Electronic foreign exchange markets and level passage events of multivariate subordinators*

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Summary. We formulate a model for electronic foreign exchange markets suggesting subordinators to represent sellers' and buyers' offers. Its analysis naturally leads to studying level passage events. The classical level passage event concerns the joint law of the time, height and jump size observed when a real-valued stochastic process first exceeds a given level h. We provide an up to date treatment when this process is a subordinator, and extend these results to multivariate subordinators. More precisely, given a multivariate subordinator, we describe the events when certain components pass individual levels.

Keywords. Subordinators, passage times, Poisson point process techniques, foreign exchange market model

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1 Introduction

Electronic foreign exchange markets collect offers to sell/buy quantities of a currency to individual minimal/maximal prices. Transactions take place as soon as a buyer's price is higher than a seller's price. What remains are the unmet offers that are commonly represented graphically as in Figure 1. We are interested in these two price-quantity processes which indicate to a potential buyer/seller what offers one can currently realise on the market. We suggest to model these processes by subordinators, i.e. monotonic

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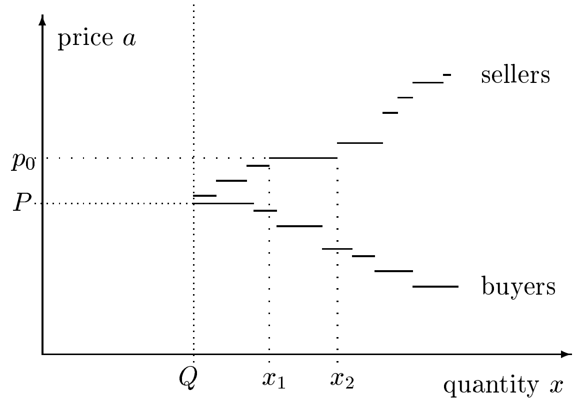

Figure 1: Offers on the market

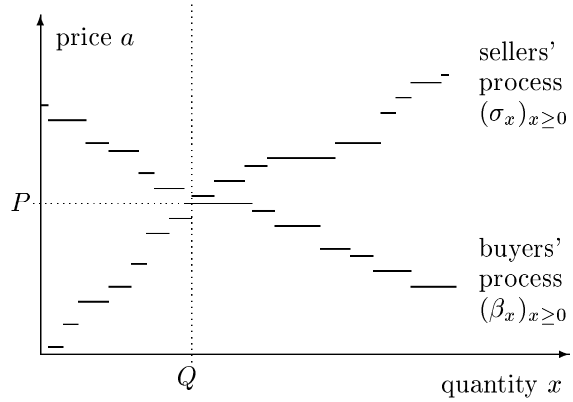

Figure 2: + realised transactions

processes with stationary independent increments. We also model the virtual left parts of the processes that correspond to the realised transactions, as shown in Figure 2.

Mathematically, the crossing in Figure 2 is related to the level passage event as studied in the end of the 1960s by Gusak [5] and Kesten [6] for subordinators $X = (X_a)_{a\geq 0}$, and more general Lévy processes. These studies involved times

$$T_h = \inf\{a \geq 0 : X_a > h\}, \qquad h \geq 0, \tag{1}$$

and their associated heights $X(T_h)$. Gusak's and Kesten's cumbersome approximation techniques are no longer necessary as is seen in more recent accounts like Section III.2 in Bertoin's book [2] where Poisson point process techniques are applied to the process of jumps of X. In a preliminary section we use these techniques to provide a complete characterisation of the level passage event at fixed levels h or independent exponential levels $\tau(q)$. By 'complete' we mean that we give the joint law not only of T_h and $X(T_h)$ but also $X(T_h-)$, and in the case of an exponential level the level itself. Clearly this also determines quantities known as undershoot, overshoot, jump size at the passage time etc.

The main purpose of this article is the generalisation to a multivariate setting. On the one hand, this allows to deal with the passage of two subordinators that move towards each other as in Figure 2. On the other hand, we describe level passage events of several dependent subordinators. For precise statements we refer to Sections 2, 3 and 4, where we treat the univariate case, one passage event involving several subordinators and several passage events for multivariate subordinators, respectively.

Let us here describe the economic model in some more detail. We consider the two trajectories in Figure 2 as being realisations of stochastic processes $(\sigma_x)_{x\geq 0}$ and $(\beta_x)_{x\geq 0}$. A step of σ at height p_0 and of length $q = x_2 - x_1$ corresponds to a seller who would like to sell q units of the currency at a price at least p_0, and buyers are collected similarly by β. Therefore, σ_x is the price at which x units are supplied to the market. β_x is the price at which x units are demanded by the market. For simplicity, we assume $\sigma_0 = 0$ and $\beta_0 > 0$. Our primary interest is in the 'crossing' quantity and prices

$$Q := \inf\{x \geq 0 : \sigma_x > \beta_x\} \qquad \text{and} \qquad P := \beta_Q \leq \sigma_Q = P'$$

representing the quantity actually traded and the range of the (collective) market price.

More precisely, our model consists in specifying σ and β to be right-continuous inverses of subordinators - sometimes also called local time processes since they arise as local times of Markov processes, cf. e.g. Rogers and Williams [10]. In other words, we choose

$$S_a = \inf\{x \geq 0 : \sigma_x > a\}, \quad a \geq 0,$$

to be any (increasing) subordinator starting from zero, and $(B_0 - B_a)_{0 \leq a \leq \beta_0}$, where

$$B_a = \inf\{x \geq 0 : \beta_x < a\}, \quad 0 \leq a \leq \beta_0,$$

another such process - with finite lifetime but this is of no importance, since we focus on the crossing event that always happens before. The crossing price and quantity can be given in terms of the two subordinators as

$$P = \inf\{a \geq 0 : B_a - S_a < 0\} = \inf\{a \geq 0 : S_a + (B_0 - B_a) > B_0\}$$
$$\text{and} \qquad Q = S_P \geq B_P = Q'.$$

We assume S and B independent. Then $X_a = S_a + (B_0 - B_a)$ is another subordinator and $P = T_h$ its first passage 'time' over the level $h = B_0$ given in (1) whereas Q and Q' are related to the passage 'height' of X. We obtain their explicit joint law as an application of theoretical results in Section 3. The multivariate analogues of Section 4 were obtained in the aim to model a temporal evolution, but more work has to be done in this direction to obtain a reasonably flexible dynamic. This seems to necessarily lead away from closed form expressions as they are presented here. We postpone the further discussion of the model to Section 5 where we also address the interpretation of parameters.

2 Preliminaries on the level passage event of a single one-dimensional subordinator

Let $X = (X_a)_{a \geq 0}$ be a subordinator, i.e. an increasing process with stationary independent increments, whose law is represented by the Lévy-Khintchine formula for its Laplace exponent

$$-\frac{\ln E(e^{-qX_a})}{a} = \Phi(q) = bq + \int_{(0,\infty)} (1 - e^{-qx})\Pi(dx), \tag{2}$$

where we refer to $b \geq 0$ as the drift coefficient and to Π as the Lévy measure. Π is required to integrate $1 \wedge x$ on $(0, \infty)$. Bertoin [3] is a standard reference, also chapter III in Bertoin [2]. Then X has the following structure

$$X_a = ba + \sum_{0 \leq s \leq a} \Delta X_s, \qquad a \geq 0, \qquad \text{where} \quad \Delta X_a = X_a - X_{a-}$$

is the process of jumps of X which is a Poisson point process with intensity measure Π. Note that the integrability condition allows Π to be infinite 'at the origin', i.e. $\Pi(\varepsilon, \infty) \uparrow \infty$ for $\varepsilon \downarrow 0$. If so, this means that X has infinitely many jumps in every interval, but these jumps are summable. If $b = 0$, X is called a pure jump subordinator.

Independent exponential random variables $\tau = \tau(\alpha) \sim Exp(\alpha)$ are often very useful. An example of their occurrence are resolvents

$$V^\alpha(dx) = \int_0^\infty e^{-\alpha a} P(X_a \in dx) da \left(= \frac{1}{\alpha} P(X_{\tau(\alpha)} \in dx) \right), \qquad \alpha \geq 0. \tag{3}$$

It is an immediate consequence that their Laplace transforms are $(\alpha + \Phi(q))^{-1}$. We shall meet resolvents in the context of hitting times

$$H_h = \inf\{a \geq 0 : X_a = h\} \in (0, \infty], \qquad h \geq 0.$$

If $b = 0$, then $H_h = \infty$ a.s. If $b > 0$, then V^α admits a density v^α w.r.t. Lebesgue measure which is continuous on $(0, \infty)$ and satisfies $v^\alpha(0) = v^\alpha(0+) = 1/b$. Furthermore,

$$E\left(\exp\{-\alpha H_h\}\right) = b v^\alpha(h). \tag{4}$$

Cf. Corollary II.18 and Theorem III.5 in [2].

Concerning the passage of a level $h \geq 0$, three basic and three associated quantities are of interest, namely the passage time T_h, the passage height $X(T_h)$ and the jump size $\Delta X(T_h)$ of the level passage, the overshoot o_h, the undershoot u_h and the pre-passage height $X(T_h-)$; mathematically, these are

$$T_h = \inf\{a \geq 0 : X_a > h\}$$
$$o_h = X(T_h) - h, \quad u_h = h - X(T_h-), \quad \Delta_h = \Delta X(T_h) = X(T_h) - X(T_h-) = o_h + u_h.$$

If X is a.s. strictly increasing, i.e. if Π is an infinite measure or $b > 0$, then a.s. for all $h \geq 0$, T_h coincides with $\inf\{a \geq 0 : X_a \geq h\}$.

