_\mu(r)$ where

$$A_\mu(r) = 2\beta \ell_T(B_T(\rho, r)) + \sum_{\sigma \in S'_\mu \cap B_T(\rho,r)} a_\sigma(\mu).$$

Thus,

$$\mathbb{E}\left[s^{N_{\mu,\lambda}(r)}\right] = \mathbb{E}\left[\exp(-cA_\mu(r)(1-s))\right]. \tag{51}$$

■

5 The Lévy forest.

5.1 Construction of the Lévy forest.

In this section, we study the increasing limit of $\mathcal{F}_\lambda$ as $\lambda \to \infty$, and properties of the limit. Let us consider an (a, ψ)-growth process started at the discrete tree with edge lengths T denoted by $(\mathcal{F}_\lambda, d_\lambda, \rho)$, $\lambda \in [0, \infty)$. Set $\mathcal{F}_\infty = \bigcup \mathcal{F}_\lambda$ and define a metric d on $\mathcal{F}_\infty$ by $d(\sigma, \sigma') = d_\lambda(\sigma, \sigma')$ if $\sigma, \sigma' \in \mathcal{F}_\lambda$. We denote by $(\mathcal{F}, d)$ the completion of $(\mathcal{F}_\infty, d)$.

Theorem 5.1 *Assume that (3) holds. Almost surely, $(\mathcal{F}, d, \rho)$ is a locally compact rooted real tree and*

$$\delta(\overline{\mathcal{F}}_\lambda, \overline{\mathcal{F}}) \underset{\lambda\to\infty}{\longrightarrow} 0 .$$

Remark 5.1 If (3) does not holds, then the popupation may become extinct but in an infinite time and therefore the underlying genealogical tree cannot be locally compact.

Proof : Thanks to Lemma 3.5, it is sufficient to prove that for any $r \in (0,\infty)$ a.s. the collection of closed balls $(B_{\mathcal{F}_\lambda}(\rho, r); \lambda \geq 0)$ is Cauchy when λ goes to infinity with respect to the Hausdorff distance d_{Haus} on compact sets of $(\mathcal{F}, d)$. Set

$$\Xi_{\mu,\lambda}(r) := d_{\text{Haus}}\left(B_{\mathcal{F}_\mu}(\rho, r)\, ,\, B_{\mathcal{F}_\lambda}(\rho, r)\right) .$$

Since $\Xi_{\mu,\lambda}(r)$ is non-decreasing in λ and non-increasing in μ, we only have to prove that for any $t > 0$

$$\lim_{\mu\to\infty} \sup_{\lambda\geq\mu} \mathbb{P}\left(\Xi_{\mu,\lambda}(r) \leq t\right) \;=\; 1 . \tag{52}$$

We first need to introduce some notation: let $(\mathcal{T}_i^o, i \in I)$ be the connected components of the open set $\mathcal{F}_\lambda \backslash T$ in $\mathcal{F}_\lambda$. Denote by σ_i the vertex of T on which $\mathcal{T}_i^o$ is grafted and set $\mathcal{T}_i = \mathcal{T}_i^o \cup \{\sigma_i\}$. Then, the $(\mathcal{T}_i, d, \sigma_i)$'s are compact rooted real trees and

$$\mathcal{F}_\lambda = T \circledast_{i\in I} (\sigma_i, \mathcal{T}_i).$$

Let $\mu \in [0, \lambda]$ and let $i \in I$. Set $\mathcal{T}_i' = \mathcal{T}_i \cap \mathcal{F}_\mu$ and denote by $(\mathcal{T}_{i,j}^o, j \in J(i))$ the connected components of $\mathcal{T}_i \backslash \mathcal{T}_i'$. Denote by $\sigma_{i,j}$ the vertex of $\mathcal{T}_i'$ on which $\mathcal{T}_{i,j}^o$ is grafted and set $\mathcal{T}_{i,j} = \mathcal{T}_{i,j}^o \cup \{\sigma_{i,j}\}$. Clearly, the $(\mathcal{T}_{i,j}, d, \sigma_{i,j})$'s are compact rooted real trees. Observe that

$$\mathcal{F}_\lambda \backslash \mathcal{F}_\mu = \bigcup_{\substack{i\in I,\\ j\in J(i)}} \mathcal{T}_{i,j}^o \;\text{ and }\; \mathcal{F}_\mu = T \circledast_{i\in I} (\sigma_i, \mathcal{T}_i').$$

Thus

$$\mathcal{F}_\lambda = \mathcal{F}_\mu \circledast_{\substack{i\in I,\\ j\in J(i)}} (\sigma_{i,j}, \mathcal{T}_{i,j}).$$

To simplify notations, we set

$$h_{i,j} := h(\mathcal{T}_{i,j}) = \sup\{d(\sigma_{i,j}, \sigma)\, ,\, \sigma \in \mathcal{T}_{i,j}\}$$

and $I(r) := \{i \in I \,:\, d(\rho, \sigma_i) \leq r\}$. Then, the previous observations imply

$$\Xi_{\mu,\lambda}(r) \leq \max\{h_{i,j}\;,\;\; i \in I(r),\, j \in J(i)\}. \tag{53}$$

Now deduce from Proposition 4.3

$$\left(\overline{Q}^a_{0,\mu}(T), \overline{Q}^a_{\mu,\lambda}(Q^a_{0,\mu}(T))\right) \overset{(d)}{=} \left(\overline{\mathcal{F}}_\mu, \overline{\mathcal{F}}_\lambda\right) .$$

So if we set

$$N_{0,\lambda}(r) = \#I(r)\;,\;\; N_i = \#J(i) \;\text{ and }\; N_{\mu,\lambda}(r) = \sum_{i\in I(r)} N_i\;,$$

then we get the following:

(a) Conditionally on N_i , $i \in I$, the trees $\overline{\mathcal{T}}_{i,j}$, $i \in I(r)$, $j \in J(i)$, are independent GW$(\xi_{\mu,\lambda}, \psi'(\psi^{-1}(\lambda)))$-real trees.

(b) Conditionally on $N_{0,\lambda}(r)$, $(\overline{\mathcal{T}}'_i, \overline{\mathcal{T}}_i)$, $i \in I(r)$ are i.i.d. pairs of trees distributed as $(\overline{\mathcal{T}}', \overline{\mathcal{T}})$ where $\mathcal{T}$ is a GW$(\xi_{0,\lambda}, \psi'(\psi^{-1}(\lambda)))$-real tree and $\mathcal{T}'$ is obtained from $\mathcal{T}$ as the black subtree resulting from a $(1-\mu/\lambda)$-Bernoulli leaf colouring.

Thus, if we set $K_{\mu,\lambda}(t) = \mathbb{P}(\max\{h_{i,j}, i \in I(r), j \in J(i)\} \le t)$, we deduce from (a)

$$K_{\mu,\lambda}(t) = \mathbb{E}\left[\exp(-v_{\mu,\lambda}(t) N_{\mu,\lambda}(r))\right], \tag{54}$$

where $\exp(-v_{\mu,\lambda}(t)) := \mathbb{P}(h(\mathcal{T}'') \le t)$ and $\mathcal{T}''$ is a GW$(\xi_{\mu,\lambda}, \psi'(\psi^{-1}(\lambda)))$-real tree. Then, (23) applied to $\varphi_{\mu,\lambda}$ and a simple change of variable imply that $v_{\mu,\lambda}(t)$ satisfies the following equation

$$\int_{(\psi^{-1}(\lambda)-\psi^{-1}(\mu))(1-e^{-v_{\mu,\lambda}(t)})}^{\psi^{-1}(\lambda)-\psi^{-1}(\mu)} \frac{dx}{\psi(\psi^{-1}(\mu)+x)-\mu} = t. \tag{55}$$

We now need to compute the distribution of $N_{\mu,\lambda}(r)$ and accordingly the distribution of the N_i, $i \in I(r)$. If $(\mathcal{T}', \mathcal{T})$ are as in (b), then denote by M the number of red trees grafted on $\mathcal{T}'$. Note that M is possibly equal to 1 if $\mathcal{T}'$ is reduced to the point tree $\{\rho\}$, that is if $\mathcal{T}$ is completely red. Set $\kappa(s) = \mathbb{E}[s^M]$. According to (13), κ satisfies

$$\varphi_{0,\lambda}(\kappa(s)) - \kappa(s) = \varphi_{0,\lambda}(sg(p)) - sg(p) - (\varphi_{0,\lambda}(g(p)) - g(p)),$$

where we recall that $1 - p = \mu/\lambda$ and

$$g(p) = \mathbb{E}[p^{\#\mathrm{Lf}(\mathcal{T})}] = 1 - \frac{\psi^{-1}(\mu)-\gamma}{\psi^{-1}(\lambda)-\gamma}.$$

A straightforward computation implies:

$$(\psi^{-1}(\lambda)-\gamma)(1-\kappa(s)) = \psi^{-1}\left[\psi\left((\psi^{-1}(\lambda)-\psi^{-1}(\mu))(1-s)+\psi^{-1}(\mu)\right)-\mu\right]-\gamma. \tag{56}$$

Then, by (a) and (b) we get:

$$\mathbb{E}\left[\exp(-v_{\mu,\lambda}(t) N_{\mu,\lambda}(r))\right] = \mathbb{E}\left[\kappa(e^{-v_{\mu,\lambda}(t)})^{N_{0,\lambda}(r)}\right].$$

Recall the notation of Remark 4.10: we take here $\mu = 0$ and therefore we set

$$A_0(r) = 2\beta \ell_T(B_T(\rho,r)) + \sum_{\sigma \in S'_0 \cap B_T(\rho,r)} a_0(\sigma).$$

Thus, by (51)

$$K_{\mu,\lambda}(t) = \mathbb{E}\left[\exp(-A_0(r)\,(\psi^{-1}(\lambda)-\gamma)(1-\kappa(e^{-v_{\mu,\lambda}(t)})))\right].$$

Deduce from (55) that

$$\lim_{\lambda\to\infty} (\psi^{-1}(\lambda)-\psi^{-1}(\mu))(1-e^{-v_{\mu,\lambda}(t)}) = w_\mu(t) < \infty$$

which satisfies

$$\int_{w_\mu(t)}^{\infty} \frac{dx}{\psi(\psi^{-1}(\mu)+x)-\mu} = t \; .$$

Notice that here we use (3).Thus

$$\lim_{\lambda\to\infty} K_{\mu,\lambda}(t) = \mathbb{E}\left[\exp\left(-A_0(r)\psi^{-1}\left(\psi\left(w_\mu(t)+\psi^{-1}(\mu)\right)-\mu\right)-\gamma\right)\right].$$

