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RIGHT INVERSES OF LÉVY PROCESSES: THE EXCURSION MEASURE IN THE GENERAL CASE

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Abstract
This article is about right inverses of Lévy processes as first introduced by Evans in the symmetric case and later studied systematically by the present authors and their co-authors. Here we add to the existing fluctuation theory an explicit description of the excursion measure away from the (minimal) right inverse. This description unifies known formulas in the case of a positive Gaussian coefficient and in the bounded variation case. While these known formulas relate to excursions away from a point starting negative continuously, and excursions started by a jump, the present description is in terms of excursions away from the supremum continued up to a return time. In the unbounded variation case with zero Gaussian coefficient previously excluded, excursions start negative continuously, but the excursion measures away from the right inverse and away from a point are mutually singular. We also provide a new construction and a new formula for the Laplace exponent of the minimal right inverse.

1 Introduction

Evans [5] defined a *(full) right inverse* of a Lévy process $X=(X_t,t\ge 0)$ to be any increasing process $K=(K_x,x\ge 0)$ such that $X_{K_x}=x$ for all $x\ge 0$. A *partial right inverse* [8] is any increasing process $K=(K_x,0\le x<\xi_K)$ such that $X_{K_x}=x$ for all $0\le x<\xi_K$ for some (random) $\xi_K>0$. The existence of partial right inverses is a local path property that has been completely characterised [4, 5, 8] in terms of the Lévy-Khintchine triplet (a,σ^2,Π) of the Lévy process X, i.e. $a\in\mathbb{R}$, $\sigma^2\ge 0$ and Π measure on $\mathbb{R}$ with $\Pi(\{0\})=0$ and $\int_{\mathbb{R}}(1\wedge y^2)\Pi(dy)<\infty$ such that

$$\mathbb{E}\left(e^{i\lambda X_t}\right)=e^{-t\psi(\lambda)},\qquad \text{where } \psi(\lambda)=-ia\lambda+\frac{1}{2}\sigma^2\lambda^2+\int_{\mathbb{R}}\left(1-e^{i\lambda y}+i\lambda y1_{\{|y|\le 1\}}\right)\Pi(dy).$$

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We recall Evans' construction of right inverses. Recursively define for each $n \geq 0$ times

$$T_0^{(n)} = 0, \qquad T_{k+1}^{(n)} = \inf\left\{t \geq T_k^{(n)} : X_t = (k+1)2^{-n}\right\}, \quad k \geq 1,$$

and a process $K_x^{(n)} = T_k^{(n)}$, $k2^{-n} \leq x < (k+1)2^{-n}$, $k \geq 0$. Then, a pathwise argument shows that, if the limit

$$K_x = \inf_{y>x} \sup_{n\geq 0} K_y^{(n)} \tag{1}$$

is finite for $0 \leq x < \xi_K$ and $\xi_K > 0$, it is the *minimal* partial right inverse: for all *right-continuous* partial right inverses $(U_x, 0 \leq x < \xi')$, we have $\xi' \leq \xi_K$ and $U_x \geq K_x$ for all $0 \leq x < \xi'$.
Where right inverses exist, the minimal right-continuous right inverse is a subordinator, and for partial right inverses (and ξ_K maximal), a subordinator run up to an independent exponential time ξ_K. In the sequel, we focus on this minimal right-continuous (partial) right inverse and denote its Laplace exponent by

$$\rho(q) = -\ln\left(\mathbb{E}\left(e^{-qK_1}; \xi_K > 1\right)\right) = \kappa_K + \eta_K q + \int_{(0,\infty)} \left(1 - e^{-qt}\right) \Lambda_K(dt). \tag{2}$$

Evans [5] showed further that the reflected process $Z = X - L$ is a strong Markov process, where $L_t = \inf\{x \geq 0 : K_x > t\}$, $0 \leq t < K_{\xi_K}$, $L_t = \xi_K$, $t \geq K_{\xi_K}$; and L is a local time process of Z at zero. In analogy with the classical theory of excursions away from the supremum (see below), there is an associated excursion theory that studies the Poisson point process $(e_x^Z, 0 \leq x < \xi)$ of *excursions away from the right inverse*, where

$$e_x^Z(r) = Z_{K_{x-}+r}, \quad 0 \leq r \leq \Delta K_x = K_x - K_{x-}, \qquad e_x^Z(r) = 0, \quad r \geq \Delta K_x.$$