We shall also treat the situation when h is replaced by an independent exponential height $\tau = \tau(q)$ with parameter $q > 0$. Then we can give the joint law of these quantities in terms of the Laplace exponent Φ of X. The central result of this section is the following. It is formulated in three different ways: a) as a distributional identity for a fixed level h, b) as a distributional identity for an exponential level $\tau(q)$ and c) as a Laplace transform identity for an exponential level which can also be viewed as a double Laplace transform of the fixed level distributions.

Theorem 1a) *For all $h > 0$*

$$\begin{aligned} &P(T_h \in da, X(T_h-) \in dy, \Delta_h \in dz) \\ &\quad = \delta_h(dy) P(H_h \in da) \delta_0(dz) + 1_{\{0 \leq y \leq h < y+z\}} P(X_a \in dy) da \Pi(dz). \end{aligned}$$

Theorem 1b) *For all $q > 0$*

$$\begin{aligned} &P(T_{\tau(q)} \in da, X(T_{\tau(q)}-) \in dy, \Delta_{\tau(q)} \in dz) \\ &\quad = P(H_y \in da) q e^{-qy} dy \delta_0(dz) + e^{-qy} P(X_a \in dy) da (1 - e^{-qz}) \Pi(dz). \end{aligned}$$

Theorem 1c) *For all $q > 0$, $\kappa > -q$, $\lambda > -q - \kappa$, $\alpha > -\Phi(q + \kappa + \lambda)$, and $\nu \geq 0$*

$$E\left(\exp\left\{-\kappa \tau(q) - \alpha T_{\tau(q)} - \lambda X(T_{\tau(q)}-) - \nu \Delta_{\tau(q)}\right\}\right) = \frac{q\left(\Phi(q + \kappa + \nu) - \Phi(\nu)\right)}{(q + \kappa)\left(\alpha + \Phi(q + \kappa + \lambda)\right)}.$$

Proof: a) We first generalise Proposition III.2 in [2] to include the time component. His argument, using the Poisson point process technique, yields on $\{0 \le y \le h < y+z, 0 < a < \infty\}$

$$P(T_h \in da, X(T_h-) \in dy, \Delta_h \in dz) = P(X_a \in dy) da \Pi(dz).$$

The full argument can also be found as a special case in the proofs of generalisations in this paper.

The term for $z = 0$ is obtained as follows:

$$P(T_h \in da, X(T_h-) \in dy, \Delta_h = 0) = \delta_h(y) P(T_h \in da, X(T_h) = h) = \delta_h(y) P(H_h \in da).$$

b) We calculate for bounded real functions f, g, and ℓ

$$\begin{aligned}
&E\left(f(T_\tau) g\left(X(T_\tau-)\right) \ell(\Delta_\tau)\right) \\
&\quad = \int_0^\infty q e^{-qh} \int_{[0,\infty)} f(a) g(h) \ell(0) P(H_h \in da) dh \\
&\qquad + \int_0^\infty q e^{-qh} \int_0^\infty \int_{[0,h]} \int_{(h-y,\infty)} f(a) g(y) \ell(z) \Pi(dz) P(X_a \in dy) da\, dh \\
&\quad = \int_0^\infty \int_{[0,\infty)} f(a) g(y) \ell(0) P(H_y \in da) q e^{-qy} dy \\
&\qquad + \int_{(0,\infty)} \int_0^\infty \int_{[0,\infty)} f(a) g(y) \ell(z) e^{-qy} (1 - e^{-qz}) P(X_a \in dy) da \Pi(dz).
\end{aligned}$$

c) We deduce from a) by integration. Assume first $\kappa = 0$. From (3) and (4) we obtain

$$\begin{aligned}
E\left(\exp\left\{-\alpha T_\tau - \lambda X(T_\tau) - \nu \Delta_\tau\right\} 1_{\{\Delta_\tau = 0\}}\right) &= \int_0^\infty q e^{-(q+\lambda)h} E\left(e^{-\alpha H_h}\right) dh \\
&= \frac{bq}{\alpha + \Phi(q+\lambda)}.
\end{aligned}$$

The calculation of the second term involves Fubini's theorem and the Lévy-Khintchine representation (2) only:

$$\begin{aligned}
&E\left(\exp\left\{-\alpha T_\tau - \lambda X(T_\tau) - \nu \Delta_\tau\right\} 1_{\{\Delta_\tau > 0\}}\right) \\
&\quad = \int_0^\infty q e^{-qh} \int_0^\infty e^{-\alpha a} \int_{[0,h]} e^{-\lambda y} \int_{(h-y,\infty)} e^{-\nu z} \Pi(dz) P(X_a \in dy) da\, dh \\
&\quad = \int_0^\infty e^{-\alpha a} \int_{[0,\infty)} e^{-\lambda y} \int_{(0,\infty)} e^{-\nu z} \int_y^{y+z} q e^{-qh} dh \Pi(dz) P(X_a \in dy) da \\
&\quad = \frac{\Phi(q+\nu) - \Phi(\nu) - bq}{\alpha + \Phi(q+\lambda)}.
\end{aligned}$$

Finally, we deduce

$$E\left(\exp\left\{-\kappa \tau(q) - \alpha T_{\tau(q)} - \lambda X(T_{\tau(q)}-) - \nu \Delta_{\tau(q)}\right\}\right)$$

$$
\begin{aligned}
&= q\int_0^\infty e^{-qh}e^{-\kappa h}E\left(\exp\{-\alpha T_h - \lambda X(T_h-) - \nu\Delta_h\}\right)dh \\
&= \frac{q}{q+\kappa}E\left(\exp\left\{-\alpha T_{\tau(q+\kappa)} - \lambda X(T_{\tau(q+\kappa)}-) - \nu\Delta_{\tau(q+\kappa)}\right\}\right) \\
&= \frac{q\left(\Phi(q+\kappa+\nu)-\Phi(\nu)\right)}{(q+\kappa)\left(\alpha+\Phi(q+\kappa+\lambda)\right)}.
\end{aligned}
$$

□

The complete result concerning c) now follows as a corollary. We also spell out explicitly some more marginals.

Corollary 1 *(i) We have for all $q, \kappa, \alpha, \beta, \gamma, \lambda, \mu, \nu$ such that the right hand side of the following formula is defined*

$$
\begin{aligned}
&E\left(\exp\{-\kappa\tau - \alpha T_\tau - \beta u_\tau - \gamma o_\tau - \lambda X(T_\tau-) - \mu X(T_\tau) - \nu\Delta_\tau\}\right) \\
&\qquad\qquad = \frac{q\left(\Phi(q+\kappa+\mu+\beta+\nu)-\Phi(\gamma+\mu+\nu)\right)}{(q+\kappa+\beta-\gamma)\left(\alpha+\Phi(q+\kappa+\mu+\lambda)\right)}.
\end{aligned}
$$

(ii) In particular $T_\tau \sim Exp(\Phi(q))$, $(T_\tau, X(T_\tau-))$ is independent of $(u_\tau, o_\tau, \Delta_\tau)$ and

$$
\begin{aligned}
E\left(\exp\{-\alpha T_\tau\}\right) &= \frac{\Phi(q)}{\alpha+\Phi(q)} \\
E\left(\exp\{-\mu X(T_\tau)\}\right) &= \frac{\Phi(q+\mu)-\Phi(\mu)}{\Phi(q+\mu)} \\
E\left(\exp\{-\nu\Delta_\tau\}\right) &= \frac{\Phi(q+\nu)-\Phi(\nu)}{\Phi(q)} \\
E\left(\exp\{-\gamma o_\tau\}\right) &= \frac{q\left(\Phi(q)-\Phi(\gamma)\right)}{(q-\gamma)\Phi(q)} \\
E\left(\exp\{-\beta u_\tau\}\right) &= \frac{q\Phi(q+\beta)}{(q+\beta)\Phi(q)} \\
E\left(\exp\{-\lambda X(T_\tau-)\}\right) &= \frac{\Phi(q)}{\Phi(q+\lambda)} \\
E\left(\exp\{-\alpha T_\tau - \lambda X(T_\tau-)\}\right) &= \frac{\Phi(q)}{\alpha+\Phi(q+\lambda)} \\
E\left(\exp\{-\beta u_\tau - \gamma o_\tau\}\right) &= \frac{q\left(\Phi(q+\beta)-\Phi(\gamma)\right)}{(q+\beta-\gamma)\Phi(q)}.
\end{aligned}
$$

The $(T_h, X(T_h))$ statement is originally due to Gusak [5] and Kesten [6], even in a more general Lévy process setting. For related results at exponential heights cf. Bertoin [2], Exercise VI.1, Theorem VII.4, and [3] Lemma 1.11.

We shall now study more thoroughly the independence of $(T_\tau, X(T_\tau-))$ from (u_τ, o_τ). It can in fact be strengthened as follows.