Finally observe that

$$t = \int_{w_\mu(t)}^{\infty} \frac{dx}{\psi(\psi^{-1}(\mu)+x)-\mu} = \int_{\psi(\psi^{-1}(\mu)+w_\mu(t))-\mu}^{\infty} \frac{dy}{y\psi'(\psi^{-1}(\mu+y))},$$

which implies

$$\lim_{\mu\to\infty} \psi(\psi^{-1}(\mu)+w_\mu(t))-\mu = 0$$

by dominated convergence. Thus

$$\lim_{\mu\to\infty}\lim_{\lambda\to\infty} K_{\mu,\lambda}(t) = 1.$$

It proves (52), which completes the proof of the theorem. ■

Remark 5.2 Assume that the (a,ψ)-Lévy growth process $(\mathcal{F}_\lambda, ||\cdot||_1, 0)$, $\lambda\in[0,\infty)$ is $\mathbb{T}_{\ell_1}$-valued. The proof actually implies that a.s.

$$\mathbf{d}(\mathcal{F}_\lambda,\mathcal{F}) \underset{\lambda\to\infty}{\longrightarrow} 0 \; .$$

Notation 5.1

- The random locally compact rooted real tree obtained as a limit of an (a,ψ)-Lévy growth process starting at T is called an (a,ψ)-*Lévy forest starting at* T and we shall sometimes denote such a random tree by the symbol $Q^a_{0,\infty}(T)$. We also denote by $\overline{Q}^a_{0,\infty}(T)$ its isometry class.
- We call (a,ψ)-*Lévy forest* the random tree $Q^a_{0,\infty}(\mathcal{F}_0)$, where $\mathcal{F}_0$ is a $\mathrm{GW}(\xi_0,\psi'(\gamma),a\gamma)$-real forest that is independent of the random variables used to define the growth process. We denote by $P^a(d\overline{T})$ the distribution on $\mathbb{T}$ of $\overline{Q}^a_{0,\infty}(\mathcal{F}_0)$.
- Let $\mu\geq 0$. Observe that ψ_μ satisfies the assumptions of Theorem 5.1. We denote the limit of the (a,ψ_μ)-growth process started at T by the symbol $Q^a_{\mu,\infty}(T)$.
- Observe that 0 is the only root of $\psi_\mu(x)=0$. So an (a,ψ_μ)-Lévy forest is the limit of an (a,ψ_μ)-growth process started a the tree reduced to a point. We denote the distribution of the isometry class of an (a,ψ_μ)-Lévy forest by $P^a_\mu(d\overline{T})$. If $\gamma>0$, then $P^a_0\neq P^a$.
- We shall also consider the following random trees. Let $\mathcal{T}_\mu$ be a $\mathrm{GW}(\xi_\mu,\psi'(\psi^{-1}(\mu)))$-real tree and let $\mathcal{T}_{\mu,\lambda}$ be $\mathrm{GW}(\xi_{\mu,\lambda},\psi'(\psi^{-1}(\lambda)))$-real tree. We denote by $P_\mu(d\overline{T})$ the distribution on $\mathbb{T}$ of $\overline{Q}^{a=0}_{\mu,\infty}(\mathcal{T}_\mu)$ and we denote by $P_{\mu,\lambda}(d\overline{T})$ the distribution on $\mathbb{T}$ of $\overline{Q}^{a=0}_{\lambda,\infty}(\mathcal{T}_{\mu,\lambda})$. Now observe that $P^0_\mu=\delta_{\overline{\{\rho\}}}$ and thus $P^0_\mu\neq P_\mu$ (recall that $\overline{\{\rho\}}$ stands for the isometry class of the tree reduced to a point).

Let us end this subsection by two useful observations: first note that Proposition 4.3 combined with Theorem 5.1 with ψ_μ imply that for any discrete tree with edge lengths T, we have

$$\overline{Q}^a_{\lambda,\infty}\left(Q^a_{\mu,\lambda}(T)\right) \stackrel{(d)}{=} \overline{Q}^a_{\mu,\infty}(T) \tag{57}$$

(here the extra random variables used to define $Q^a_{\lambda,\infty}$ are chosen independent of $Q^a_{\mu,\lambda}(T)$). Then, recall notation $\Delta^a_{\mu,\lambda}$ from the previous section. Apply Theorem 5.1 with ψ_μ to get

$$\Delta^a_{\mu,\lambda} \underset{\lambda\to\infty}{\longrightarrow} P^a_\mu \tag{58}$$

weakly in the space of probability measures on $\mathbb{T}$.

5.2 The mass measure.

Let $a \geq 0$ and let $(\mathcal{F}_\lambda; \lambda \geq 0)$ be an (a, ψ)-Lévy growth process. *We assume that the $\mathcal{F}_\lambda$ are embedded in $\ell_1(\mathbb{N})$* and we denote by $\mathcal{F}$ the limit of this growth process in $\mathbb{T}_{\boldsymbol{\ell_1}}$. We also denote by $\mathbf{m}_\lambda$ the empirical distribution of the leaves of $\mathrm{Lf}(\mathcal{F}_\lambda)$:

$$\mathbf{m}_\lambda = \sum_{\sigma\in \mathrm{Lf}(\mathcal{F}_\lambda)\backslash \mathrm{Lf}(T)} \delta_\sigma \ . \tag{59}$$

Theorem 5.2 *There exists a random measure* $\mathbf{m}$ *on* $\ell_1(\mathbb{N})$ *such that*

(i) *Almost surely the convergence*

$$\lambda^{-1}\mathbf{m}_\lambda \underset{\lambda\to\infty}{\longrightarrow} \mathbf{m}$$

holds for the vague topology of Radon measures on $\ell_1(\mathbb{N})$*;*

(ii) *Almost surely the topological support of* $\mathbf{m}$ *is* $\mathcal{F}$*;*

(iii) *Let* $\mathcal{P} = \{(\sigma_j, U_j)\ ,\ j \in J)\}$ *be a Cox process on* $\ell_1(\mathbb{N}) \times [0, \infty)$ *with random intensity* $\mathbf{m}(d\sigma) \otimes du$*. For any* $\lambda \geq 0$ *denote by* $\mathcal{F}'_\lambda$ *the subtree of* $\mathcal{F}$ *spanned by* 0 *and the set of vertices* $\{\sigma_j\ ;\ j \in J\ ,\ U_j \leq \lambda\}$*:*

$$\mathcal{F}'_\lambda = \bigcup\{[\![0, \sigma_j]\!]\ ;\ j \in J\ ,\ U_j \leq \lambda\}.$$

Then,

$$(\overline{\mathcal{F}}_\lambda; \lambda \geq 0) \stackrel{(d)}{=} (\overline{\mathcal{F}}'_\lambda; \lambda \geq 0).$$

Remark 5.3 The measure $\mathbf{m}$ is concentrated on the leaves of $\mathcal{F}$ since by definition $\mathbf{m}(\mathcal{F}_\infty) = 0$ and since $\mathcal{F}\backslash\mathrm{Lf}(\mathcal{F}) \subset \mathcal{F}_\infty$.

Proof: Let us prove (i). By standard density arguments, it is sufficient to prove that for any non-negative continuous function f on $\ell_1(\mathbb{N})$ with compact support there exists a non-negative finite random variable $\mathbf{m}(f)$ such that we a.s. have

$$\lambda^{-1} < \mathbf{m}_\lambda, f > \underset{\lambda\to\infty}{\longrightarrow} \mathbf{m}(f). \tag{60}$$

Fix $\mu \geq 0$. We denote by $\mathcal{T}_i^o$, $i \in I(\mu)$, the connected components of $\mathcal{F} \backslash \mathcal{F}_\mu$ and we denote by σ_i the vertex of $\mathcal{F}_\mu$ on which $\mathcal{T}_i^o$ is grafted and we set

$$\mathcal{T}_i = \{\sigma_i\} \cup \mathcal{T}_i^o \quad \text{and} \quad \mathcal{T}_i(\lambda) = \mathcal{T}_i \cap \mathcal{F}_\lambda \ , \ \lambda \geq \mu \ .$$

Set for any $\lambda \geq \mu$

$$\mathbf{m}_\lambda^{\mathcal{T}_i} = \sum_{\sigma \in \mathrm{Lf}(\mathcal{T}_i(\lambda))} \delta_\sigma,$$

with the conventions that if $\mathcal{T}_i(\lambda) = \{\sigma_i\}$ then $\mathrm{Lf}(\mathcal{T}_i(\lambda)) = \emptyset$ and $\mathbf{m}_\lambda^{\mathcal{T}_i} = 0$. Then, for any $\lambda_2 \geq \lambda_1 \geq \mu$

$$\lambda_2^{-1} < \mathbf{m}_{\lambda_2}, f > -\lambda_1^{-1} < \mathbf{m}_{\lambda_1}, f >= T_1 + T_2 + T_3,$$

where

$$T_1 = \left(\lambda_2^{-1} - \lambda_1^{-1}\right) < \mathbf{m}_\mu, f > ,$$

$$T_2 = \sum_{i \in I(\mu)} < \lambda_2^{-1} \mathbf{m}_{\lambda_2}^{\mathcal{T}_i} - \lambda_1^{-1} \mathbf{m}_{\lambda_1}^{\mathcal{T}_i} \ , \ f - f(\sigma_i) > ,$$

$$T_3 = \sum_{i \in I(\mu)} f(\sigma_i) \left(< \lambda_2^{-1} \mathbf{m}_{\lambda_2}^{\mathcal{T}_i} > - < \lambda_1^{-1} \mathbf{m}_{\lambda_1}^{\mathcal{T}_i} > \right).$$