Specifically, we denote by n^Z its intensity measure on the space $(E, \mathscr{E})$ of excursions

$$E = \{\omega \in D : \omega(s) = 0 \text{ for all } s \geq \zeta(\omega) = \inf\{r > 0 : \omega(r) = 0\}\}$$

equipped with the restriction sigma-algebra $\mathscr{E}$ induced by the Borel sigma-algebra $\mathscr{D}$ associated with Skorohod's topology on the space $D = D([0,\infty), \mathbb{R})$ of càdlàg paths $\omega\colon [0,\infty) \to \mathbb{R}$. The entrance laws $n_r^Z(dy) = n^Z(\{\omega \in E : \omega(r) \in dy, \zeta(\omega) > r\})$ are characterised by their Fourier-Laplace transform

$$\int_0^\infty e^{-qr} \int_{\mathbb{R}} e^{i\lambda y} n_r^Z(dy) dr = \frac{\rho(q) - i\lambda}{q + \psi(\lambda)} - \eta_K, \tag{3}$$

see [5, 8]. The sub-stochastic semi-group within these excursions is the usual killed semi-group

$$P_t^\dagger(y, dz) = \mathbb{P}_y^\dagger(\{\omega \in D : \omega(t) \in dz, \zeta(\omega) > t\}), \qquad \text{with } \mathbb{P}_y^\dagger = \mathbb{P}((X_{t\wedge\zeta(X)}, t \geq 0) \in \cdot \,|X_0 = y)$$

as canonical measure on $E \subset D$ of the distribution of X starting from y and frozen when hitting zero. More explicit expressions for n^Z are available from [8] in two cases: when $\sigma^2 > 0$, then

$$n^Z(d\omega) = \frac{2}{\sigma^2} n^X(d\omega; \omega(s) < 0 \text{ for all } 0 < s < \varepsilon \text{ and some } \varepsilon > 0) \tag{4}$$

is proportional to the intensity measure n^X of excursions of X away from zero restricted to those starting negative; in the bounded variation case (BV), $\sigma^2 = 0$ and $\int_{\mathbb{R}} (1 \wedge |y|)\Pi(dy) < \infty$,

$$n^Z(d\omega) = \frac{1}{b} \int_{\mathbb{R}} \mathbb{P}_y^\dagger(d\omega)\Pi(dy), \quad \text{where necessarily } b = a - \int_{\mathbb{R}} y 1_{\{|y|\leq 1\}}\Pi(dy) > 0. \tag{5}$$

Recall now from [4, 5, 8] that a Lévy process X possesses a partial right inverse if and only if

$$\sigma^2 > 0 \quad \text{or} \quad \text{BV, } b > 0,\ \Pi((0,\infty)) < \infty \quad \text{or} \quad \text{not BV, } \int_0^1 \frac{x^2\Pi(dx)}{(\int_0^x \int_y^1 \Pi(-\infty,-s)ds\,dy)^2} < \infty.$$

In the present paper, we describe n^Z for a general Lévy process that possesses a partial right inverse. This seems to answer the final open question [4] related to the notion of right inverse of a Lévy process. However, this study can also be seen in the light of more general subordination [7] of the form $X_{T_x} = Y_x$, where (T,Y) is a bivariate Lévy process, increasing in the T-component. To formulate our main result, we recall some classical fluctuation theory [1, 2]. With any Lévy process X we associate the ascending ladder time and ladder height processes (τ,H), a bivariate subordinator such that $H_x = X_{\tau_x} = \overline{X}_{\tau_x}$ visits all suprema $\overline{X}_t = \sup\{X_s, 0 \le s \le t\}$, $t \ge 0$. If $X_t \to -\infty$ as $t \to \infty$, then $\tau = (\tau_x, 0 \le x < \xi)$ is a subordinator run up to an exponential time ξ. We write the Laplace exponent of (τ,H) in Lévy-Khintchine form as

$$k(\alpha,\beta) = -\ln\left(\mathbb{E}\left(e^{-\alpha\tau_1-\beta H_1};\xi > 1\right)\right) = \kappa + \eta\alpha + \delta\beta + \int_{[0,\infty)^2}\left(1-e^{-\alpha s-\beta y}\right)\Lambda(ds,dy). \qquad (6)$$