Proposition 1 *$(X_a)_{0\le a<T_\tau}$ is independent from (u_τ, o_τ).*

Proof: This is once more due to the Poisson point process property of the process $J_a = X_a - X_{a-}$ of jumps of X. Choose an arbitrary functional f on the path space, $\beta > 0$ and $\gamma > 0$. Assume first $b = 0$. Then

$$
\begin{aligned}
& E\left(f\left((X_s)_{0\le s<T_\tau}\right)\exp\{-\beta u_\tau - \gamma o_\tau\}\right) \\
&= E\left(\sum_{a\ge 0} f\left((X_s)_{0\le s<a}\right)\exp\{-\beta(\tau - X_{a-}) - \gamma(X_{a-} + J_a - \tau)\}\, 1_{\{X_{a-}\le x, J_a > \tau - X_{a-}\}}\right) \\
&= \int_0^\infty E\left(f\left((X_s)_{0\le s<a}\right) 1_{\{X_{a-}\le\tau\}}\int_{(0,\infty)} e^{-\beta(\tau - X_{a-})-\gamma(X_{a-}+z-\tau)} 1_{\{z>\tau - X_{a-}\}}\Pi(dz)\right) da \\
&= \int_0^\infty \int_{[0,\infty)} E\left(f\left((X_s)_{0\le s<a}\right)\middle|\, X_{a-} = y\right) e^{-(\gamma-\beta)y} \\
&\qquad \int_{(0,\infty)} e^{-\gamma z}\int_y^{y+z} q e^{-qh} e^{-(\beta-\gamma)h} dh \Pi(dz) P(X_{a-}\in dy) da \\
&= \left(\int_0^\infty \int_{[0,\infty)} E\left(f\left((X_s)_{0\le s<a}\right)\middle|\, X_{a-} = y\right) e^{-qy} P(X_{a-}\in dy) da\right) \\
&\qquad \frac{q}{q+\beta-\gamma}\int_{(0,\infty)}\left(\left(1 - e^{-(q+\beta)z}\right) - \left(1 - e^{-\gamma z}\right)\right)\Pi(dz)
\end{aligned}
$$

shows the asserted independence. If $b > 0$, denoting the range of X by $\mathcal{R}$, we obtain from the second line an additional term, cf. Corollary IV.6 in [2]

$$
\begin{aligned}
E\left(f\left((X_s)_{0\le s<T_\tau}\right) 1_{\{\Delta_\tau = 0\}}\right) &= \int_0^\infty q e^{-qh} E\left(f\left((X_s)_{0\le s<T_h}\right) 1_{\{h\in\mathcal{R}\}}\right) dh \\
&= E\left(\int_0^\infty f\left((X_s)_{0\le s<T_h}\right) e^{-qX(T_h-)} qb\, dT_h\right) \\
&= E\left(\int_0^\infty f\left((X_s)_{0\le s<a}\right) e^{-qX_{a-}} qb\, da\right) \\
&= qb\int_0^\infty E\left(f\left((X_s)_{0\le s<a}\right) e^{-qX_{a-}}\right) da
\end{aligned}
$$

which cancels with an additional occurring from the last line above due to

$$
\int_{(0,\infty)}\left(\left(1 - e^{-(q+\beta)z}\right) - \left(1 - e^{-\gamma z}\right)\right)\Pi(dz) = \Phi(q+\beta) - \Phi(\gamma) - b(q+\beta-\gamma).
$$

□

The next two sections extend the results presented here to two different multivariate settings. The first is to consider $X = X^{(1)} + \ldots + X^{(m)}$ as the sum of m *independent* subordinators $X^{(1)}, \ldots, X^{(m)}$. The additional information asked for are then the individual heights at T_h and which of the components jumps at the passage time. The special case $m = 2$ appears in the foreign exchange market model.

The second way to generalise to a multivariate setting is by taking a *multivariate* subordinator $X = (X^{(1)}, \ldots, X^{(n)})$ with *dependent* components, and simultaneously study the times and associated quantities when the components $X^{(j)}$ pass levels h_j, $j = 1, \dots n$. The most general example of a multivariate subordinator is obtained by chosing its deterministic drift vector and the Lévy measure which is the measure gouverning the rates and height distributions of its Poissonian jumps like in the one-dimensional case (2). More particular examples have been discussed by Barndorff-Nielsen *et al.* [1]. Specifically, multivariate subordinators can be obtained by superposition, i.e. linear transformation of a vector of independent subordinators, or by subordination, i.e. e.g. time-changing a vector of independent subordinators by another independent subordinator.

3 Level passage for a sum of independent subordinators

Let $X = X^{(0)} + X^{(1)} + \ldots + X^{(m)}$ be a subordinator built from a deterministic drift $X_a^{(0)} = ba$, and m independent pure jump subordinators $X^{(1)}, \dots, X^{(m)}$. We study

$$T_h = \inf\{a \geq 0 : X_a > h\}, \qquad h \geq 0,$$

the associated individual heights $X^{(i)}(T_h-)$, $i = 1, \dots, m$, and the jump occuring at T_h - there is only at most one jump since independent subordinators have no common jump times a.s., cf. e.g. Revuz and Yor [9], Proposition XII.(1.5). We use the obvious notation $\Phi^{(i)}$, $\Pi^{(i)}$,... for the Laplace exponent, Lévy measure,... of $X^{(i)}$, $i = 0, \dots, m$. Clearly,

$$\Phi(q) = \sum_{i=0}^{m} \Phi^{(i)}(q) \qquad \text{and} \qquad \Pi(dz) = \sum_{i=1}^{m} \Pi^{(i)}(dz).$$

3.1 Which subordinator performs the passage?

In this subsection we shall not bother about the laws of the quantities introduced but only how the passage takes place. Specifically, we calculate the probabilities that the drift ($i = 0$) or a jump ($i = 1, \dots, m$) of $X^{(i)}$ makes X pass the level. We denote these events by

$$A^{(i)}(h) = \{\Delta X^{(i)}(T_h) > 0\}, \quad i = 1, \dots, m, \quad A^{(0)}(h) = \left(\bigcup_{i=1}^{m} A^{(i)}(h)\right)^c.$$

The 0-resolvent measure $V = V^0$ of X, cf. (3), is called the renewal measure. Recall that $b > 0$ entails the existence of a Lebesgue density v, continuous on $(0, \infty)$. We also introduce the tails $\bar{\Pi}^{(i)}(x) = \Pi^{(i)}(x, \infty)$ of the Lévy measures of $X^{(i)}$.

Proposition 2 *a) For all $h > 0$*

$$P(A^{(0)}(h)) = bv(h) \quad \text{and} \quad P(A^{(i)}(h)) = \bar{\Pi}^{(i)} * V(h), \quad i = 1, \dots, m,$$

where $*$ is the convolution of a function with a measure.

b) For $\tau = \tau(q)$, $q > 0$,

$$P(A^{(i)}(\tau)) = \frac{\Phi^{(i)}(q)}{\Phi(q)}, \quad i = 0, \dots, m.$$

Proof: a) The first is the probability that X hits h, cf. (4).

For the second probability we repeat the proof of Proposition III.2 in [2] to obtain on $\{0 \le x \le h < x+z\}$

$$P\left(X(T_h-) \in dx, X^{(i)}(T_h) - X^{(i)}(T_h-) \in dz\right) = \Pi^{(i)}(dz)V(dx)$$

and therefore

$$P\left(X^{(i)}(T_h) - X^{(i)}(T_h-) > 0\right) = \int_{[0,h]} \bar{\Pi}^{(i)}(h-x)V(dx) = \bar{\Pi}^{(i)} * V(h).$$

b) We integrate the formulas in a) w.r.t. the exponential law and use the facts that the Laplace transforms of V and $\bar{\Pi}^{(i)}(h)dh$ are $1/\Phi(q)$ and $\Phi^{(i)}(q)/q$, respectively, cf. [2] section III.1. □

The special case $m = 2$ can be interpreted differently. Also, we see that supposing the only drift term is in $X^{(0)}$ is for notational ease.

Example 1 (Passage of two subordinators moving towards each other) Let Y and Z be two independent subordinators, with drift coefficient b_{Y+Z}, Lévy measures Π_Y and Π_Z, etc. associated in the obvious notation. Consider now Y as it is but $h - Z$ the subordinator starting from h and moving downwards, towards Y. The passage event of $X = Y + Z$ over level h is the event that Y and $h - Z$ cross. The preceding proposition yields

a) for all $h > 0$

$$\begin{aligned} P(A_0(h)) &= P\left(Y(T_h-){=}Y(T_h){=}h{-}Z(T_h){=}h{-}Z(T_h-)\right) = b_{Y+Z}v_{Y+Z}(h) \\ P(A_Y(h)) &= P\left(Y(T_h-){<}h{-}Z(T_h){=}h{-}Z(T_h-){<}Y(T_h)\right) = \bar{\Pi}_Y * V_{Y+Z}(h) \\ P(A_Z(h)) &= P\left(h{-}Z(T_h){<}Y(T_h-){=}Y(T_h){<}h{-}Z(T_h-)\right) = \bar{\Pi}_Z * V_{Y+Z}(h). \end{aligned}$$

and b) for $\tau = \tau(q)$, $q > 0$,

$$\begin{aligned} P(A_0(\tau)) &= P\left(Y(T_\tau-){=}Y(T_\tau){=}\tau{-}Z(T_\tau){=}\tau{-}Z(T_\tau-)\right) = \frac{qb_{Y+Z}}{\Phi_{Y+Z}(q)} \\ P(A_Y(\tau)) &= P\left(Y(T_\tau-){<}\tau{-}Z(T_\tau){=}\tau{-}Z(T_\tau-){<}Y(T_\tau)\right) = \frac{\Phi_{Y-b_Y I}(q)}{\Phi_{Y+Z}(q)} \\ P(A_Y(\tau)) &= P\left(\tau{-}Z(T_\tau){<}Y(T_\tau-){=}Y(T_\tau){<}\tau{-}Z(T_\tau-)\right) = \frac{\Phi_{Z-b_Z I}(q)}{\Phi_{Y+Z}(q)}, \end{aligned}$$

where I denotes the identity process $I_t = t$.

3.2 The joint law of the passage variables

We now turn to the joint law of the involved quantities. These are principally the passage time T_h, the pre-passage heights $X^{(i)}(T_h-)$, $i=1,\ldots,m$, the jump size Δ_h and the information which subordinator performs the passage, i.e. which of the events $A^{(i)}$, $i=0,\ldots,m$, occurs. When h is replaced by an independent exponential height τ, this height is added to the list. Note however, that on $A^{(0)}$ there is no jump at time T_τ, hence $\tau = X^{(0)}(T_\tau-)+\ldots+X^{(m)}(T_\tau-) = X^{(0)}(T_\tau)+\ldots+X^{(m)}(T_\tau)$ is determined by $X^{(0)}(T_\tau-)=bT_\tau, X^{(1)}(T_\tau-),\ldots,X^{(m)}(T_\tau-)$. From these principal quantities we can derive e.g. overshoot, undershoot and passage heights by linear transformations.