We set

$$M(\lambda) := \lambda^{-1} < \mathbf{m}_\lambda, \mathbf{1}_{B(0,r)} >= \lambda^{-1} \# \{\sigma \in \mathrm{Lf}(\mathcal{F}_\lambda) \ : \ ||\sigma||_1 \leq r\},$$

where r is such that $f(\sigma) = 0$ if $||\sigma||_1 > r$. We also define for any $\lambda \geq \mu$

$$M_{\mu,f}(\lambda) := \sum_{i \in I(\mu)} f(\sigma_i) \lambda^{-1} < \mathbf{m}_\lambda^{\mathcal{T}_i} > \ .$$

Lemma 5.3 *There exist two finite random variables $M_{\mu,f}(\infty)$ and $M(\infty)$ such that a.s.*

$$M_{\mu,f}(\lambda) \underset{\lambda \to \infty}{\longrightarrow} M_{\mu,f}(\infty) \quad \text{and} \quad M(\lambda) \underset{\lambda \to \infty}{\longrightarrow} M(\infty).$$

Proof of the lemma: For any $\lambda \geq \mu$, denote by $\mathcal{G}_\lambda$ the sigma-field generated by $\mathcal{F}_\mu$, the random variables $(\sigma_i, \overline{\mathcal{T}}_i(\lambda'); \lambda' \geq \lambda)$, $i \in I(\mu)$, and the $\mathbb{P}$-null sets. Set also

$$I(\mu, \lambda) = \{i \in I(\mu) \ : \ \mathcal{T}_i(\lambda) \neq \{\sigma_i\}\}.$$

Clearly for any $\lambda \geq \lambda' \geq \mu$, we have $\mathcal{G}_\lambda \subset \mathcal{G}_{\lambda'}$ and $M_{\mu,f}(\lambda)$ is $\mathcal{G}_\lambda$-measurable. Moreover the random variable $< \mathbf{m}_\lambda^{\mathcal{T}_i} >$ only depends on $\mathcal{G}_\lambda$ via $\overline{\mathcal{T}}_i(\lambda)$. Then, observe that for any $\lambda \geq \lambda'$ conditionally on $\mathcal{F}_\mu$ and on $I(\mu, \lambda)$, the trees $(\overline{\mathcal{T}}_i(\lambda'), \overline{\mathcal{T}}_i(\lambda))$, $i \in I(\mu, \lambda)$, are independent and distributed as $(\overline{\mathcal{T}}_b, \overline{\mathcal{T}})$ where $\mathcal{T}$ is a $\mathrm{GW}(\xi_{\mu,\lambda}, \psi'(\psi^{-1}(\lambda)))$-real tree and where $\mathcal{T}_b$ is the black subtree of $\mathcal{T}$ resulting from a $(1 - (\lambda' - \mu)/(\lambda - \mu))$-Bernoulli leaf colouring. Therefore, conditional on $\overline{\mathcal{T}}$, $\#\mathrm{Lf}(\mathcal{T}_b)$ has a binomial distribution with parameters $\#\mathrm{Lf}(\mathcal{T})$ and $(\lambda' - \mu)/(\lambda - \mu)$. Accordingly

$$\mathbb{E}\left[\#\mathrm{Lf}(\mathcal{T}_b) | \overline{\mathcal{T}}\right] = \frac{\lambda' - \mu}{\lambda - \mu} \#\mathrm{Lf}(\mathcal{T}).$$

Then, deduce from the latter observations that

$$\begin{aligned}\mathbb{E}\left[M_{\mu,f}(\lambda') \middle| \mathcal{G}_\lambda \right] &= \sum_{i\in I(\mu)} f(\sigma_i)\lambda^{-1}\mathbb{E}\left[<\mathbf{m}_{\lambda'}^{\mathcal{T}_i}> \middle| \overline{\mathcal{T}}_i(\lambda) \right] \\ &= \frac{\lambda'-\mu}{\lambda-\mu}\cdot\frac{\lambda}{\lambda'} M_{\mu,f}(\lambda).\end{aligned}$$

Thus, $M = (\frac{\lambda}{\lambda-\mu}M_{\mu,f}(\lambda); \lambda \geq \mu)$ is a non-negative backward martingale with respect to $(\mathcal{G}_\lambda; \lambda \geq \mu)$. A similar result holds for $(\frac{\lambda}{\lambda-\mu}M(\lambda); \lambda \geq \mu)$. Therefore, these two backward martingales converge to two limits in $[0,\infty]$ denoted by resp. $M_{\mu,f}(\infty)$ and $M(\infty)$. Since $\lambda/(\lambda-\mu)$ converges to 1 when λ goes to infinity, it implies the two convergences of the lemma. It remains to show that these two limiting random variables are a.s. finite.

To that end, observe that

$$M_{\mu,f}(\lambda) \leq \frac{||f||_\infty}{\lambda} \sum_{i\in I(\mu)} \mathbf{1}_{[0,r]}(||\sigma_i||_1)\ \#\mathrm{Lf}(\mathcal{T}_i(\lambda)). \tag{61}$$

Then, recall that conditionally on $\mathcal{F}_\mu$ and $I(\mu,\lambda)$, the trees $\overline{\mathcal{T}}_i(\lambda)$, $i \in I(\mu,\lambda)$ are independent with the same distribution as $\overline{\mathcal{T}}$. Fix $\theta > 0$. Use Remark 4.5, take $s = e^{-\theta/\lambda}$ and replace ψ by ψ_μ in (32), to get

$$g(e^{-\theta/\lambda}) = \mathbb{E}\left[e^{-\frac{\theta}{\lambda}\#\mathrm{Lf}(\mathcal{T})}\right] = 1 - \frac{\psi_\mu^{-1}((1-e^{-\theta/\lambda})(\lambda-\mu))}{\psi_\mu^{-1}(\lambda-\mu)} .$$

Set $N_{\mu,\lambda}(r) = \#\{i \in I(\mu,\lambda)\ :\ ||\sigma_i||_1 \leq r\}$. Then the previous observation implies

$$\mathbb{E}\left[\exp\left(-\theta\,\lambda^{-1}\sum_{i\in I(\mu,\lambda)} \mathbf{1}_{[0,r]}(||\sigma_i||_1)\ \#\mathrm{Lf}(\mathcal{T}_i(\lambda))\right)\right] = \mathbb{E}\left[\left(g(e^{-\frac{\theta}{\lambda}})\right)^{N_{\mu,\lambda}(r)}\right] .$$

Now use Remark 4.10 to get

$$\mathbb{E}\left[\left(g(e^{-\frac{\theta}{\lambda}})\right)^{N_{\mu,\lambda}(r)}\right] = \mathbb{E}\left[\exp(-A_\mu(r)\psi_\mu^{-1}((1-e^{-\frac{\theta}{\lambda}})(\lambda-\mu)))\right].$$

Thus,

$$\lim_{\lambda\to\infty}\mathbb{E}\left[\left(g(e^{-\theta/\lambda})\right)^{N_{\mu,\lambda}(r)}\right] = \mathbb{E}\left[e^{-A_\mu(r)\psi_\mu^{-1}(\theta)}\right].$$

Then, by (61)

$$\begin{aligned}\mathbb{E}\left[e^{-\theta M_{\mu,f}(\infty)}\right] &= \lim_{\lambda\to\infty}\mathbb{E}\left[\exp\left(-\theta M_{\mu,f}(\lambda)\right)\right] \\ &\geq \lim_{\lambda\to\infty}\mathbb{E}\left[\exp\left(-\frac{\theta||f||_\infty}{\lambda}\sum_{i\in I(\mu)}\mathbf{1}_{[0,r]}(||\sigma_i||_1)\#\mathrm{Lf}(\mathcal{T}_i(\lambda))\right)\right] \\ &= \mathbb{E}\left[e^{-A_\mu(r)\psi_\mu^{-1}(\theta||f||_\infty)}\right].\end{aligned}$$

Since the right member of the last inequality tends to 1 when θ goes to 0, so does the first member, which implies that $M_{\mu,f}(\infty)$ is a.s. finite. A similar argument works for $M(\infty)$. This completes the proof of the lemma. ■

Let us fix $\Omega' \subset \Omega$ such that $\mathbb{P}(\Omega') = 1$ and such that the following limits hold

$$\lim_{\lambda\to\infty} \mathbf{d}(\mathcal{F}_\lambda, \mathcal{F}) = 0\ , \quad \lim_{\lambda\to\infty} M_{\mu,f}(\lambda) = M_{\mu,f}(\infty) \quad \text{and} \quad \lim_{\lambda\to\infty} M(\lambda) = M(\infty).$$

We fix $\omega \in \Omega'$. Let $\epsilon > 0$. For any $\eta > 0$ we denote the modulus of uniform continuity of f by $w(f,\eta) := \sup\{|f(\sigma) - f(\sigma')|;\ ||\sigma - \sigma'||_1 \leq \eta\}$.

(a) We choose η such that

$$3M(\infty)\, w(f,\eta) \ \leq\ \epsilon.$$

(b) We choose μ large enough such that

$$\mathbf{d}(\mathcal{F}_\mu, \mathcal{F}) \ \leq\ \eta$$

(c) Then, we choose λ large enough such that for any $\lambda_1, \lambda_2 \geq \lambda$

$$|M_{\mu,f}(\lambda_1) - M_{\mu,f}(\lambda_2)| < \epsilon \quad , \quad M(\lambda_1) + M(\lambda_2) \leq 3M(\infty)$$

and

$$|T_1| = \left|\lambda_2^{-1} - \lambda_1^{-1}\right| < \mathbf{m}_\mu, f > \leq\ \epsilon.$$

Then, by (b) we have

$$< \lambda_2^{-1}\mathbf{m}_{\lambda_2}^{\mathcal{T}_i} - \lambda_1^{-1}\mathbf{m}_{\lambda_1}^{\mathcal{T}_i}\ ,\ f - f(\sigma_i) > \leq\ w(f,\eta)\left(\lambda_2^{-1} < \mathbf{m}_{\lambda_2}^{\mathcal{T}_i} > + \lambda_1^{-1} < \mathbf{m}_{\lambda_1}^{\mathcal{T}_i} >\right).$$

Thus, by (a) and (c)

$$\begin{aligned} |T_2| \ &\leq\ (M(\lambda_1) + M(\lambda_2))\, w(f,\eta) \\ &\leq\ 3M(\infty)\, w(f,\eta) \ \leq\ \epsilon. \end{aligned}$$

Now observe that $T_3 = M_{\mu,f}(\lambda_1) - M_{\mu,f}(\lambda_2)$. By (c) we get $|T_3| \leq \epsilon$. Thus we have proved that for any $\omega \in \Omega'$ and any $\epsilon > 0$, we can find a sufficiently large λ such that

$$\sup_{\lambda_1,\lambda_2 \geq \lambda} \left|\lambda_2^{-1} < \mathbf{m}_{\lambda_2}, f > - \lambda_1^{-1} < \mathbf{m}_{\lambda_1}, f >\right| \ \leq\ 3\epsilon,$$

which implies (60) and then (i) of Theorem 5.2.