It was shown in [8] that $\delta > 0$ whenever there exists a partial right inverse. *In the sequel, we will always normalise the ascending ladder processes so that* $\delta = 1$. Also, when partial right inverses exist, then $\mathbb{P}(T_{\{z\}} < \infty) > 0$ for all $z > 0$, where $T_{\{z\}} = \inf\{t \ge 0 : X_t = z\}$. In particular, the q-resolvent measure

$$U^q(dz) = \int_0^\infty e^{-qt}\mathbb{P}(X_t \in dz)dt$$

then admits a bounded density $u^q(z)$ that is continuous except possibly for a discontinuity at zero in the bounded variation case, see [6, Theorem 43.19]. Now $R = X - \overline{X}$ is a strong Markov process with τ as its inverse local time; its excursions, with added height ΔH_x at freezing,

$$e_x^R(r) = R_{\tau_{x-}+r}, \quad 0 \le r < \Delta\tau_x = \tau_x - \tau_{x-}, \qquad e_x^R(r) = \Delta H_x, \quad r \ge \Delta\tau_x,$$

form a Poisson point process whose intensity measure we denote by $\widetilde{n}^R$. For $\omega \in D$, we write $\zeta^+(\omega) = \inf\{r > 0 : \omega(r) > 0\}$. For $\omega_1 \in D$ with $\zeta^+(\omega_1) < \infty$ and $\omega_2 \in D$, we concatenate

$$\omega = \omega_1 \oplus \omega_2, \qquad \text{where } \omega(r) = \omega_1(r), \quad 0 \le r < \zeta^+(\omega_1), \qquad \omega(\zeta^+(\omega_1)+r) = \omega_2(r), \quad r \ge 0.$$

Theorem 1. *Let X be a Lévy process that possesses a partial right inverse. Then the excursion measures n^Z away from the right inverse and $\widetilde{n}^R$ away from the supremum are related as*

$$n^Z(d\omega) = (\widetilde{n}^R \oplus \mathbb{K})(d\omega), \qquad (7)$$

where the stochastic kernel

$$\mathbb{K}(\omega_1,d\omega_2) = \mathbb{P}^\dagger_{\omega_1(\zeta^+(\omega_1))}(d\omega_2), \quad \text{if } \zeta^+(\omega_1) < \infty, \qquad \mathbb{K}(\omega_1,d\omega_2) = \delta_0 \quad \text{otherwise},$$

associates to a path ω_1 that passes positive a Lévy process path ω_2 that starts at the first positive height of ω_1 and is frozen when reaching zero, and where $(\widetilde{n}^R \oplus \mathbb{K})(d\omega)$ is the image measure of $\mathbb{K}(\omega_1,d\omega_2)\widetilde{n}^R(d\omega_1)$ under concatenation $(\omega_1,\omega_2) \mapsto \omega = \omega_1 \oplus \omega_2$.

For the case $\sigma^2 = 0$ and X of unbounded variation, it was noted in [4] that a.e. excursion under n^X starts positive and ends negative, while a.e. excursion under $\widetilde{n}^R$ starts negative; by Theorem 1, the two measures n^Z and n^X are therefore singular, in contrast to (4) in the case $\sigma^2 > 0$. When X is of bounded variation, the discussion in [8, Section 5.3] easily yields the compatibility of (7) and (5); it can happen that a jump in the ladder height process ΔH_x occurs without an excursion away from the supremum, $\Delta\tau_x = 0$, and so n^Z can charge paths with $\omega(0) > 0$, while in the unbounded variation case we have $\omega(0) = 0$ for n^Z-a.e. $\omega \in E$.
Other descriptions of n^Z follow: let $\widetilde{n}^R_t(dz) = \widetilde{n}^R(\{\omega \in E : \omega(t) \in dz, \zeta^+(\omega) \wedge \zeta(\omega) > t\})$.

Corollary 2. *In the setting of Theorem 1, the entrance laws of n^Z are given by*

$$n^Z_t(dz) = \widetilde{n}^R_t(dz) + \int_{[0,t]\times(0,\infty)} P^\dagger_{t-s}(y, dz)\Lambda(ds, dy), \tag{8}$$

and the semi-group of n^Z, or rather $n^Z(\{\omega \in E : (\omega(t), 0 \le t < \zeta(\omega)) \in \cdot\}$, is $(P^\dagger_t(y, dz), t \ge 0)$.