In this setting we formulate the main theorem of this section.

Theorem 2a) *For all $h>0$ and $j=1,\ldots,m$*

$$\begin{aligned}
&P(T_h\in da, X^{(i)}(T_h-)\in dx_i, i=1,\ldots,m, \Delta_h\in dz, A^{(0)})\\
&\qquad = P(X_a^{(i)}\in dx_i, i=1,\ldots,m|X_a=h)P(H_h\in da)\delta_0(dz)\\
&P(T_h\in da, X^{(i)}(T_h-)\in dx_i, i=1,\ldots,m, \Delta_h\in dz, A^{(j)})\\
&\qquad = 1_{\{x_1+\ldots+x_m\le h<x_1+\ldots+x_m+z\}}\left(\prod_{i=1}^m P(X_a^{(i)}\in dx_i)\right) da\Pi^{(j)}(dz).
\end{aligned}$$

Theorem 2b) *For all $q>0$ and $j=1,\ldots,m$*

$$\begin{aligned}
&P(T_\tau\in da, X^{(i)}(T_\tau-)\in dx_i, i=1,\ldots,m, \Delta_\tau\in dz, A^{(0)})\\
&\qquad = \left(\prod_{i=1}^m e^{-qx_i}P(X_a^{(i)}\in dx_i)\right) da\, bq\delta_0(dz)\\
&P(T_\tau\in da, X^{(i)}(T_\tau-)\in dx_i, i=1,\ldots,m, \Delta_\tau\in dz, A^{(j)})\\
&\qquad = \left(\prod_{i=1}^m e^{-qx_i}P(X_a^{(i)}\in dx_i)\right) da(1-e^{-qz})\Pi^{(j)}(dz).
\end{aligned}$$

Theorem 2c) *For all $q>0$, $\kappa\ge 0$, $\alpha\ge 0$ $\xi_1\ge 0$,..., $\xi_m\ge 0$, $\nu\ge 0$, $j=0,\ldots,m$*

$$\begin{aligned}
&E\left(\exp\left\{-\kappa\tau-\alpha T_\tau-\xi_1X^{(1)}(T_\tau-)-\ldots-\xi_mX^{(m)}(T_\tau-)-\nu\Delta_\tau\right\}1_{A^{(j)}}\right)\\
&\qquad = \frac{q\left(\Phi^{(j)}(q+\kappa+\nu)-\Phi^{(j)}(\nu)\right)}{(q+\kappa)\left(\alpha+\Phi^{(1)}(q+\kappa+\xi_1)+\ldots+\Phi^{(m)}(q+\kappa+\xi_m)\right)}.
\end{aligned}$$

Proof: a) The first law is immediate since $\{H_h=a\}=\{X_a+Y_a=h\}$ for all $h>0$, $a>0$.

The second is obtained like in Theorem 1b) by verifying for f, g, ℓ bounded measurable functions via the compensation formula for the Poisson point process $J_a^{(j)}=X_a^{(j)}-X_{a-}^{(j)}$

of jumps of $X^{(j)}$. Note that this process ignores the jumps of $X^{(i)}$ for $i \neq j$.

$$
\begin{aligned}
&E\left(f(T_h)g(X^{(1)}(T_h-),\ldots,X^{(m)}(T_h-))\ell(\Delta_h)1_{A^{(j)}}\right)\\
&\quad= E\left(\sum_{a\geq 0} f(a)g(X^{(1)}_{a-},\ldots,X^{(m)}_{a-})\ell(J^{(j)}_a)1_{\{X_{a-}\leq h, J_a>h-X_{a-}\}}\right)\\
&\quad= \int_0^\infty f(a)E\left(g(X^{(1)}_{a-},\ldots,X^{(m)}_{a-})1_{\{X_{a-}\leq h\}}\int_{(0,\infty)}\ell(z)1_{\{z>h-X_{a-}\}}\Pi^{(j)}(dz)\right)da\\
&\quad= \int_0^\infty \int_{0\leq x_1+\ldots+x_m\leq h<x_1+\ldots+x_m+z} f(a)g(x_1,\ldots,x_m)\ell(z)\left(\prod_{i=1}^m P(X^{(i)}_a\in dx)\right)\Pi^{(j)}(dz)da.
\end{aligned}
$$

b) We verify the first law by fixing f and g arbitrary bounded measurable functions and using Corollary IV.6 in [2] like in proof of our Proposition 1:

$$
\begin{aligned}
&E\left(f(T_\tau)g(X^{(1)}(T_\tau-),\ldots,X^{(m)}(T_\tau-))1_{\{X(T_\tau-)=\tau\}}\right)\\
&\quad= \int_0^\infty qe^{-qh}E\left(f(T_h)g(X^{(1)},\ldots,X^{(m)}(T_h-))1_{\{X(T_h-)=h\}}\right)dh\\
&\quad= E\left(\int_0^\infty qe^{-qX(T_h-)}f(T_h)g(X^{(1)}(T_h-),\ldots,X^{(m)}(T_h-))1_{\{h\in\mathcal{R}\}}dh\right)\\
&\quad= E\left(\int_0^\infty qe^{-qX(T_h-)}f(T_h)g(X^{(1)}(T_h-),\ldots,X^{(m)}(T_h-))b\,dT_h\right)\\
&\quad= E\left(\int_0^\infty qe^{-qX_{a-}}f(a)g(X^{(1)}_{a-},\ldots,X^{(m)}_{a-})b\,da\right)\\
&\quad= \int_0^\infty E\left(qe^{-qX_{a-}}f(a)g(X^{(1)}_{a-},\ldots,X^{(m)}_{a-})\right)b\,da\\
&\quad= \int_0^\infty\int_{[0,\infty)^m} f(a)g(x_1,\ldots,x_m)qbe^{-q(x_1+\ldots+x_m)}\left(\prod_{i=1}^m P(X^{(i)}_a\in dx_i)\right)da.
\end{aligned}
$$

The second formula follows from a) by Fubini's theorem.

c) From b) we calculate for $j=0$

$$
\begin{aligned}
&E\left(\exp\left\{-\alpha T_\tau-\xi_1X^{(1)}(T_\tau-)-\ldots-\xi_mX^{(m)}(T_\tau-)-\nu\Delta_\tau\right\}1_{A^{(0)}}\right)\\
&\quad= \int_0^\infty\int_{[0,\infty)^m} qe^{-q(x_1+\ldots+x_m)}e^{-\alpha a-\xi_1x_1-\ldots-\xi_mx_m}P(X^{(1)}_a\in dx_1)\ldots P(X^{(m)}_a\in dx_m)b\,da\\
&\quad= \int_0^\infty \exp\left\{-\left(\alpha+\Phi^{(1)}(q+\xi_1)+\ldots+\Phi^{(m)}(q+\xi_m)\right)a\right\}qb\,da\\
&\quad= \frac{qb}{\alpha+\Phi^{(1)}(q+\xi_1)+\ldots+\Phi^{(m)}(q+\xi_m)}
\end{aligned}
$$

and for $j=1,\ldots,m$

$$
E\left(\exp\left\{-\alpha T_\tau-\xi_1X^{(1)}(T_\tau-)-\ldots-\xi_mX^{(m)}(T_\tau-)-\nu\Delta_\tau\right\}1_{A^{(j)}}\right)
$$

$$= \left(\int_0^\infty \int_{[0,\infty)^m} e^{-q(x_1+\ldots+x_m)} e^{-\alpha a - \xi_1 x_1 - \ldots - \xi_m x_m} P(X_a^{(1)} \in dx_1) \ldots P(X_a^{(m)} \in dx_m) da\right)$$
$$\left(\int_{(0,\infty)} e^{-\nu z}(1-e^{-qz})\Pi^{(j)}(dz)\right)$$
$$= \frac{\Phi^{(j)}(q+\nu) - \Phi^{(j)}(\nu)}{\alpha + \Phi^{(1)}(q+\xi_1) + \ldots + \Phi^{(m)}(q+\xi_m)}.$$

In both cases, τ can be included in the joint transform in the same way as for Theorem 1c). □

Example 2 (Example 1 continued) In the situation of Example 1 we have e.g.

$$\begin{aligned} &P(T_h \in da, Y(T_h-) \in dy, Z(T_h-) \in dz, \Delta_h \in d\zeta, A_0) \\ &\qquad = P(Y_a \in dy, Z_a \in dz | Y_a + Z_a = h) P(H_h \in da) \delta_0(d\zeta) \\ &P(T_h \in da, Y(T_h-) \in dy, Z(T_h-) \in dz, \Delta_h \in d\zeta, A_Y) \\ &\qquad = 1_{\{y+z \le h < y+z+\zeta\}} P(Y_a \in dy) P(Z_a \in dz) da \Pi_Y(d\zeta) \\ &P(T_h \in da, Y(T_h-) \in dy, Z(T_h-) \in dz, \Delta_h \in d\zeta, A_Z) \\ &\qquad = 1_{\{y+z \le h < y+z+\zeta\}} P(Y_a \in dy) P(Z_a \in dz) da \Pi_Z(d\zeta) \end{aligned}$$

Again, one can easily derive from Theorem 2c) the joint Laplace transform of all involved quantities like in Corollary 1. We leave the details to the reader but mention, that $(T_\tau, X^{(1)}(T_\tau-), \ldots, X^{(m)}(T_\tau-))$ is independent of (u_τ, o_τ), and this extends to

Proposition 3 *$(X_a^{(0)}, \ldots, X_a^{(m)})_{a<T_\tau}$ and $(u_\tau, o_\tau, 1_{A^{(0)}}, \ldots, 1_{A^{(m)}})$ are independent.*

Proof: The proof of Proposition 1 is easily adapted. □

Furthermore, we can characterize the law of $(T_\tau, X^{(1)}(T_\tau-), X^{(m)}(T_\tau-))$ as follows.