Let us prove (ii). To that end, set for any $\theta \in [0,\infty)$ and any $\lambda > \mu$

$$h_{\mu,\lambda}(\theta) = \frac{\psi_\mu^{-1}(\theta + \lambda - \mu) - \psi_\mu^{-1}(\theta)}{\psi_\mu^{-1}(\lambda - \mu)}.$$

We need the following lemma

Lemma 5.4 *Conditionally on $\mathcal{F}_\mu$ and on $I(\mu,\lambda)$, the random variables $\mathbf{m}(\mathcal{T}_i)$, $i \in I(\mu,\lambda)$ are i.i.d. and the Laplace transform of their conditional distribution is $h_{\mu,\lambda}$.*

Proof of the lemma: Since for any $i \in I(\mu,\lambda)$, $\mathbf{m}(\{\sigma_i\})=0$, it is easy to check that

$$(\lambda')^{-1} < \mathbf{m}^{\mathcal{T}_i}_{\lambda'} > \underset{\lambda'\to\infty}{\longrightarrow} \mathbf{m}(\mathcal{T}_i). \tag{62}$$

Now observe that almost surely for $\lambda' \geq \lambda \geq \mu$, conditionally on $\mathcal{F}_\mu$ and on $I(\mu,\lambda)$, the trees $\overline{\mathcal{T}_i}(\lambda')$, $i \in I(\mu,\lambda)$, are independent and distributed as $\overline{\mathcal{T}}$ where $\mathcal{T}$ stands for a $\mathrm{GW}(\xi_{\mu,\lambda'},\psi'(\psi^{-1}(\lambda')))$-real tree conditioned on not being completely red after a $(1-\frac{\lambda-\mu}{\lambda'-\mu})$-Bernoulli leaf colouring. Denote by $\mathcal{T}_b$ the black subtree resulting from such a colouring. An elementary computation based on Remark 4.5 and (32) implies that

$$\mathbb{E}\left[s^{\#\mathrm{Lf}(\mathcal{T})} \Big| \mathcal{T}_b \neq \{\rho\}\right] = \frac{\psi_\mu^{-1}\left((1-s)\lambda'+s\lambda-\mu\right)-\psi_\mu^{-1}\left((1-s)(\lambda'-\mu)\right)}{\psi_\mu^{-1}\left(\lambda-\mu\right)}.$$

Take $s=\exp(-\theta/\lambda')$ and then observe that the right member converges to $h_{\mu,\lambda}(\theta)$ when λ' goes to infinity. This completes the proof of the lemma. ■

End of the proof of the theorem: Since ψ_μ^{-1} is concave, we get

$$h_{\mu,\lambda}(\theta) \;\leq\; \frac{1}{\psi_\mu^{-1}(\lambda-\mu)}.\frac{\lambda-\mu}{\psi'_\mu(\psi_\mu^{-1}(\theta))} \underset{\theta\to\infty}{\longrightarrow} 0.$$

Thus for any $\lambda > \mu$, $\mathbf{m}(\mathcal{T}_i) > 0$, $i \in I(\mu,\lambda)$ a.s. It implies that a.s. $\mathbf{m}(\mathcal{T}_i) > 0$ for every $i \in I(\mu)$. Then a.s. for every $\mu \geq 0$ the topological support of $\mathbf{m}$ has a non-trivial intersection with each of the connected components of $\mathcal{F}\backslash\mathcal{F}_\mu$, which implies (ii).

Let us prove (iii). Since the process $(\mathcal{F}'_\lambda;\lambda\geq 0)$ is obviously Bernoulli leaf colouring consistent, we only have to prove that for a fixed $\mu>0$, we have

$$\overline{\mathcal{F}}'_\mu \overset{(d)}{=} \overline{\mathcal{F}}_\mu. \tag{63}$$

Conditionally on $(\mathcal{F}_\lambda;\lambda\geq 0)$, let V_σ , $\sigma \in \bigcup_{\lambda\geq 0}\mathrm{Lf}(\mathcal{F}_\lambda)$ be i.i.d. $[0,1]$-uniform random variables. Set for any $\lambda \geq \mu$

$$\mathcal{N}_{\mu,\lambda} = \sum_{\sigma\in\mathrm{Lf}(\mathcal{F}_\lambda)} \mathbf{1}_{[0,\mu/\lambda]}(V_\sigma)\,\delta_{(\sigma,\lambda V_\sigma)}.$$

Denote by $\mathcal{M}(\ell_1(\mathbb{N}))$ the set of Radon measures of $\ell_1(\mathbb{N})$ and equip it with a metric compatible with the vague topology. Let K be a measurable non-negative function on $\mathbb{T}_{\ell_1}\times\mathcal{M}(\ell_1(\mathbb{N}))$ and let f be a non-negative continuous function on $\ell_1(\mathbb{N})\times[0,\mu]$ with compact support. Set

$$\mathcal{E}_\lambda = \mathbb{E}\left[K(\mathcal{F},\mathbf{m})e^{-<\mathcal{N}_{\mu,\lambda},f>}\right].$$

First observe that

$$\mathcal{E}_\lambda = \mathbb{E}\left[K(\mathcal{F},\mathbf{m})\exp\left(\sum_{\sigma\in\mathrm{Lf}(\mathcal{F}_\lambda)}\log\left(1-\frac{1}{\lambda}\int_0^\mu du\left(1-e^{-f(\sigma,u)}\right)\right)\right)\right].$$

Note that all the product and sums involved in the latter expression are finite since f has compact support. Let $r>0$ be such that $f(\sigma,u)=0$ for all σ such that $||\sigma||_1 \geq r$ and all $u\in[0,\mu]$. We now use the elementary inequality

$$0 \leq -\log(1-x) - x \leq \frac{x^2}{2(1-x)} \; , \quad x\in[0,1)$$

to get

$$\left| \sum_{\sigma\in \mathrm{Lf}(\mathcal{F}_\lambda)} \log\left(1-\frac{1}{\lambda}\int_0^\mu du \left(1-e^{-f(u,\sigma)}\right)\right) + \int_0^\mu du \int \frac{1}{\lambda}\mathbf{m}_\lambda(d\sigma)\left(1-e^{-f(u,\sigma)}\right)\right|$$
$$\leq \frac{1}{2\lambda}\frac{\mu^2}{1-\mu/\lambda}\frac{1}{\lambda}\mathbf{m}_\lambda\left(B_{\ell_1(\mathbb{N})}(0,r)\right) \; .$$

The first point of the Theorem then implies that

$$\lim_{\lambda\to\infty}\mathcal{E}_\lambda = \mathbb{E}\left[K(\mathcal{F},\mathbf{m})\exp\left(-\int_0^\mu du\int \mathbf{m}(d\sigma)\left(1-e^{-f(u,\sigma)}\right)\right)\right].$$

This implies that the following joint convergence

$$(\mathcal{F},\mathbf{m},\mathcal{N}_{\mu,\lambda}) \xrightarrow[\lambda\to\infty]{} (\mathcal{F},\mathbf{m},\mathcal{N}_{\mu,\infty}) \tag{64}$$

holds in distribution on $\mathbb{T}_{\ell_1}\times\mathcal{M}(\ell_1(\mathbb{N}))\times\mathcal{M}(\ell_1(\mathbb{N})\times[0,\mu])^2$; here $\mathcal{N}_{\mu,\infty}$ stands for a Cox process on $\ell_1(\mathbb{N})\times[0,\mu]$ with random intensity $\mathbf{m}(d\sigma)\otimes\mathbf{1}_{[0,\mu]}(x)dx$. Using Skorohod's representation theorem, we assume that (64) holds a.s. (for convenience we keep denoting the random variables in the same way). For any $\lambda\in[0,\infty)\cup\{\infty\}$, we denote by $\mathcal{P}_{\mu,\lambda}$ the set of $\sigma\in\ell_1(\mathbb{N})$ for which there exists $U\in[0,\mu]$ such that (σ,U) is an atom of $\mathcal{N}_{\mu,\lambda}$. We also introduce the subtree $\mathcal{F}_{\mu,\lambda}$ of $\mathcal{F}$ spanned by 0 and the points of $\mathcal{P}_{\mu,\lambda}$:

$$\mathcal{F}_{\mu,\lambda} := \bigcup_{\sigma\in\mathcal{P}_{\mu,\lambda}} [\![0,\sigma]\!] \; .$$

Clearly

$$\overline{\mathcal{F}}_{\mu,\infty} \overset{(d)}{=} \overline{\mathcal{F}}'_{\mu}. \tag{65}$$

Next deduce from Lemma 4.1 and from the definition of $\mathcal{N}_{\mu,\lambda}$ that the distribution of $\overline{\mathcal{F}}_{\mu,\lambda}$ does not depend on λ and is equal to Δ^a_μ. Observe now that for any $r>0$ such that $||\sigma||_1\neq r$ if $\sigma\in\mathcal{P}_{\mu,\infty}$, (64) implies

$$d_{Haus}\left(\mathcal{P}_{\mu,\lambda}\cap B(0,r),\mathcal{P}_{\mu,\infty}\cap B(0,r)\right) \xrightarrow[\lambda\to\infty]{} 0.$$

(Recall that d_{Haus} stands for the Hausdorff distance on the compact sets of $\ell_1(\mathbb{N})$). Next, set for any $r>0$ and any $\lambda\in[\mu,\infty)\cup\{\infty\}$

$$\mathcal{F}_{\mu,\lambda}(r) = \bigcup\{[\![0,\sigma]\!] \; ; \; \sigma\in\mathcal{P}_{\mu,\lambda}\cap B(0,r)\}.$$

or any $\sigma,\sigma'\in\mathcal{F}$

$$d_{Haus}\left([\![0,\sigma]\!],[\![0,\sigma']\!]\right)\leq ||\sigma-\sigma'||_1.$$