We also record an expression for the Laplace exponent ρ of the partial right inverse K.

Theorem 3. *Let X be a Lévy process which possesses a partial right inverse. Then*

$$\rho(q) = \kappa + \eta q + \int_{[0,\infty)\times[0,\infty)} \left(1 - e^{-qs}\frac{u^q(-y)}{u^q(0)}\right)\Lambda(ds, dy), \quad \text{with } u^q(0) = u^q(0+), \tag{9}$$

where $\kappa \ge 0$, $\eta \ge 0$ and Λ are as in (6), respectively, the killing rate and the drift coefficient of the ascending ladder time process and the Lévy measure of the bivariate ladder subordinator (τ, H). In particular, the characteristics $(\kappa_K, \eta_K, \Lambda_K)$ of K in (2) are given by

$$\kappa_K = \kappa + \int_{[0,\infty)^2} \mathbb{P}(T_{\{-y\}} = \infty)\Lambda(ds, dy), \qquad \eta_K = \eta,$$

$$\Lambda_K(dt) = \int_{[0,\infty)\times[0,\infty)} \mathbb{P}(s + T_{\{-y\}} \in dt; T_{\{-y\}} < \infty)\Lambda(ds, dy). \tag{10}$$

We stress that $\Lambda(\{0\}, dy)$ is the zero measure unless X can jump into its new supremum from the position of its current supremum. The latter can happen only when the ascending ladder time has a positive drift $\eta > 0$, i.e. in particular only when X is of bounded variation.
Let us note a simple consequence of Theorem 3 which can also be seen directly using more elementary arguments: $\mathbb{P}(\xi_K > x, K_x \le t) > 0$ for some $t > 0$ implies $\mathbb{P}(\xi_K > x) = 1$.

Corollary 4. *A recurrent Lévy process has a partial right inverse iff it has a full right inverse.*

We proceed as follows. Sections 2 and 3 contain proofs of Theorem 3 and Corollary 4 exploiting recent developments [3] on joint laws of first passage variables, and, respectively, of Theorem 1 and Corollary 2. In an appendix, we introduce an alternative construction of right inverses and indicate an alternative proof of Theorem 3.

2 The Laplace exponent of K; proof of Theorem 3

Before we formulate and prove some auxiliary results, we introduce some notation. Denote by $\mathscr{U}(ds, dy) = \int_0^\infty \mathbb{P}(\tau_x \in ds, H_x \in dy; \xi > x)dx$ the potential measure of the bivariate ascending ladder subordinator (τ, H), by $T_x^+ = \inf\{t \geq 0 : X_t \in (x, \infty)\}$ the first passage time across level $x > 0$, by $\overline{X}_t = \sup_{0 \leq s \leq t} X_s$ the supremum process, by $\overline{G}_t = \sup\{s \leq t : X_s = \overline{X}_t \text{ or } X_{s-} = \overline{X}_t\}$ the time of the last visit to the supremum and by $O_x = X_{T_x^+} - x \geq 0$ the overshoot over level x. Then on $\{(t, s, w, y) : t \geq 0, s \geq 0, w > 0, 0 \leq y \leq x\}$, we have

$$\mathbb{P}(T_x^+ - \overline{G}_{T_x^+ -} \in dt, \overline{G}_{T_x^+ -} \in ds, O_x \in dw, x - \overline{X}_{T_x^+ -} \in dy; T_x^+ < \infty) = \Lambda(dt, du + y)\mathscr{U}(x - dy, ds), \tag{11}$$

by a corollary of the quintuple law of Doney and Kyprianou [3].
Suppose that X possesses a partial right inverse. Recall construction (1). A crucial quantity there is the hitting time of levels, $T_{\{x\}}$. A key observation for our developments is that

$$T_{\{x\}} = T_x^+ + \widetilde{T}_{\{-O_x\}} \qquad \text{a.s. on } \{T_x^+ < \infty\}, \tag{12}$$

where $\widetilde{T}_{\{-O_x\}} = \inf\{t \geq 0 : \widetilde{X}_t = -O_x\}$ for $\widetilde{X} = (X_{T_x^+ + t} - X_{T_x^+}, t \geq 0)$ independent of (T_x^+, O_x). For $q > 0$, let