Proposition 4 *$(T_\tau, X^{(1)}(T_\tau-), \ldots, X^{(m)}(T_\tau-))$ is infinitely divisible with zero drift coefficient and Lévy measure $a^{-1}e^{-qx_1}P(X_a^{(1)} \in dx_1) \ldots e^{-qx_m}P(X_a^{(m)} \in dx_m)da$.*

Proof: By Theorem 2b), the law in question is given by

$$\begin{aligned} &P\left(T_\tau \in da, X^{(1)}(T_\tau-) \in dx_1, \ldots, X^{(m)}(T_\tau-) \in dx_m\right) \\ &\quad = \Phi(q) e^{-qx_1} P(X_a^{(1)} \in dx_1) \ldots e^{-qx_m} P(X_a^{(m)} \in dx_m) da \\ &\quad = \Phi(q) e^{-a\Phi(q)} P(X_a^{(1,q)} \in dx_1) \ldots P(X_a^{(m,q)} \in dx_m) da \\ &\quad = P\left(\tau(\Phi(q)) \in da, X^{(1,q)}\left(\tau\left(\Phi(q)\right)\right) \in dx_1, \ldots, X^{(m,q)}\left(\tau\left(\Phi(q)\right)\right) \in dx_m\right), \end{aligned}$$

where $X^{(j,q)}$ is constructed from $X^{(j)}$ by an exponential density transformation, cf. Sato [11] Section 33, which is also called temperation, discountation or exponential tilting. The importance of this reformulation is that now the exponential random variable $\tau(\Phi(q))$ is independent of $(X^{(1,q)}, \ldots, X^{(m,q)})$, and we may apply Lemma VI.7 of Bertoin [2] to obtain the Lévy measure

$$\begin{aligned} &a^{-1}e^{-a\Phi(q)} P\left(X_a^{(1,q)} \in dx_1, \ldots, X_a^{(m,q)} \in dx_m\right) da \\ &\quad = a^{-1}e^{-qx_1} P(X_a^{(1)} \in dx_1) \ldots e^{-qx_m} P(X_a^{(m)} \in dx_m) da. \end{aligned}$$

□

Finally, let us give two examples where joint densities can be calculated explicitly. Again, we consider the two-dimensional case, for simplicity. The extensions to more dimensions are obvious.

Example 3 (Gamma subordinators) Let $X_a \sim \Gamma(\vartheta_X a, m_X)$ and $Y_a \sim \Gamma(\vartheta_Y a, m_Y)$. Then we have

$$\frac{P(X_a \in dx)}{dx} = \frac{(xm_X)^{\vartheta_X a}}{\Gamma(\vartheta_X a)}\frac{e^{-m_X x}}{x}, \quad \frac{\Pi_X(dz)}{dz} = \vartheta_X \frac{e^{-m_X z}}{z}, \quad \Phi_X(q) = \vartheta_X \ln\left(1 + \frac{q}{m_X}\right)$$

and for Y likewise. Then by Theorem 2a), $(T_h, X(T_h-), Y(T_h-), \Delta_h)$ has on $[0,\infty)^4$ the joint density

$$1_{\{x+y\leq h\leq x+y+z\}}\frac{m_X^{\vartheta_X a} m_Y^{\vartheta_Y a}}{\Gamma(\vartheta_X a)\Gamma(\vartheta_Y a)} x^{-\vartheta_X a - 1} y^{-\vartheta_Y a - 1} z^{-1} e^{-m_X x - m_Y y}\left(\vartheta_X e^{-m_X z} + \vartheta_Y e^{-m_Y z}\right)$$

with some further simplifications if $\vartheta_X = \vartheta_Y$ and/or $m_X = m_Y$. In particular, if $m_X = m_Y$, $(1_{A_X}, 1_{A_Y})$ is independent of $(T_\tau, X(T_\tau-), Y(T_\tau-), \Delta_\tau)$ and

$$P(A_X) = \frac{\vartheta_X}{\vartheta_X + \vartheta_Y} = r, \quad P(A_Y) = \frac{\vartheta_Y}{\vartheta_X + \vartheta_Y} = 1 - r.$$

This is, in fact the case whenever $b_X = b_Y = 0$ and $(1-r)\Pi_X = r\Pi_Y$, not only in the Gamma case.

Example 4 (Inverse Gaussian subordinators) Let $X_a \sim IG(\delta_X a, \gamma_X)$ and $Y_a \sim IG(\delta_Y a, \gamma_Y)$. This means

$$\frac{P(X_a \in dx)}{dx} = \frac{\delta_X a}{\sqrt{2\pi}} e^{\gamma_X \delta_X a}\frac{e^{-\frac{1}{2}\gamma_X^2 x - \frac{1}{2}\delta_X^2 a^2 x^{-1}}}{x^{3/2}}$$

$$\frac{\Pi_X(dz)}{dz} = \delta_X \frac{\gamma_X}{\sqrt{2\pi}}\frac{e^{-\gamma_X^2 z}}{z^{3/2}}, \quad \Phi_X(q) = \delta_X\left(\sqrt{\gamma_X^2 + 2q} - \gamma_X\right)$$

and for $\gamma_X = \gamma_Y = \gamma$, this yields the following as density for $(T_h, X(T_h-), Y(T_h-), \Delta_h)$ on $[0,\infty)^4$

$$1_{\{x+y\leq h\leq x+y+z\}}\frac{\delta_X\delta_Y\gamma a^2}{(2\pi)^{3/2}} e^{\gamma(\delta_X+\delta_Y)a}\frac{e^{-\frac{1}{2}\gamma^2(x+y+z) - \frac{1}{2}\delta_X^2 a^2 x^{-1} - \frac{1}{2}\delta_Y^2 a^2 y^{-1}}}{x^{3/2}y^{3/2}z^{3/2}}\left(\delta_X + \delta_Y\right).$$

4 General setting of a multivariate subordinator

The law of multivariate subordinators, i.e. stochastic processes $X = (X^{(1)}, \ldots, X^{(n)})$ with stationary independent increments in $[0,\infty)^n$, are characterised by the analogue of (2), the so-called multidimensional Lévy-Khintchine formula of their joint Laplace exponent

$$-\frac{\ln E(e^{-<q,X_a>})}{a} = \Phi(q) = <b,q> + \int_{[0,\infty)^n}(1 - e^{-<q,x>})\Pi(dx), \qquad q\in[0,\infty)^n, \quad (5)$$

where $<a,b> = a_1b_1 + \ldots + a_nb_n$ is the Euclidean inner product of $\mathbb{R}^n$. The vector $b = (b_1, \ldots, b_n)$ is called the drift vector and Π, as in the one-dimensional case called the Lévy measure, does not charge $\{0\}$ but may charge hyperplanes and coordinate axes. The analogous integral condition on Π is that it integrates $1 \wedge |x|$ where $|x|^2 = <x,x>$ defines the Euclidean norm. Again,

$$X_a = ab + \sum_{0 \le s \le a} \Delta X_s, \qquad a \ge 0, \qquad \text{where} \quad \Delta X_a = X_a - X_{a-},$$

and $(\Delta X_a)_{a \ge 0}$ is a Poisson point process with intensity measure Π. As a consequence of a well-known result from the theory of Poisson point processes, X has independent components if and only if Π is concentrated on the coordinate axes, cf. e.g. Revuz and Yor [9], Proposition XII.(1.7). In the sequel we shall suppose $b = 0$ in order not to overload the presentation with unnecessary technicalities. It should be clear from the preceding sections that the case $b > 0$ can be dealt with in a similar way, leading to some more subcases and extra terms. Since the remaining zero-drift process is only moving by jumps we call it a pure jump subordinator.

Level passage events can now be studied in each component, i.e. we can define

$$T_{h_j}^{(j)} = \inf\{a \ge 0 : X_a^{(j)} > h_j\}, \qquad j = 1, \ldots, n,$$

and ask for the probabilistic structure of the vector $(T_{h_1}^{(1)}, \ldots, T_{h_n}^{(n)})$ and associated heights and jumps of the components at these passage times. We also replace $h_1, \ldots, h_n$ by independent exponential heights.

4.1 The case of a bivariate subordinator

First, we focus on the bivariate case and indicate generalisations later. This is convenient since the two-dimensional case captures all intrinsic difficulties whereas the large number of random variables to be characterised in higher dimensions causes notational difficulties. Let therefore (X, Y) be a bivariate subordinator, and

$$T_h^X = \inf\{a \ge 0 : X_a > h\} \qquad \text{and} \qquad T_k^Y = \inf\{a \ge 0 : Y_a > k\}$$

two level passage times. We use the notation $\Pi_{(X,Y)}$, Π_X and Π_Y for the Lévy measures of (X,Y), X and Y, etc. Then we have the following result.