Thus, we get

$$d_{Haus}\left(\mathcal{F}_{\mu,\lambda}(r),\mathcal{F}_{\mu,\infty}(r)\right)\leq d_{Haus}\left(\mathcal{P}_{\mu,\lambda}\cap B(0,r),\mathcal{P}_{\mu,\infty}\cap B(0,r)\right).$$

Then for any $r>0$ such that $||\sigma||_1\neq r$ if $\sigma\in\mathcal{P}_{\mu,\infty}$,

$$d_{Haus}\left(\mathcal{F}_{\mu,\lambda}(r),\mathcal{F}_{\mu,\infty}(r)\right)\underset{\lambda\to\infty}{\longrightarrow}0. \tag{66}$$

Let $r>0$ be such that $||\sigma||_1\neq r$ if $\sigma\in\mathcal{P}_{\mu,\infty}$. Since $\mathcal{P}_{\mu,\infty}$ has no limit point, we can find $\eta\in(0,1)$ such that

$$\mathcal{P}_{\mu,\infty}\cap\left(B(0,r+\eta)\backslash B(0,r-\eta)\right)=\emptyset. \tag{67}$$

For the same reason, there is only a finite number of connected components $C_1,\ldots,C_k$ of $\mathcal{F}\backslash B(0,r)$ containing at least one point of $\mathcal{P}_{\mu,\infty}$. For any $1\leq i\leq k$, denote by σ_i the point of $\mathcal{F}$ on which C_i is grafted (observe that $||\sigma_i||_1=r$). Then,

$$\mathcal{F}_{\mu,\infty}\cap B(0,r)=\mathcal{F}_{\mu,\infty}(r)\bigcup_{i=1}^{k}[\![0,\sigma_i]\!] \tag{68}$$

Set $R=\max_{1\leq i\leq k}\min\{||\sigma||_1\ ,\ \sigma\in\mathcal{P}_{\mu,\infty}\cap C_i\}$. Observe that $R>r+\eta$ and that for any $r'>R$, we have

$$\begin{aligned}\mathcal{F}_{\mu,\infty}(r')\cap B(0,r) &= \mathcal{F}_{\mu,\infty}(r)\bigcup_{i=1}^{k}[\![0,\sigma_i]\!] &(69)\\ &= \mathcal{F}_{\mu,\infty}\cap B(0,r). &(70)\end{aligned}$$

Now for any $\lambda>\mu$ such that

$$d_{Haus}\left(\mathcal{P}_{\mu,\lambda}\cap B(0,R+1),\mathcal{P}_{\mu,\infty}\cap B(0,R+1)\right)<\eta/2,$$

the connected components of $\mathcal{F}\backslash B(0,r)$ containing at least one point of $\mathcal{P}_{\mu,\lambda}$ are exactly $C_1,\ldots,C_k$. Thus,

$$\begin{aligned}\mathcal{F}_{\mu,\lambda}(R+1)\cap B(0,r) &= \mathcal{F}_{\mu,\lambda}(r)\bigcup_{i=1}^{k}[\![0,\sigma_i]\!] &(71)\\ &= \mathcal{F}_{\mu,\lambda}\cap B(0,r). &(72)\end{aligned}$$

Then by (68) (69) and (71),

$$d_{Haus}\left(\mathcal{F}_{\mu,\lambda}\cap B(0,r),\mathcal{F}_{\mu,\infty}\cap B(0,r)\right)\leq d_{Haus}\left(\mathcal{F}_{\mu,\lambda}(R+1),\mathcal{F}_{\mu,\infty}(R+1)\right)\underset{\lambda\to\infty}{\longrightarrow}0.$$

This combined with (66) implies that a.s.

$$\delta\left(\overline{\mathcal{F}}_{\mu,\lambda},\overline{\mathcal{F}}_{\mu,\infty}\right)\underset{\lambda\to\infty}{\longrightarrow}0.$$

Since the distribution of the $\overline{\mathcal{F}}_{\mu,\lambda}$ is constant and equal to Δ^a_μ , it implies that $\overline{\mathcal{F}}_{\mu,\infty}$ is also distributed according to Δ^a_μ, which proves (65) and which completes the proof of (iii) and the proof of the Theorem by (63).

■

Remark 5.4 (Connection with previous works in [24, 25, 11, 12]) Lévy forests have first been defined in the subcritical or critical case via the coding by a process $H = (H_t, t \geq 0)$ introduced by Le Gall and Le Jan in [24] called the ψ-height process. This process is obtained from a Lévy process $X = (X_t, t \geq 0)$ with Laplace exponent ψ, by the following approximation procedure: for every $t \geq 0$, the following limit in probability exists

$$H_t = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t ds \, \mathbf{1}_{\{X_s \leq I_t^s + \varepsilon\}},$$

where we have set $I_t^s := \inf_{s \leq r \leq t} X_r$ (this approximation is a consequence of Lemma 1.1.3 in [11]). Set $T_a = \inf\{t \geq 0 \ : \ X_t = -a\}$. Then, the process $(H_t, 0 \leq t \leq T_a)$ represents the "contour" of the tree $(\mathcal{F}, d, \rho)$ in the following sense. For any $s, s' \in [0, T_a]$, set

$$d(s, s') = H_s + H_{s'} - 2 \inf_{s \wedge s' \leq u \leq s \vee s'} H_u$$

and introduce the equivalence relation $s \sim s'$ iff $d(s, s') = 0$. Then Theorem 2.1 in [12] asserts that

$$(\mathcal{F} \, , \; d \, , \; \rho) := \Big([0, T_a]/\sim \, , \; d \, , \; \widetilde{0} \Big)$$

is a compact random real rooted tree; for any $s \in [0, T_a]$, denote by $\widetilde{s}$ the $\sim$-isometry class of s. Let us explain why $\overline{\mathcal{F}}$, defined in this way, is an (a, ψ)-Lévy forest. Let $\mathcal{P} = \{(t_i, r_i), i \in J\}$ be a Poisson point process on $[0, \infty)^2$ with intensity the Lebesgue measure. For any $\lambda \geq 0$ we set

$$\mathcal{F}(\lambda) = \bigcup \{ [\![\rho, \widetilde{t_i}]\!] \; ; \; i \in J \ : \ r_i \leq \lambda \; ; \; t_i \leq T_a \}.$$

Obviously the family of real trees $(\mathcal{F}(\lambda); \lambda \geq 0)$ is consistent under Bernoulli leaf colouring and Theorem 3.2.1 [11] asserts that $(\mathcal{F}(\lambda), d, \rho)$ is a $\mathrm{GW}(\xi_\lambda, \psi'(\psi^{-1}(\lambda)), a\psi^{-1}(\lambda))$ -real rooted forest Thus, $(\mathcal{F}(\lambda); \lambda \geq 0)$ is an (a, ψ)-growth process. Besides, it is clear from the construction that a.s.

$$\lim_{\lambda \to \infty} \delta(\mathcal{F}(\lambda), \mathcal{F}) = 0.$$

Moreover, if we take $(\mathcal{F}(\lambda); \lambda \geq 0)$ in Theorem 5.2, the mass distribution is clearly the image of the Lebesgue measure on the line by the canonical projection associated with $\sim$. We refer to [12] for discussion of various geometric properties of Lévy forests.

Remark 5.5 The construction of the mass measure on a Lévy tree given in [12] only relies on the metric structure of the Lévy tree and not on a particular coding (see the remark before Theorem 4.4 in [11]). We failed to give a proof that is well-suited to our approach that $\mathbf{m}$ is actually a deterministic functional of its topological support $\mathcal{F}$.

5.3 Excursion measure of Lévy trees.

Fix $a > 0$ and consider an (a, ψ)-Lévy forest $\mathcal{F}$. Denote by $\mathcal{T}_i^o$, $i \in J$, the connected components of $\mathcal{F} \backslash \{\rho\}$ and set for any $i \in J$, $\mathcal{T}_i = \{\rho\} \cup \mathcal{T}_i^o$. The main goal of this section is to define a Borel measure $\Theta(d\overline{\mathcal{T}})$ on $\mathbb{T}$ such that the following proposition holds:

Proposition 5.5 *The point measure*

$$\mathcal{N}(d\overline{\mathcal{T}}) := \sum_{i \in J} \delta_{\overline{\mathcal{T}}_i}(d\overline{\mathcal{T}})$$

is a Poisson point measure on $\mathbb{T}$ *with intensity* $a\,\Theta(d\overline{\mathcal{T}})$.

Before proving this proposition, recall the notation $P_\mu(d\overline{\mathcal{T}})$ and $P_{\mu,\lambda}(d\overline{\mathcal{T}})$ from Subsection 5.1. We first establish

Claim:
$$P_\lambda = \left(1 - \frac{\psi^{-1}(\mu)}{\psi^{-1}(\lambda)}\right) P_{\mu,\lambda} + \frac{\psi^{-1}(\mu)}{\psi^{-1}(\lambda)} P_\mu . \tag{73}$$

Proof of the claim: Let $\mathcal{T}_\mu$ and $\mathcal{T}_{\mu,\lambda}$ be as in the last point of Notation 5.1. Perform a $(1-\mu/\lambda)$-Bernoulli leaf colouring on $\mathcal{T}_\lambda$. Recall that the probability that $\mathcal{T}_\lambda$ is completely red is $1-\psi^{-1}(\mu)/\psi^{-1}(\lambda)$. Moreover, conditionally on this event, $\overline{\mathcal{T}}_\lambda$ is distributed as $\overline{\mathcal{T}}_{\mu,\lambda}$ and conditionally on the complementary event, $\overline{\mathcal{T}}_\lambda$ is distributed as $\overline{Q}_{\mu,\lambda}(\mathcal{T}_\mu)$. Then, flip a coin with probability $1-\psi^{-1}(\mu)/\psi^{-1}(\lambda)$ to be head. If it is head, then set $\mathcal{T}' = \mathcal{T}_{\mu,\lambda}$; otherwise set $\mathcal{T}' = Q_{\mu,\lambda}(\mathcal{T}_\mu)$. The previous observations imply that $\overline{\mathcal{T}}'$ and $\overline{\mathcal{T}}_\lambda$ have the same distribution. Accordingly, $\overline{Q}_{\lambda,\infty}(\mathcal{T}_\lambda)$ and $\overline{Q}_{\lambda,\infty}(\mathcal{T}')$ have the same distribution. Use now (57) with $T = \mathcal{T}_\mu$ to get

$$\overline{Q}^a_{\lambda,\infty}\left(Q^a_{\mu,\lambda}(\mathcal{T}_\mu)\right) \quad \overset{(d)}{=} \quad \overline{Q}^a_{\mu,\infty}(\mathcal{T}_\mu) \; .$$