$$z_n = 2^n \mathbb{E}\left(1 - e^{-qT_{\{2^{-n}\}}}\right) = 2^n \mathbb{E}\left(1 - e^{-q(T_{2^{-n}}^+ + \widetilde{T}_{\{-O_{2^{-n}}\}})}\right),$$

with the convention that $e^{-\infty} = 0$ and $T_{2^{-n}}^+ + \widetilde{T}_{\{O_{2^{-n}}\}} = \infty$ on $\{T_{2^{-n}}^+ = \infty\}$. As was already exploited by Evans [5], construction (1) allows us to express

$$\rho(q) = -\ln\left(\lim_{n\to\infty} \mathbb{E}\left(e^{-qK_1^{(n)}}\right)\right) = -\ln\left(\lim_{n\to\infty}\left(1 - \frac{z_n}{2^n}\right)^{2^n}\right) = \lim_{n\to\infty} z_n, \tag{13}$$

because $K_1^{(n)}$ is the sum of 2^n independent random variables with the same distribution as $T_{\{2^{-n}\}}$. To calculate this limit, we will use (12) and also decompose z_n, as follows, setting

$$\widehat{z}_n = 2^n \mathbb{E}\left(\left(1 - e^{-qT_{\{2^{-n}\}}}\right) 1_{\{O_{2^{-n}} > 0, T_{2^{-n}}^+ < \infty\}}\right) \quad \text{and} \quad \widetilde{z}_n = z_n - \widehat{z}_n. \tag{14}$$

Lemma 5. *Let X be a Lévy process of unbounded variation which possesses a partial right inverse. Then*

$$\lim_{n\to\infty} \widehat{z}_n = \int_{(t,h)\in(0,\infty)^2} \left(1 - e^{-qt} \frac{u^q(-h)}{u^q(0)}\right) \Lambda(dt, dh). \tag{15}$$

Proof. According to [6, Theorems 43.3, 43.19 and 47.1], the resolvent density u^q is bounded and continuous for all $q > 0$ and $\mathbb{E}(e^{-qT_{\{x\}}}) = u^q(x)/u^q(0)$ for all $x \in \mathbb{R}$. We use (11) to obtain

$$\begin{aligned}
\mathbb{E}\left(\left(1 - e^{-q(T_x^+ + \widetilde{T}_{\{-O_x\}})}\right) 1_{\{O_x > 0\}}\right) &= \mathbb{E}\left(\left(1 - e^{-q(\overline{G}_{T_x^+ -} + T_x^+ - \overline{G}_{T_x^+ -} + \widetilde{T}_{\{-O_x\}})}\right) 1_{\{O_x > 0\}}\right) \\
&= \int_{(s,y)\in(0,\infty)\times[0,x]} \int_{(t,w)\in(0,\infty)^2} \left(1 - e^{-q(s+t)} \frac{u^q(-w)}{u^q(0)}\right) \Lambda(dt, dw + y)\mathscr{U}(ds, x - dy) \\
&= \int_{(t,h)\in(0,\infty)^2} \int_{(s,y)\in(0,\infty)\times[0,x\wedge h]} \left(1 - e^{-q(s+t)} \frac{u^q(-h+y)}{u^q(0)}\right) \mathscr{U}(ds, x - dy)\Lambda(dt, dh).
\end{aligned}$$

Therefore it will be sufficient to show that as n tends to infinity, we have the convergence

$$\begin{aligned}
2^n &\int_{(t,h)\in(0,\infty)^2} \int_{(s,y)\in(0,\infty)\times[0,2^{-n}\wedge h]} \left(1 - e^{-q(t+s)} \frac{u^q(-h+y)}{u^q(0)}\right) \mathscr{U}(ds, 2^{-n} - dy)\Lambda(dt, dh) \\
&\longrightarrow \int_{(t,h)\in(0,\infty)^2} \left(1 - e^{-qt} \frac{u^q(-h)}{u^q(0)}\right) \Lambda(dt, dh).
\end{aligned} \tag{16}$$