Theorem 3a) *Let (X, Y) be a bivariate pure jump subordinator and $h > 0$ and $k > 0$ two levels. Then*

$$\begin{aligned}
&P\left(T_h^X = T_k^Y, T_h^X \in da, X\left(T_h^X-\right) \in dx, Y\left(T_k^Y-\right) \in dy, \Delta X\left(T_h^X\right) \in db, \Delta Y\left(T_k^Y\right) \in dc\right) \\
&\quad = 1_{\{x \le h < x+b, y \le k < y+c\}} P(X_a \in dx, Y_a \in dy) da \Pi_{(X,Y)}(db, dc) \\
&P\left(T_h^X < T_k^Y, T_h^X \in da, X\left(T_h^X-\right) \in dx, Y\left(T_h^X-\right) \in dy, \Delta X\left(T_h^X\right) \in db, \Delta Y\left(T_h^X\right) \in dc,\right. \\
&\qquad\qquad \left. T_k^Y - T_h^X \in d\tilde{a}, Y\left(T_k^Y-\right) - Y\left(T_h^X\right) \in d\tilde{y}, \Delta Y\left(T_k^Y\right) \in d\tilde{c}\right) \\
&\quad = 1_{\{x \le h < x+b, y+c+\tilde{y} \le k < y+c+\tilde{y}+\tilde{c}\}} P(X_a \in dx, Y_a \in dy) da \Pi_{(X,Y)}(db, dc) \\
&\qquad\qquad P(Y_{\tilde{a}} \in d\tilde{y}) d\tilde{a} \Pi_Y(d\tilde{c}).
\end{aligned}$$

Theorem 3b) *Let (X,Y) be a bivariate pure jump subordinator and $\tau \sim Exp(q)$ and $\sigma \sim Exp(p)$ two independent exponential random variables. Then*

$$\begin{aligned}
&P\left(T^X_\tau = T^Y_\sigma, T^X_\tau \in da, X\left(T^X_\tau-\right) \in dx, Y\left(T^Y_\sigma-\right) \in dy, \Delta X\left(T^X_\tau\right) \in db, \Delta Y\left(T^Y_\sigma\right) \in dc\right)\\
&\quad = e^{-qx-py}\left(1-e^{-qb}\right)\left(1-e^{-pc}\right)P(X_a \in dx, Y_a \in dy)da\Pi_{(X,Y)}(db,dc)\\
&P\left(T^X_\tau < T^Y_\sigma, T^X_\tau \in da, X\left(T^X_\tau-\right) \in dx, Y\left(T^X_\tau-\right) \in dy, \Delta X\left(T^X_\tau\right) \in db, \Delta Y\left(T^X_\tau\right) \in dc,\right.\\
&\qquad\qquad\qquad \left. T^Y_\sigma - T^X_\tau \in d\tilde{a}, Y\left(T^Y_\sigma-\right) - Y\left(T^X_\tau\right) \in d\tilde{y}, \Delta Y\left(T^Y_\sigma\right) \in d\tilde{c}\right)\\
&\quad = e^{-qx-(y+c+\tilde{y})p}\left(1-e^{-qb}\right)\left(1-e^{-p\tilde{c}}\right)P(X_a \in dx, Y_a \in dy)da\Pi_{(X,Y)}(db,dc)\\
&\qquad\qquad\qquad P(Y_{\tilde{a}} \in d\tilde{y})d\tilde{a}\Pi_Y(d\tilde{c}).
\end{aligned}$$

Theorem 3c) *In the situation of Theorem 3b)*

$$\begin{aligned}
&E\left(\exp\left\{-\alpha T^X_\tau - \xi X_{T^X_\tau-} - \eta Y_{T^Y_\sigma-} - \beta\Delta X_{T^X_\tau} - \gamma\Delta Y_{T^Y_\sigma}\right\}1_{\{T^X_\tau = T^Y_\sigma\}}\right)\\
&\quad = \frac{\Phi_{(X,Y)}(\beta,p+\gamma)+\Phi_{(X,Y)}(q+\beta,\gamma)-\Phi_{(X,Y)}(\beta,\gamma)-\Phi_{(X,Y)}(q+\beta,p+\gamma)}{\alpha+\Phi_{(X,Y)}(q+\xi,p+\eta)}\\
&E\left(\exp\left\{-\alpha T^X_\tau - \xi X_{T^X_\tau-} - \beta\Delta X_{T^X_\tau} - \tilde{\alpha}T^Y_\sigma - \tilde{\eta}Y_{T^Y_\sigma-} - \tilde{\gamma}\Delta Y_{T^Y_\sigma}\right\}1_{\{T^X_\tau < T^Y_\sigma\}}\right)\\
&\quad = \frac{\Phi_{(X,Y)}(q+\beta,p+\tilde{\eta})-\Phi_{(X,Y)}(\beta,p+\tilde{\eta})}{\alpha+\tilde{\alpha}+\Phi_{(X,Y)}(q+\xi,p+\tilde{\eta})} \times \frac{\Phi_Y(p+\tilde{\gamma})-\Phi_Y(\tilde{\gamma})}{\tilde{\alpha}+\Phi_Y(p+\tilde{\eta})}.
\end{aligned}$$

Proof: a) For the first formula note that the pure jump subordinators X and Y can pass a level simultaneously only if they jump at the same time. Let $J_a = (\Delta X_a, \Delta Y_a)$ be the jump process of (X,Y). The Poisson point process argument, cf. the proof of Theorem 2a), yields the result.

For the second formula the argument is iterated, once for a first jump that makes X pass its level h, then for a second jump that makes Y pass its level. The first application of the compensation formula for Poisson point processes is replaced by an application of Maisonneuve's analogous formula for exit systems, which generalises the compensation formula to include information on the post-point process, cf. Théorème XX.49 of Dellacherie *et al.* [4]. We introduce the post-T^X_h processes $\tilde{Y}_a = Y(T^X_h + a)$ and $\tilde{J}_a = J(T^X_h + a)$ which are independent of $(X_a, Y_a)_{a \le T^X_h}$ and have the same law as Y and J, and we denote the conditional law given $(X_a, Y_a)_{a \le T^X_h}$ by $\tilde{P}$. The second application of the compensation formula is the argument of Theorem 1a) applied to the process $\tilde{Y}$ and the level $k - Y_{a-} - c$ under $\tilde{E}$. Specifically, we calculate for bounded measurable functions $f, g, \ell, \tilde{f}, \tilde{g}$ and $\tilde{\ell}$

$$\begin{aligned}
&E\Big(f(T^X_h)g(X(T^X_h-), Y(T^X_h-))\ell(\Delta X(T^X_h), \Delta Y(T^X_h))\\
&\qquad\qquad \tilde{f}(T^Y_k - T^X_h)\tilde{g}(Y(T^Y_k-) - Y(T^X_h))\tilde{\ell}(\Delta Y(T^Y_k))1_{T^X_h < T^Y_k}\Big)\\
&\quad = E\left(\sum_{a\ge 0} f(a)g(X_{a-}, Y_{a-})\ell(J_a)\tilde{f}(T^Y_k - a)\tilde{g}(Y(T^Y_k-) - Y_a)\tilde{\ell}(\Delta Y(T^Y_k))\right.\\
&\qquad\qquad \left. 1_{\{X_{a-} \le h < X_{a-} + J^X_a\}}1_{\{Y_{a-} + J^Y_a \le k\}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty f(a) E\Bigg(g(X_{a-}, Y_{a-}) 1_{\{X_{a-}\le h\}} \int_{[0,\infty)^2} \ell(b,c) 1_{\{b > h - X_{a-}\}} 1_{\{c \le k - Y_{a-}\}} \\
&\qquad\qquad \tilde f(T^{\tilde Y}_{k-Y_{a-}-c}) \tilde g(\tilde Y(T^{\tilde Y}_{k-Y_{a-}-c}-)) \tilde\ell(\Delta \tilde Y(T^{\tilde Y}_{k-Y_{a-}-c})) \Pi_{(X,Y)}(db,dc)\Bigg) da \\
&= \int_{[0,\infty)^2} \int_0^\infty f(a)\ell(b,c) E\Bigg(X_{a-}, Y_{a-}) 1_{\{X_{a-}\le h\}} 1_{\{b > h - X_{a-}\}} 1_{\{c \le k - Y_{a-}\}} \\
&\qquad\qquad \tilde E\left(\tilde f(T^{\tilde Y}_{k-Y_{a-}-c}) \tilde g(\tilde Y(T^{\tilde Y}_{k-Y_{a-}-c}-)) \tilde\ell(\Delta \tilde Y(T^{\tilde Y}_{k-Y_{a-}-c}))\right)\Bigg) da \Pi_{(X,Y)}(db,dc) \\
&= \int_{[0,\infty)^2} \int_0^\infty f(a)\ell(b,c) E\Bigg(g(X_{a-}, Y_{a-}) 1_{\{X_{a-}\le h\}} 1_{\{b > h - X_{a-}\}} 1_{\{c \le k - Y_{a-}\}} \\
&\qquad\qquad \int_0^\infty \int_{\tilde y \le k - Y_{a-} - c < \tilde y + \tilde c} \tilde f(\tilde a) \tilde g(\tilde y) \tilde \ell(\tilde c) \tilde P(\tilde Y_{\tilde a} \in d\tilde y) \tilde \Pi_{\tilde Y}(d\tilde c) d\tilde a \Bigg) da \Pi_{(X,Y)}(db,dc) \\
&= \int_{\tilde c > 0} \int_{\tilde a = 0}^\infty \int_{b \ge 0, c \ge 0} \int_{a=0}^\infty \int_{x \le h < x+b,\, y+c+\tilde y \le k < y+c+\tilde y + \tilde c} f(a) g(x,y) \ell(b,c) \tilde f(\tilde a) \tilde g(\tilde y) \tilde \ell(\tilde c) \\
&\qquad\qquad P(X_a \in dx, Y_a \in dy) da \Pi_{(X,Y)}(db,dc) P(Y_{\tilde a} \in d\tilde y) d\tilde a \Pi_Y(d\tilde c).
\end{aligned}
$$

b) The formulas are easily derived from a) by integrating w.r.t. the exponential laws of τ and σ.

c) Again, this is elementary integration. Just note that we restricted attention to the core quantities of each level passage event. As a first step we do the representation in terms of the quantities used in b), for the second formula e.g.

$$
\begin{aligned}
& E\left(\exp\left\{-\alpha T^X_\tau - \xi X_{T^X_\tau -} - \beta \Delta X_{T^X_\tau} - \tilde\alpha T^Y_\sigma - \tilde\eta Y_{T^Y_\sigma -} - \tilde\gamma \Delta Y_{T^Y_\sigma}\right\} 1_{\{T^X_\tau < T^Y_\sigma\}}\right) \\
&\quad = E\Big(\exp\Big\{-(\alpha+\tilde\alpha) T^X_\tau - \xi X_{T^X_\tau -} - \tilde\eta Y_{T^X_\tau -} - \beta \Delta X_{T^X_\tau} - \tilde\gamma \Delta Y_{T^X_\tau} \\
&\qquad\qquad -\tilde\alpha\left(T^Y_\sigma - T^X_\tau\right) - \tilde\eta\left(Y_{T^Y_\sigma -} - Y_{T^X_\tau}\right) - \tilde\gamma \Delta Y_{T^Y_\sigma}\Big\} 1_{\{T^X_\tau < T^Y_\sigma\}}\Big)
\end{aligned}
$$

The proof is concluded like in Theorem 1c). □

Clearly, Theorem 3a)-3c) is not given in its most general form. One can include information on X at T^Y_σ. Also, in part c) the laws of σ and τ can be included for a full description. This works in the same way as for Theorem 2c).