This, combined with the previous observation imply the claim. ■

Let $\lambda_0 = 0 < \lambda_1 < \lambda_2, \dots$ be any increasing sequence going to infinity. We define the excursion measure by

$$\Theta(d\overline{\mathcal{T}}) = \gamma P_0(d\overline{\mathcal{T}}) + \sum_{n\geq 0} (\psi^{-1}(\lambda_{n+1}) - \psi^{-1}(\lambda_n)) P_{\lambda_n,\lambda_{n+1}}(d\overline{\mathcal{T}}).$$

Recall that $\gamma = \psi^{-1}(\lambda_0)$. Let us first prove that $\Theta(d\overline{\mathcal{T}})$ does not depend on $(\lambda_n; n \geq 0)$ and more precisely for any non-negative measurable function K on $\mathbb{T}$, let us prove that

$$< \Theta, K >= \lim_{\lambda\to\infty} \uparrow < \psi^{-1}(\lambda) P_\lambda, K > . \tag{74}$$

Proof of (74): (73) implies that $\lambda \to < \psi^{-1}(\lambda)P_\lambda, K >$ is non-decreasing. Thus, the limit in (74) exists in $[0,\infty]$. Denote this limit by $L(K)$ and observe that

$$\psi^{-1}(\lambda_n) < P_{\lambda_n}, K >= \gamma P_0(d\overline{\mathcal{T}}) + \sum_{k=0}^{n-1} (\psi^{-1}(\lambda_{k+1}) - \psi^{-1}(\lambda_k)) < P_{\lambda_k,\lambda_{k+1}}, K > .$$

Thus, by letting n go to infinity, we get $< \Theta, K >= L(K)$, which proves (74). ■

Proof of the proposition: Fix $\lambda > 0$ and define

$$J_\lambda = \{ i \in J \; : \; \mathcal{T}_i \cap \mathcal{F}_\lambda \neq \{\rho\} \}.$$

Set $\mathcal{T}_i(\lambda) = \mathcal{T}_i \cap \mathcal{F}_\lambda$ for any $i \in J_\lambda$. From the construction of the growth process, we deduce that $\# J_\lambda$ is a Poisson random variable with parameter $a\psi^{-1}(\lambda)$ and that conditionally on J_λ, the $\overline{\mathcal{T}}_i(\lambda)$, $i \in J_\lambda$, are i.i.d. and distributed as the isometry class of a $\mathrm{GW}(\xi_\lambda, \psi'\psi^{-1}(\lambda))$-real tree. Now observe that for any $i \in J_\lambda$ the tree $\mathcal{T}_i$ is obtained as the limit of a growth process started at $\mathcal{T}_i(\lambda)$. Then $\overline{\mathcal{T}}_i$ and $\overline{Q}_{\lambda,\infty}(\mathcal{T}_i(\lambda))$ have the same distribution. Thus conditionally on

J_λ the $\overline{\mathcal{T}}_i$, $i \in J_\lambda$, are independent and distributed according to P_λ and for any non-negative measurable function K on $\mathbb{T}$, we have

$$\begin{aligned}\mathbb{E}\left[\exp\left(-\sum_{i\in J_\lambda} K(\overline{\mathcal{T}}_i)\right)\right] &= \mathbb{E}\left[\left(\int P_\lambda(d\overline{\mathcal{T}})e^{-K(\overline{\mathcal{T}})}\right)^{\#J_\lambda}\right] \\ &= \mathbb{E}\left[\exp\left(-a\psi^{-1}(\lambda)\int P_\lambda(d\overline{\mathcal{T}})\left(1-e^{-K(\overline{\mathcal{T}})}\right)\right)\right] .\end{aligned}$$

Now, observe that

$$< \mathcal{N}, K >= \lim_{\lambda\to\infty} \uparrow \sum_{i\in J_\lambda} K(\overline{\mathcal{T}}_i) ,$$

which completes the proof by (74) and by the dominated convergence theorem. ■

Recall that the height $h(T)$ of a rooted real tree (T, d, ρ) is the (possibly infinite) real number $\sup\{d(\rho,\sigma) \ , \ \sigma \in T\}$. Observe that $h(T)$ is invariant up to isometry so it makes sense to define $h(\overline{T})$ as the height of any representative of $\overline{T}$. It is easy to check here that a.s.

$$\lim_{\lambda\to\infty} \uparrow h(\overline{\mathcal{F}}_\lambda) = h(\overline{\mathcal{F}}) . \tag{75}$$

Recall from (23) that the probability that the height of a single $\mathrm{GW}(\xi_\lambda, \psi'(\psi^{-1}(\lambda)))$-real tree is greater that x is $e^{-v_\lambda(x)}$ where $v_\lambda(x)$ satifies

$$\psi^{-1}(\lambda)\int_0^{e^{-v_\lambda(x)}} \frac{du}{\psi((1-u)\psi^{-1}(\lambda))} = x.$$

Then, (75) and a simple computation imply that

$$\mathbb{P}\left(h(\overline{\mathcal{F}}) \le x\right) = \exp(-av(x)) ,$$

where v satisfies the equation

$$\int_{v(x)}^{\infty} \frac{du}{\psi(u)} = x . \tag{76}$$

Now observe that $h(\overline{\mathcal{F}}) = \sup\{h(\overline{\mathcal{T}}_i) \ , \ i \in J\}$. Thus Proposition 5.5 implies that

$$\Theta\left(h(\overline{\mathcal{T}}) > x\right) = v(x). \tag{77}$$

Remark 5.6 Observe that Proposition 5.5 and (77) imply that a.s.

$$a = \lim_{\epsilon\to 0} \frac{1}{v(\epsilon)} \#\{i \in J \ : \ h(\overline{\mathcal{T}}_i)\}.$$

Notation 5.2 The measure Θ is called the *ψ-excursion measure.* The terminology comes from the fact that in the critical or subcritical case when the Lévy forest is coded by a ψ-height process as explained in Remark 5.4, Θ is the distribution of the tree coded by one excursion above 0 of the height process. In the last section we shall use the notation Θ_λ for the ψ_λ-excursion measure.

5.4 Decomposition of the Lévy forest along the ancestral tree of a Poisson sample

Fix $\mu_0 \geq 0$ and $a \geq 0$. Consider a $\mathbb{T}_{\ell_1}$-valued (a,ψ)-growth process $(\mathcal{F}_\lambda; \lambda \geq 0)$ denote by $\mathcal{F}$ the limit of this growth process. Recall that $\overline{\mathcal{F}}_{\mu_0}$ is distributed as the isometry class of the ancestral subtree of a Poisson sampling on $\mathcal{F}$ with intensity $\mu_0 \, . \, \mathbf{m}$. The aim of this subsection is to compute the distribution of $\mathcal{F}$ conditionally on $\mathcal{F}_{\mu_0}$, as the reconstruction procedure does in the discrete case. To avoid technicalities and to make easier the statement of this decomposition we also assume that

$$\mathcal{F}_{\mu_0} = Q^a_{0,\mu_0}(\mathcal{F}_0), \tag{78}$$

where the extra random variables used to define Q^a_{0,μ_0} are chosen independent of $\mathcal{F}_0$. Before stating the main result, we need to introduce some notation: Denote by Gr the set of points on which the connected components of $\mathcal{F}\backslash\mathcal{F}_{\mu_0}$ are grafted; for any $\sigma \in \mathrm{Gr}$, denote by $F^o_i(\sigma)$, $i \in J(\sigma)$, the connected components of $\mathcal{F}\backslash\mathcal{F}_{\mu_0}$ that are grafted on σ and set

$$F_\sigma = \{\sigma\} \cup \{F^o_i(\sigma), i \in J(\sigma)\}.$$

Observe that F_σ is a closed and connected set. Next, let us introduce the sets of points

$$S^1_{\mu_0} = \{\sigma \in \mathrm{Gr}\backslash(\mathrm{Br}(\mathcal{F}_{\mu_0}) \cup \{\rho\}) \ : \ \#J(\sigma) \geq 2\}$$

and

$$S^2_{\mu_0} = \{\sigma \in \mathrm{Gr}\backslash(\mathrm{Br}(\mathcal{F}_{\mu_0}) \cup \{\rho\}) \ : \ \#J(\sigma) = 1\}.$$

Note that some branching points of $\mathcal{F}_{\mu_0}$ may not be in Gr. We then set

$$S_{\mu_0} = S^1_{\mu_0} \cup S^2_{\mu_0} \cup \mathrm{Br}(\mathcal{F}_{\mu_0}) \cup \{\rho\},$$

and if $\sigma \in (\mathrm{Br}(\mathcal{F}_{\mu_0}) \cup \{\rho\})\backslash\mathrm{Gr}$, then we set $F_\sigma = \{\sigma\}$.