First fix $(t,h)\in(0,\infty)^2$ and consider the bounded and continuous function $f:[0,\infty)^2\to[0,\infty)$ given by

$$f(s,y)=1-e^{-q(t+s)}\frac{u^q(-h+y)}{u^q(0)} \tag{17}$$

and the measures $\vartheta_{n,h}(ds,dy)=1_{[0,2^{-n}\wedge h]}(y)2^n\mathscr{U}(ds,2^{-n}-dy)$ on $[0,\infty)^2$. Since H has unit drift, we have that $\mathbb{P}(H_t\geq t \text{ for all } t\geq 0)=1$ and so for all $\varepsilon>0$, we have

$$\begin{aligned}\lim_{n\to\infty}\vartheta_{n,h}(([0,\varepsilon]\times[0,\varepsilon])^c)&=\lim_{n\to\infty}2^n\int_0^\infty\mathbb{P}\left(\tau_x>\varepsilon,H_x\leq 2^{-n}\right)dx\\&=\lim_{n\to\infty}2^n\int_0^{2^{-n}}\mathbb{P}\left(\tau_x>\varepsilon,H_x\leq 2^{-n}\right)dx\leq\lim_{n\to\infty}\mathbb{P}(\tau_{2^{-n}}>\varepsilon)=0,\end{aligned}$$

whereas $H_t/t\to 1$ a.s., as $t\to 0$, implies that $\mathbb{P}(H_{2^{-n}(1-\varepsilon)}\leq 2^{-n})\geq 1-\varepsilon$ for n sufficiently large, and so

$$1\geq 2^n\int_0^{2^{-n}}\mathbb{P}(\tau_x\geq 0,H_x\leq 2^{-n})dx\geq 2^n\int_0^{2^{-n}(1-\varepsilon)}\mathbb{P}(H_x\leq 2^{-n})dx\geq(1-\varepsilon)^2.$$

This shows convergence $\vartheta_{n,h}(ds,dy)=1_{[0,2^{-n}\wedge h]}(y)2^n\mathscr{U}(ds,2^{-n}-dy)\to\delta_{(0,0)}$ weakly as n tends to infinity, where $\delta_{(0,0)}$ is the Dirac measure in $(0,0)$; in particular

$$\int_{[0,\infty)^2}f(s,y)\vartheta_{n,h}(ds,dy)\to f(0,0). \tag{18}$$

To deduce (16), and hence (15), from (18), we use the Dominated Convergence Theorem and for this purpose we show that

$$f_n(t,h)=\int_{(s,y)\in(0,\infty)\times[0,2^{-n}\wedge h]}\left(1-e^{-q(t+s)}\frac{u^q(-h+y)}{u^q(0)}\right)\mathscr{U}(ds,2^{-n}-dy) \tag{19}$$

is bounded above by a Λ-integrable function. We shall prove the bound

$$f_n(t,h)\leq(1-e^{-qt})+\left(k(q,0)+\frac{1}{2}k(0,\rho(q))\right)(1\wedge h)+\left(1-\frac{u^q(-h)}{u^q(0)}\right), \tag{20}$$

where we recall from (6) that k is the Laplace exponent of (τ,H). The integrand in (19) is

$$\begin{aligned}f(s,y)&=(1-e^{-qt})+e^{-qt}(1-e^{-qs})+e^{-q(t+s)}\left(1-\frac{u^q(-h+y)}{u^q(0)}\right)\\&\leq(1-e^{-qt})+(1-e^{-qs})+\left(1-\frac{u^q(-h+y)}{u^q(0)}\right),\end{aligned}$$

three terms, where $f(s,y)$ is defined in (17). First note that as before since H has unit drift

$$\vartheta_{n,h}([0,\infty)^2)=2^n\int_0^{2^{-n}}P(H_t\leq 2^{-n})dt\leq 1.$$

Therefore

$$\int_{(s,y)\in[0,\infty)^2}(1-e^{-qt})\vartheta_{n,h}(ds,dy)\leq 1-e^{-qt}. \tag{21}$$

For the second term we use $\mathbb{P}(H_u - H_t \geq u - t \text{ for all } u \geq t \geq 0) = 1$ to write

$$
\begin{aligned}
&\int_{[0,\infty)^2} (1-e^{-qs})\vartheta_{n,h}(ds,dy) \\
&\quad \leq \int_{s\in[0,\infty)} (1-e^{-qs}) 2^n \int_{r\in[0,\infty)} \mathbb{P}\left(\tau_r \in ds, H_r \in [2^{-n}-h, 2^{-n}]\right) dr \\
&\quad = 2^n \int_{[0,2^{-n}]} \mathbb{E}\left(\left(1-e^{-q\tau_r}\right) 1_{\{H_r \in [2^{-n}-h,2^{-n}]\}}\right) dr \\
&\quad \leq 2^n \int_{[0,2^{-n}]} \mathbb{E}\left(\left(1-e^{-q\tau_{2^{-n}}}\right) 1_{\{H_r \in [2^{-n}-h,2^{-n}]\}}\right) dr \\
&\quad \leq 2^n (h\wedge 1)\left(1-e^{-2^{-n}k(q,0)}\right) \leq (h\wedge 1)k(q,0).
\end{aligned} \tag{22}
$$