To stress the importance of the first formulas in each part of Theorem 3a)-3c), we state

Corollary 2 $P(T^X_\tau = T^Y_\sigma) = 0$ *if and only if* X *and* Y *are independent.*

This is also obvious from the fact that a bivariate subordinator has independent components if and only if they have no common jump times if and only if they pass levels at the same time with positive probability. Note that the situation is the same if we admit drift components because σ is a.s. different from the value of Y at the passage time of X over level τ.

Also, we can deduce the analogue of Proposition 2 giving the probabilities of the subcases.

Corollary 3 *For independent $\tau \sim Exp(q)$ and $\sigma \sim Exp(p)$ we have*

$$P(T_\tau^X = T_\sigma^Y) = \frac{\Phi(0,p) + \Phi(q,0) - \Phi(q,p)}{\Phi(q,p)}$$

$$P(T_\tau^X < T_\sigma^Y) = \frac{\Phi(q,p) - \Phi(0,p)}{\Phi(q,p)} \qquad \textit{and} \qquad P(T_\tau^X > T_\sigma^Y) = \frac{\Phi(q,p) - \Phi(q,0)}{\Phi(q,p)}$$

where $\Phi = \Phi_{(X,Y)}$.

There are also formulas for fixed h and k, but apart from

$$P(T_h^X = T_k^Y) = \bar{\Pi}_{(X,Y)} * V_{(X,Y)}(h,k) \qquad \text{with} \qquad \bar{\Pi}(x,y) = \Pi_{(X,Y)}((x,\infty) \times (y,\infty))$$

these are not so nice. One could write

$$P(T_h^X < T_k^Y) = \overline{\bar{\Pi}_Y * V_Y * \Pi_{(X,Y)}} * V_{(X,Y)}(h,k)$$

where the convolutions and tail operators are to be interpreted in different ways. The first convolution convolves a function of one variable with the one-dimensional measure V_Y. The middle convolution convolves a function of one variable with the second component of the measure $\Pi_{(X,Y)}$. The last convolution convolves a function of two variables with the two-dimensional measure $V_{(X,Y)}$. The tail operators always operate on one-dimensional measures, here.

4.2 Examples

A source of examples for multivariate subordinators is Barndorff-Nielsen *et al.* [1]. In particular, in our bivariate situation, we discuss briefly the constructions by subordination and by superposition.

Example 5 (Bivariate subordinators by subordination) Let A, B, C be three independent purely atomic subordinators and (X,Y) the bivariate subordinator obtained by $X = A \circ C$, $Y = B \circ C$. Then

$$\Phi_{(X,Y)}(\xi,\eta) = \Phi_C\left(\Phi_A(\xi) + \Phi_B(\eta)\right)$$

Also joint law and Lévy measure are easily seen to have the explicit forms

$$P(X_t \in dx, Y_t \in dy) = \int_{(0,\infty)} P(A_c \in dx) P(B_c \in dy) P(C_t \in dc)$$

$$\Pi_{(X,Y)}(da,db) = \int_{(0,\infty)} P(A_c \in da) P(B_c \in db) \Pi_C(dc)$$

This yields for the formulas in the theorem e.g.

$$\begin{aligned} &E\left(\exp\left\{-\theta T_\tau^X - \xi X_{T_\tau^X -} - \eta Y_{T_\sigma^Y -} - \alpha \Delta X_{T_\tau^X} - \beta \Delta Y_{T_\sigma^Y}\right\} 1_{\{T_\tau^X = T_\sigma^Y\}}\right) \\ &= \frac{\Phi_C\left(\Phi_A(\alpha) + \Phi_B(p+\beta)\right) + \Phi_C\left(\Phi_A(q+\alpha) + \Phi_B(\beta)\right)}{\theta + \Phi_C\left(\Phi_A(\xi) + \Phi_B(\eta)\right)} \\ &\quad - \frac{\Phi_C\left(\Phi_A(\alpha) + \Phi_B(\beta)\right) + \Phi_C\left(\Phi_A(q+\alpha) + \Phi_B(p+\beta)\right)}{\theta + \Phi_C\left(\Phi_A(\xi) + \Phi_B(\eta)\right)} \end{aligned}$$

It is also possible to include information on C into the formulas since (X, Y, C) is a subordinator as well. However, this is only a special case in the next section on subordinators in dimensions higher than two.

Example 6 (Bivariate subordinators by superposition) Let A, B, C be three independent subordinators and $X = A + C$, $Y = B + rC$. Then

$$\Phi_{(X,Y)}(\xi, \eta) = \Phi_A(\xi) + \Phi_B(\eta) + \Phi_C(\xi + r\eta).$$

Also

$$P(X_t \in dx, Y_t \in dy) = \int_{(0,\infty)} P(A_t + c \in dx)P(B_t + rc \in dy)P(C_c \in dc)$$
$$\Pi_{(X,Y)}(da, db) = \delta_0(db)\Pi_A(da) + \Pi_B(db)\delta_0(da) + \delta_{ra}(db)\Pi_C(da).$$

As already mentioned in [1], there is a connection to processes of Ornstein-Uhlenbeck type (OU processes). Specifically, a positive stationary OU process can be used to construct families of (dependent) subordinators.

Example 7 (Relation to OU processes) Let $(Z(a,t))_{a\geq 0, t\in \mathbb{R}}$ be a random field which contains stationary OU processes for fixed $a \geq 0$ and subordinators for fixed $t \in \mathbb{R}$. Then (X, Y) for $X_a = Z(a, 0)$ and $Y_a = Z(a, t)$ is a bivariate subordinator with Laplace exponent

$$\Phi_{(X,Y)}(\xi, \eta) = \Phi_{stat}(\xi + \eta e^{-ct}) + \Phi_{stat}(\eta) - \Phi_{stat}(\eta e^{-ct}).$$

where Φ_{stat} is the stationary univariate Laplace exponent of the subordinators, c is the correlation parameter of the OU processes. This is a special case of the superposition example.

In the same way, one can study continuous-state branching process (CB processes). The family of associated subordinators can be interpreted as describing an underlying genealogy. Cf. Le Gall [8].

Example 8 (Relation to CB processes) Let $(Z(a,t))_{a\geq 0, t\geq 0}$ be a family of CB processes started in a, $a \geq 0$, such that one has a family of subordinators for fixed t. Like in Example 7, (X, Y) for $X_a = Z(a, s)$ and $Y_a = Z(a, s+t) = \tilde{Z}(X_a, t)$ is a bivariate subordinator with Laplace exponent

$$\Phi_{(X,Y)}(\xi, \eta) = \Phi_s(\xi + \Phi_t(\eta))$$

where Φ_s is the Laplace exponent of $(Z(a,s))_{a\geq 0}$. This is a special case of the subordination example.

Of course, one can also analyse the non-stationary case of the OU example. On the other hand, CB processes have no stationary behaviour. In the critical and subcritical cases they die at a finite time, if the variation of the sample paths is a.s. unbounded, and converge to zero if the variation is bounded. In the supercritical case they explode (possibly even at a finite time).

4.3 Extensions to higher dimensions

All results can be extended to n-variate subordinators $(X^{(1)}, \ldots, X^{(n)})$ and levels $h_1, \ldots, h_n$. A presentation needs some notational simplifications since the number of cases increases quickly from the three cases which determine the order of T_h^X and T_k^Y. More precisely, let $1 \leq d \leq n$, $D_1 \cup \ldots \cup D_d = \{1, \ldots, n\}$ a partition into d disjoint subsets, and π a permutation of $\{1, \ldots, d\}$. We interpret the choice of D_r as the events $\{T_{h_i}^{(i)} = T_{h_j}^{(j)} \text{ iff } i, j \in D_r\}$, denote $T_{(r)} = T_{h_i}^{(i)}$ for $i \in D_r$, $r = 1, \ldots, d$. The choice of π is to fix the order of passage times, i.e. to describe the event $\{T_{(r)} < T_{(s)} \text{ iff } \pi(r) < \pi(s)\}$. Once $D_1, \ldots, D_d$ and π have been chosen, the formulas are composed by terms like in Theorem 3a)-3c) according as two successive times are equal or not. We leave the details to the interested reader.

Example 9 In principle, the above allows to give explicitly all finite-dimensional marginal distributions of the passage events of $(Z(a,t))_{a \geq 0, t \geq 0}$ in both the OU and CB settings of Examples 7 and 8.

5 Applications in the econometric model

Figure 3 shows how offers on electronic foreign exchange markets are represented in practice.