Recall from the end of Section 5.3 the notation Θ_{μ_0} for the ψ_{μ_0}-excursion measure and also recall from Section 5.1 the notation $P^r_{\mu_0}$. Let us denote by v_{μ_0} the function defined on $[0,\infty)$ that satisfies

$$\int_{v_{\mu_0}(t)}^{\infty} \frac{du}{\psi_{\mu_0}(u)} = t. \tag{79}$$

Theorem 5.6 *Almost surely for every $\sigma \in S_{\mu_0}$, the following limit exists and is finite*

$$a(\sigma) := \lim_{\epsilon \to 0} \frac{1}{v_{\mu_0}(\epsilon)} \#\{i \in J(\sigma) \ : \ h(F^o_i(\sigma)) > \epsilon\}.$$

Moreover, conditionally on $\mathcal{F}_{\mu_0}$ the collections of random variables

$$\mathcal{P}_1 = \{(\sigma, a(\sigma), \overline{F}_\sigma) \ , \ \sigma \in S^1_{\mu_0}\} \qquad \mathcal{P}_2 = \{(\sigma, \overline{F}_\sigma) \ , \ \sigma \in S^2_{\mu_0}\}$$

and $\mathcal{P}_3 = \{(a(\sigma), \overline{F}_\sigma), \ \sigma \in \mathrm{Br}(\mathcal{F}_{\mu_0}) \cup \{\rho\}\}$, are independent. Their conditional distributions are given by the following:

(i) *$\mathcal{P}_1$ is a Poisson point process on $\mathcal{F}_{\mu_0} \times [0,\infty) \times \mathbb{T}$ with intensity measure*

$$\ell_{\mathcal{F}_{\mu_0}}(d\sigma) \otimes e^{-r\psi^{-1}(\mu_0)}\Pi(dr) \otimes P^r_{\mu_0}(d\overline{\mathcal{T}});$$

(ii) $\mathcal{P}_2$ *is a Poisson point process on* $\mathcal{F}_{\mu_0} \times \mathbb{T}$ *with intensity measure*

$$2\beta \ell_{\mathcal{F}_{\mu_0}}(d\sigma) \otimes \Theta_{\mu_0}(d\overline{\mathcal{T}});$$

(iii) *The* $(a(\sigma), \overline{F}_\sigma)$, $\sigma \in \mathrm{Br}(\mathcal{F}_{\mu_0}) \cup \{\rho\}$ *are independent random variables; Moreover, for each* $\sigma \in \mathrm{Br}(\mathcal{F}_{\mu_0})$, *the* $[0,\infty) \times \mathbb{T}$*-valued random variables* $(a(\sigma), \overline{F}_\sigma)$ *are distributed according to*

$$\eta_{\mu_0,l}(dr) \otimes P^r_{\mu_0}(d\overline{\mathcal{T}})$$

where $l = \mathrm{n}(\sigma, \mathcal{F}_{\mu_0}) - 1$*;* $a(\rho) = a$ *and* $\overline{F_\rho}$ *is distributed according to* $P^a_{\mu_0}$*.*

Remark 5.7 Recall that $P^0_{\mu_0} = \delta_{\overline{\{\rho\}}}$. If $l = 2$ in (iii), then since

$$\eta_{\mu_0,2}(\{0\}) = \frac{2\beta}{\psi^{(2)}(\psi^{-1}(\mu_0))},$$

F_σ reduces to a point with probability $\eta_{\mu_0,2}(\{0\}) > 0$ as soon as $\beta > 0$.

Proof: Set for any $\lambda \geq \mu_0$ and any $\sigma \in S_{\mu_0}$, $F_\sigma(\lambda) = F_\sigma \cap \mathcal{F}_\lambda$ and

$$S_{\mu_0,\lambda} = \{\sigma \in S_{\mu_0} \ : \ F_\sigma(\lambda) \neq \{\sigma\}\}.$$

Deduce from Theorem 5.1 that a.s. for any $\sigma \in S_{\mu_0}$

$$\mathbf{d}(F_\sigma(\lambda), F_\sigma) \underset{\lambda\to\infty}{\longrightarrow} 0 \,. \tag{80}$$

Recall notation $\Delta_{\mu_0,\lambda}$ and $\Delta^r_{\mu_0,\lambda}$ from the end of Section 4.1. For convenience of notation, let us set

$$M_1(d\overline{\mathcal{T}}) = \int_{(0,\infty)} \Pi(dr) r e^{-r\psi^{-1}(\mu_0)} P^r_{\mu_0}(d\overline{\mathcal{T}})$$

and

$$M_2(d\overline{\mathcal{T}}) = 2\beta\Theta_{\mu_0}(d\overline{\mathcal{T}}).$$

Lemma 5.7 *For any non-negative continuous function* R *on* $\mathbb{T}$*, any* $\mu_0 \geq 0$ *and any* $\epsilon > 0$*, we have*

(a) $$2\beta\left(\psi^{-1}(\lambda) - \psi^{-1}(\mu_0)\right) \int_{\{h(\overline{\mathcal{T}})>\epsilon\}} \Delta_{\mu_0,\lambda}(d\overline{\mathcal{T}}) \left(1 - e^{-R(\overline{\mathcal{T}})}\right) \underset{\lambda\to\infty}{\longrightarrow} \int_{\{h(\overline{\mathcal{T}})>\epsilon\}} M_2(d\overline{\mathcal{T}}) \left(1 - e^{-R(\overline{\mathcal{T}})}\right).$$

(b) $$\int_{(0,\infty)} \Pi(dr) r e^{-r\psi^{-1}(\mu_0)} \int_{\{h(\overline{\mathcal{T}})>\epsilon;\overline{\mathcal{T}}\neq\overline{\rho}\}} \Delta^r_{\mu_0,\lambda}(d\overline{\mathcal{T}}) \left(1 - e^{-R(\overline{\mathcal{T}})}\right) \underset{\lambda\to\infty}{\longrightarrow} \int_{\{h(\overline{\mathcal{T}})>\epsilon\}} M_1(d\overline{\mathcal{T}}) \left(1 - e^{-R(\overline{\mathcal{T}})}\right).$$

End of the proof of the theorem: Before proving the lemma, let us complete the proof of the theorem. Let K be a non-negative continuous function on $\ell_1(\mathbb{N})\times\mathbb{T}$ such that $K(\sigma,\overline{\mathcal{T}})=0$ for every $\overline{\mathcal{T}}\in\mathbb{T}$ and every σ such that $||\sigma||_1\geq r_0$, where r_0 is a fixed positive number. First deduce from (80) that a.s. for every $\sigma\in S_{\mu_0}$

$$\lim_{\lambda\to\infty}\uparrow h(\overline{F}_\sigma(\lambda))=h(\overline{F}_\sigma).$$

Fix $\epsilon>0$. Since $\mathcal{F}$ is locally compact, there is only a finite number of F_σ's such that $h(\overline{F}_\sigma)>\epsilon$ and $||\sigma||_1\leq r_0$. Thus for any $i\in\{1,2\}$ a.s.

$$\lim_{\lambda\to\infty}\sum_{\sigma\in S_{\mu_0,\lambda}\cap S^i_{\mu_0}}K(\sigma,\overline{F}_\sigma(\lambda))\mathbf{1}_{\{h(\overline{F}_\sigma(\lambda))>\epsilon\}}=\sum_{\sigma\in S^i_{\mu_0}}K(\sigma,\overline{F}_\sigma)\mathbf{1}_{\{h(\overline{F}_\sigma)>\epsilon\}}.$$

Since we have supposed (78),

$$\mathcal{P}_1(\lambda)=\{(\sigma,\overline{F}_\sigma(\lambda)),\ \sigma\in S^1_{\mu_0}\cap S_{\mu_0,\lambda}\}\ ,\ \mathcal{P}_2(\lambda)=\{(\sigma,\overline{F}_\sigma(\lambda)),\ \sigma\in S^2_{\mu_0}\cap S_{\mu_0,\lambda}\}$$

and $\mathcal{P}_3(\lambda)=\{(\sigma,\overline{F}_\sigma(\lambda))\ ,\ \sigma\in\mathrm{Br}(T)\cup\{\rho\}\}$ are distributed as specified in Remark (4.9) with $T=\mathcal{F}_{\mu_0}$. Then deduce from Remark 4.9 (i) and (ii) and from Lemma 5.7 that

$$\mathbb{E}\left[\exp\left(-\sum_{\sigma\in S^i_{\mu_0}}K(\sigma,\overline{F}_\sigma)\mathbf{1}_{\{h(\overline{F}_\sigma)>\epsilon\}}\right)\middle|\mathcal{F}_{\mu_0},S^i_{\mu_0}\cap S_{\mu_0,\lambda}\right]$$
$$=\exp\left(-\int\ell_{\mathcal{F}_{\mu_0}}(d\sigma)\int_{\{h(\overline{\mathcal{T}})>\epsilon\}}M_i(d\overline{\mathcal{T}})\left(1-e^{-K(\sigma,\overline{\mathcal{T}})}\right)\right),$$

for $i\in\{1,2\}$. Now, let ϵ go to 0: It implies that conditionally on $\mathcal{F}_{\mu_0}$ the sets of points $\{(\sigma,\overline{F_\sigma})\,;\,\sigma\in S^i_{\mu_0}\}$, $i\in\{1,2\}$ are two independent Poisson point processes with resp. intensities $\ell_{\mathcal{F}_{\mu_0}}\otimes M_i$, $i\in\{1,2\}$.

Recall that (58) asserts that for any $r>0$, the probability measure $\Delta^r_{\mu_0,\lambda}$ on $\mathbb{T}$ weakly converges to $P^r_{\mu_0}$. This observation combined with Remark 4.9 imply that conditionally on $\mathcal{F}_{\mu_0}$ for every $\sigma\in\mathrm{Br}(\mathcal{F}_{\mu_0})$, $\overline{F}_\sigma$ is distributed according to

$$\int\eta_{\mu_0,l}(dr)P^r_{\mu_0}(d\overline{\mathcal{T}}),$$

(with $l=\mathrm{n}(\sigma,\mathcal{F}_{\mu_0})-1$), and that $\overline{F}_\rho$ is distributed according to $P^a_{\mu_0}$. Then, Remark 5.6 implies the first point of the theorem; this, combined with the previous observations, implies that conditionally on $\mathcal{F}_{\mu_0}$, $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ are distributed as specified in the theorem; then, their conditional independence is an easy consequence of the conditional independence of $\mathcal{P}_1(\lambda)$, $\mathcal{P}_2(\lambda)$ and $\mathcal{P}_3(\lambda)$ stated in Remark 4.9. This completes the proof of the theorem. ■

Proof of Lemma 5.7: Recall (41) that makes the connection between the distribution Δ_λ and $\Delta_{\mu_0,\lambda}$. By replacing ψ by ψ_{μ_0}, the first point of the lemma is then equivalent to the following limit

$$\psi^{-1}(\lambda)\int_{\{h(\overline{\mathcal{T}})>\epsilon\}}\Delta_\lambda(d\overline{\mathcal{T}})\left(1-e^{-R(\overline{\mathcal{T}})}\right)\underset{\lambda\to\infty}{\longrightarrow}\int_{\{h(\overline{\mathcal{T}})>\epsilon\}}\Theta(d\overline{\mathcal{T}})\left(1-e^{-R(\overline{\mathcal{T}})}\right).\qquad(81)$$