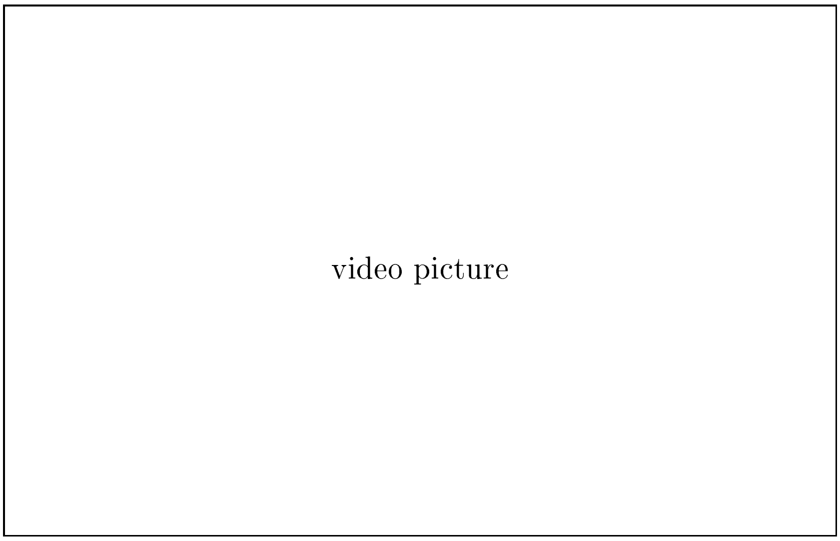

Figure 3: Snap shot of the electronic £-\$ exchange market, 13 May 2001, 12h34:56

This is one of the original pictures that are behind our Figure 1 in the Introduction. After indicating the context of this picture rather briefly, we focus on the model we set up to its description. This includes a discussion of the properties required to obtain tractability with the passage event theory developed in earlier sections of the present paper. Also, the degree of modelling freedom is explained. Finally, we mention some generalisations of the model.

5.1 Main features of electronic foreign exchange markets

Let us briefly describe how these electronic markets work. Looking at Figure 3 as a potential seller, the relevant information is what the current best buyers' offers are, the left part of the lower path (the vertical axis showing the price, lengths of horizontal lines being quantities). A potential seller has two possibilities; either he is prepared to meet the best buyers' offers - then transaction takes place, the buyers' offers are removed from the screen, sellers' offers remain unchanged; or he puts forward an offer higher than the best buyers' offers which is inserted into the sellers' path waiting for new buyers willing to meet the offer. A potential buyer has the opposite possibilities. In addition to these four types of modifications, current offers may be removed by their owners.

5.2 Representation of offers by homogeneous Poisson point processes

Let us now fix time, hence assume the situation of the snap shot, Figure 3. It is instructive to continue the paths to their left as in Figure 2 of the Introduction and think of the added part as the offers behind realised transactions. To be more precise, think of a number of sellers and buyers who present their offers. These offers are then ordered by their price and represented graphically as quantity-price paths as in Figure 2. Some buyers' prices will be higher than sellers' prices so that transactions can take place. Assume that one collective price is calculated at which all willing buyers and sellers buy and sell. This price P is so that the quantities sold and bought coincide.

Let N_S be the number of sellers and N_B the number of buyers, $P_S(s) > 0$ the minimal price at which seller s would sell his $Q_S(s)$ units of currency, $s = 1, \ldots, N_S$, $P_B(b) > 0$ the maximal price at which a buyer b would buy $Q_B(b)$ units of currency. All information is contained in

$$S_a = \sum_{s:P_S(s)\le a} Q_S(s) \qquad \text{and} \qquad B_a = \sum_{b:P_B(b)>a} Q_B(b), \qquad a \ge 0, \tag{6}$$

the number of units offered and demanded, respectively, if the agreed price was a. Obviously, S_a is increasing in a starting from zero whereas B_a decreases. The price, for which the two coincide, is

$$P = \inf\{a \ge 0 : S_a > B_a\}$$

or possibly the strict inequality replaced by a weak one or the inf replaced by a sup of the inverse inequality. We consider these differences as negligible and stick to our definition of the collective price P which is the most suitable from a mathematical point of view.

B and S are not the processes whose realisations we see in pictures like Figure 2, but almost. All we need to do is swap the coordinate axes since B and S are quantity processes indexed by price. The corresponding price processes indexed by quantity are

$$\sigma_x = \inf\{a \ge 0 : S_a > x\} \qquad \text{and} \qquad \beta_x = \inf\{a \ge 0 : B_a > x\}, \qquad x \ge 0.$$

Let us focus on S now, the sellers. Without complicating the analysis, we may allow an infinite number N_S of sellers provided their quantities are summable over the appropriate price ranges in (6). As path of a stochastic process indexed by the price parameter a, $(S_a)_{a\geq 0}$ is increasing and only moving by jumps, here given in its characterisation by the point process

$$\{(P_S(s), Q_S(s)) : s = 1, 2, \ldots\}$$

of jumps (a, y) of size y at price a. If this point process is a homogeneous Poisson point process, then $(S_a)_{a\geq 0}$ is a subordinator. The Poisson property implies an independence of sellers, i.e. the numbers of sellers with minimal price and quantity in disjoint regions are independent and have Poisson distributions. The homogeneity means that the law of the quantities is proportional to the Lebesgue size of a price region (in the sense of convolution powers, i.e. the quantity of currency in a region r times as big as another has as law the rth convolution power of the other). It is another consequence of the homogeneous Poisson property that conditionally on having n sellers in a price-quantity region, the n quantities and minimal prices are independent and respectively identically distributed. The law of the quantities can be chosen under some consistency constraints as the so-called Lévy measure of the subordinator. The minimal prices have a uniform distribution on the region. Globally, this uniform spread on the price axis may seem unsatisfactory but it is certainly acceptable locally. Note in this context that only minimal prices close to the actual price P have a direct influence on its determination, the others only influence by their quality of being low or too high.

Next, consider the buyers as modelled by B. Clearly, infinitely many may occur provided their total quantity of currency is finite. Here, we cannot allow arbitrarily high prices for non-negligible quantities since this makes the sum defining B in (6) infinite. The only possibility to allow an infinite demand is as the price tends to zero, but homogeneity considerations will prohibit us to do so. Let us look again at the point process

$$\{(P_B(b), Q_B(b)) : b = 1, 2, \ldots\}$$

of jumps (a, y) of $(B_a)_{a\geq 0}$. Here (a, y) means a jump of size $-y$ at price a. If this point process is a Poisson point process, the buyers have the same independence property as the sellers. Our aim is, of course, to establish a subordinator type property for $(B_a)_{a\geq 0}$, as well, but we cannot ask for homogeneity on the whole price axis $[0, \infty)$ since this would make the sums in (6) diverge a.s. However, there are other ways to get to a subordinator setting. Mathematically, the most convenient is to fix the total quantity B_0 of good demanded to a constant $h > 0$ or to an independent exponential random variable $\tau = \tau(q)$ with parameter (inverse mean) $q > 0$. Then we may consider homogeneity of the Poisson point process but adjust the definition of B to sum $Q_B(b)$ not over all b with $P_B(b) > a$ but only those below a random price threshold β_0.

The situation described is now the one in Examples 1 and 2, if we put $Y_a = S_a$ and $Z_a = h - B_a$. Note however, that we have no drift components here, $b_Y = b_Z = b_{Y+Z} = 0$. This leads to some simplifications. E.g., the passage cannot take place continuously, i.e. $P(A_0) = 0$. In the notation of this section and the Introduction, we obtain the explicit laws of (P, Q, Q', Δ) as a corollary to Theorem 2a)-2c). Here $Q = S_P$ is the amount of currency that sellers offer at the collective market price P, $Q' = B_P$ is the amount of currency that buyers demand at price P. As $Q \neq Q'$ a.s., either the last buyer or the last seller can only partially realise their transaction. We denote by Δ their total quantity offered or demanded.

Corollary 4 *Let B and S be the subordinators associated to buyers' and sellers' offers via (6).*

i) Assume that $B_0 = h$ is a constant, then

$$
\begin{aligned}
&P(P \in da, Q \in dx, Q' \in dy, \Delta \in dz) \\
&\qquad = 1_{\{x<y, z \geq x-y\}} P(S_a \in dx) P(B_0 - B_a - z \in dy) da \Pi_B(dz) \\
&\qquad + 1_{\{y<x, z \geq y-x\}} P(S_a + z \in dx) P(B_0 - B_a \in dy) da \Pi_S(dz).
\end{aligned}
$$

ii) Assume that $B_0 = \tau \sim Exp(q)$ is independent of $B_0 - B$ and S. Then

$$
\begin{aligned}
&E\left(\exp\left\{-\alpha P - \xi Q - \eta Q' - \nu \Delta\right\}\right) \\
&\qquad = \frac{q\left(\Phi_B(q+\nu) - \Phi_B(\nu-\eta) + \Phi_S(q+\nu+\eta+\xi) - \Phi_S(\nu+\xi)\right)}{(q+\eta)\left(\alpha + \Phi_B(q) + \Phi_S(q+\eta+\xi)\right)}
\end{aligned}
$$

We can also split the formulas like in Theorem 2a)-2c) to specify whether Δ refers to a buyer or a seller. Also, by linear transformations we obtain other related quantities.

5.3 Extensions by transformation

There are immediate extensions of the model away from homogeneity. One might consider higher intensities in a 'realistic' price range with decreasing tails or cut off close to zero and infinity. This corresponds to a deterministic time change of the subordinators. If both subordinators are transformed by the same time change, T_h and associated quantities are just transformed accordingly. Some care is needed if the time change is not infinite - the agreed price may then be outside the possible region to make sense in the model - this happens if the tails at infinity are integrable, particularly when cut off. Also, independent random time changes are a possibility. Dependent random time changes or different time changes for sellers and buyers, or replacing the subordinators by suitable increasing additive processes, lead out of the immediate range of the theory presented in this paper.

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