Recall from Section 5.3 the notation $\mathcal{T}_i$, $i \in J$, for the subtrees of $\mathcal{F}$ grafted at $\{\rho\}$. Set $\mathcal{T}_i(\lambda) = \mathcal{F}_\lambda \cap \mathcal{T}_i$, $i \in J$ and

$$J(\lambda) = \{i \in J \ : \ \mathcal{T}_i(\lambda) \neq \{\rho\}\}.$$

Since $\#J(\lambda)$ is a Poisson variable with parameter $a\psi^{-1}(\lambda)$ and since conditionally on $J(\lambda)$, the trees $\overline{\mathcal{T}}_i(\lambda)$, $i \in J(\lambda)$ are independent with the same distribution Δ_λ, we get

$$\mathbb{E}\left[\exp\left(-\sum_{i\in J(\lambda)} R(\overline{\mathcal{T}}_i(\lambda))\mathbf{1}_{\{h(\overline{\mathcal{T}}_i(\lambda))>\epsilon\}}\right)\right] \tag{82}$$

$$= \exp\left(-a\psi^{-1}(\lambda)\int_{\{h(\overline{\mathcal{T}})>\epsilon\}} \Delta_\lambda(d\overline{\mathcal{T}})\left(1-e^{-R(\overline{\mathcal{T}})}\right)\right) . \tag{83}$$

Now observe that a.s. for any $i \in J$

$$\lim_{\lambda\to\infty} \delta(\overline{\mathcal{T}}_i, \overline{\mathcal{T}}_i(\lambda)) = 0 \quad \text{and} \quad \lim_{\lambda\to\infty} \uparrow h(\overline{\mathcal{T}}_i(\lambda)) = h(\overline{\mathcal{T}}_i) . \tag{84}$$

Since $\mathcal{F}$ is locally compact, there are only finitely many $\mathcal{T}_i$'s such that $h(\overline{\mathcal{T}}_i) > \epsilon$. Thus (84) implies

$$\lim_{\lambda\to\infty} \sum_{i\in J(\lambda)} R(\overline{\mathcal{T}}_i(\lambda))\mathbf{1}_{\{h(\overline{\mathcal{T}}_i(\lambda))>\epsilon\}} = \sum_{i\in J} R(\overline{\mathcal{T}}_i)\mathbf{1}_{\{h(\overline{\mathcal{T}}_i)>\epsilon\}}, \tag{85}$$

and (a) follows from Proposition 5.5 and (82).

It remains to prove (b): An elementary computation based on (23) with $\varphi = \varphi_{\mu_0,\lambda}$ implies that

$$\Delta^r_{\mu_0,\lambda}\left(h(\overline{\mathcal{T}}) > \epsilon\right) = 1 - \exp\left(-r(\psi^{-1}(\lambda) - \psi^{-1}(\mu_0))(1 - e^{-v_{\mu_0,\lambda}(\epsilon)})\right)$$

where $v_{\mu_0,\lambda}$ satisfies the following equation

$$\int_{(\psi^{-1}(\lambda)-\psi^{-1}(\mu_0))(1-e^{-v_{\mu_0,\lambda}(\epsilon)})}^{\psi^{-1}(\lambda)-\psi^{-1}(\mu_0)} \frac{du}{\psi_{\mu_0}(u)} = \epsilon \, .$$

Thus,

$$\lim_{\lambda\to\infty} \uparrow \Delta^r_{\mu_0,\lambda}\left(h(\overline{\mathcal{T}}) > \epsilon\right) = 1 - e^{-rv_{\mu_0}(\epsilon)} = P^r_{\mu_0}\left(h(\overline{\mathcal{T}}) > \epsilon\right) \ , \tag{86}$$

where v_{μ_0} satisfies (79). By (58), for any $r > 0$ we get

$$\lim_{\lambda\to\infty} \int_{\{h(\overline{\mathcal{T}})>\epsilon\, ;\, \overline{\mathcal{T}}\neq\overline{\{\rho\}}\}} \Delta^r_{\mu_0,\lambda}(d\overline{\mathcal{T}})\left(1-e^{-R(\overline{\mathcal{T}})}\right) = \int_{\{h(\overline{\mathcal{T}})>\epsilon\, ;\, \overline{\mathcal{T}}\neq\overline{\rho}\}} P^r_{\mu_0}(d\overline{\mathcal{T}})\left(1-e^{-R(\overline{\mathcal{T}})}\right). \tag{87}$$

Now by (86)

$$\int_{\{h(\overline{\mathcal{T}})>\epsilon\, ;\, \overline{\mathcal{T}}\neq\overline{\rho}\}} \Delta^r_{\mu_0,\lambda}(d\overline{\mathcal{T}})\left(1-e^{-R(\overline{\mathcal{T}})}\right) \leq P^r_{\mu_0}\left(h(\overline{\mathcal{T}}) > \epsilon\right) = 1 - e^{-rv_{\mu_0}(\epsilon)}.$$

Now note that

$$\int_{(0,\infty)} \Pi(dr) r e^{-r\psi^{-1}(\mu_0)}(1 - e^{-rv_{\mu_0}(\epsilon)}) \ < \ \infty \ ,$$

which implies (b) by (87) and the dominated convergence theorem. This completes the proof of the lemma. ■

References

[1] R. Abraham. Un arbre aléatoire infini associé à l'excursion Brownienne. In *Sém. de Proba.*, volume XXVI, pages 374–397. Springer, Berlin, 1992.

[2] D. Aldous and J. Pitman. Tree-valued Markov chains derived from Galton-Watson processes. *Ann. Inst. H. Poincaré.*, 34:637–686, 1998.

[3] D. J. Aldous. The continuum random tree I. *Ann. Probab.*, 19:1–28, 1991.

[4] K. Athreya and P. Ney. *Branching process.* Number 196 in Grundlehren der Mathematischen Wissenschaften. Springer, 1972.

[5] N. H. Bingham. Continuous branching processes and spectral positivity. *Stochastic Process. Appl.*, 4:217–242, 1976.

[6] Y. Burago, D. Burago and S. Ivanov. *A Course in Metric Geometry*, volume 33. AMS, Boston, 2001.

[7] Hobson D.G. Marked excursions and random trees. In Springer, editor, *Lecture Notes Math.*, volume 1729 of *Séminaire de Probabilités XXXIV*, pages 289–301, 2000.

[8] A. Dress. Trees, tight extensions of metric spaces, and the cohomological dimension of certain groups: A note on combinatorial properties of metric spaces. *Adv. Math.*, 53:321–402, 1984.

[9] A. Dress, V. Moulton, and W. Terhalle. T-theory: an overview. *European J. Combin.*, 17:161–175, 1996.

[10] A. Dress and W. Terhalle. The real tree. *Adv. Math.*, 120:283–301, 1996.

[11] T. Duquesne and J-F. Le Gall. *Random Trees, Lévy Processes and Spatial Branching Processes.* Astérisque no 281, 2002.

[12] T. Duquesne and J-F. Le Gall. Probabilistic and fractal aspects of Lévy trees. *To appear in Probab. Theorey and Rel. Fields*, 2004.

[13] S. Evans. Snakes and spiders: Brownian motion on real trees. *Probab. Theory Related Fields*, 117(3):361–386, 2000.

[14] S.N. Evans, J. Pitman, and A. Winter. Rayleigh processes, real trees, and root growth with re-grafting. *To appear in Probab. Th. Rel. Fields*, 2005.

[15] S.N. Evans and A. Winter. Subtree prune and re-graft: a reversible real tree valued Markov process. *preprint*, 2005.

[16] W. Feller. *An Introduction to Probability Theory and Its Applications, Vol. II, sec. ed.* Wiley, New York., 1971.

[17] J. Geiger and Kauffmann L. The shape of large Galton-Watson trees with possibly infinite variance. *Rand. Struct. Alg.*, 25(3):311–335, 2004.

[18] M. Gromov. *Metric Structures for Riemannian and non-Riemannian Spaces.* Progress in Mathematics. Birkhuser, 1999.

[19] B.M. Hambly and T.J. Lyons. Uniqueness for the signature of a path of bounded variation and continuous analogues for the free group. *Preprint*, 2004.

[20] M. Jirina. Stochastic branching processes with continous state-space. *Czech. Math. J.*, 8:292–313, 1958.

[21] J. Lamperti. Continuous-state branching processes. *Bull. Amer. Math. Soc.*, 73:382–386, 1967.

[22] J. Lamperti. The limit of a sequence of branching processes. *Z. Wahrsch. Verw. Gebiete*, 7:271–288, 1967.

[23] J. Lamperti. Limiting distributions of branching processes. In *Fifth Berkeley Symp.*, volume II, Part 2, pages 225–241, 1967.

[24] J-F. Le Gall and Y. Le Jan. Branching processes in Lévy processes: the exploration process. *Ann. Probab.*, 26-1:213–252, 1998.

[25] J-F. Le Gall and Y. Le Jan. Branching processes in Lévy processes: Laplace functionals of snakes and superprocesses. *Ann. Probab.*, 26:1407–1432, 1999.

[26] Y. Le Jan. Superprocesses and projective limits of branching Markov processes. *Ann. Inst. H.Poincaré* , 27:91–106, 1991.

[27] R. Lyons and Y Peres. *Probability on Trees and Networks.* Cambridge University Press, in progress. Current version published on the web at http://php.indiana.edu/∼rdlyons, 2004.

[28] J. Neveu. Arbres et processus de Galton-Watson. *Ann. Inst. H. Poincaré*, 26:199–207, 1986.

[29] F. Paulin. Topologie de Gromov équivariante, structures hyperboliques et arbres réels. *Invent. Math.*, 94(1):53–80, 1988.

[30] F. Paulin. The Gromov topology on real-trees. *Topology Appl.*, 32(3):197–221, 1989.

[31] J. Pitman. *Combinatorial Stochastic Processes.* Lecture Notes for St. Flour Course. Springer, 2002.

[32] J. Pitman and M. Winkel. Growth of the Brownian forest. *to appear in Ann. Probab.*, 2005.

[33] P. Salminen. Cutting markovian trees. *Ann.Acad. Scient. Fenn. A. I. Math.*, 17:123–137, 1992.