For the third term we mimick the previous calculation to get

$$
\begin{aligned}
&\int_{[0,\infty)^2} \left(1-\frac{u^q(-h+y)}{u^q(0)}\right)\vartheta_{n,h}(ds,dy) \\
&\quad = \mathbb{E}\left(2^n \int_{r\in[0,2^{-n}\wedge h]} \left(1-\frac{u^q((-h+H_r)\wedge 0)}{u^q(0)}\right) dr\right) =: \Upsilon(h).
\end{aligned}
$$

We now exploit the fact [8, Corollary 2] that $x \mapsto e^{-\rho(q)x}u^q(-x)$ is decreasing, and also $1-e^{-x} \geq x - x^2/2$, to see, for $h \leq 2^{-n}$

$$
\begin{aligned}
\Upsilon(h) &\leq 2^n \left(h - \frac{u^q(-h)}{u^q(0)}\mathbb{E}\left(\int_0^h e^{-\rho(q)H_r} dr\right)\right) \\
&= 2^n \left(h - \frac{u^q(-h)}{u^q(0)} \frac{1}{k(0,\rho(q))}(1-e^{-hk(0,\rho(q))})\right) \\
&\leq \left(1-\frac{u^q(-h)}{u^q(0)}\right) + h\frac{1}{2}k(0,\rho(q)),
\end{aligned}
$$

and similarly for $h > 2^{-n}$,

$$
\Upsilon(h) \leq 2^n \left(2^{-n} - \frac{u^q(-h)}{u^q(0)}\left(2^{-n} + 2^{-2n}\frac{k(0,\rho(q))}{2}\right)\right).
$$

Together, this yields an upper bound for all $h \in (0,\infty)$

$$
\Upsilon(h) \leq \left(1-\frac{u^q(-h)}{u^q(0)}\right) + \frac{1}{2}k(0,\rho(q))(h\wedge 1). \tag{23}
$$

Thus (20) follows from (21), (22) and (23). In view of the fact that $\Lambda(dt,dh)$ is a Lévy measure of a subordinator the RHS of (20) will be $\Lambda(dt,dh)$-integrable if $1-u^q(-h)/u^q(0)$ is $\Lambda(dt,dh)$-integrable. First using Fubini's Theorem in (16), followed by Fatou's Lemma, because of (18) and the simple inequality

$$
1-\frac{u^q(-h)}{u^q(0)} \leq 1-e^{-qt}\frac{u^q(-h)}{u^q(0)},
$$

we get

$$\liminf_{n\to\infty} \hat{z}_n \geq \int_{(t,h)\in(0,\infty)^2} \left(1 - \frac{u^q(-h)}{u^q(0)}\right) \Lambda(dt,dh).$$

On the other hand, (13) gives $\lim_{n\to\infty} z_n = \rho(q) < \infty$. Moreover, $\widehat{z}_n \leq z_n$ and we conclude that $1 - u^q(-h)/u^q(0)$ is $\Lambda(dt,dh)$-integrable. Thus (20), together with the Dominated Convergence Theorem, implies (16) and then (15). □

Lemma 6. *Assume that X is of unbounded variation and that X possesses a partial right inverse. Then*

$$\lim_{n\to\infty} \widetilde{z}_n = \kappa + \int_{(0,\infty)} (1 - e^{-qu})\Lambda(dt,\{0\}).$$

Proof. This proof is based on [1, Theorem VI.18], which yields $\mathbb{E}(1 - e^{-qT_x^+}) = k(q,0)V^q(x)$, where $V^q(x) = \int_0^\infty \mathbb{E}(e^{-q\tau_s}; H_s \leq x)ds$. Also, $V^q(x) \sim x$ as $x \downarrow 0$, since V^q is differentiable with $v^q(0+) = 1$, by dominated convergence, as H has unit drift coefficient. Then note that

$$\begin{aligned}
\widetilde{z}_n &= 2^n \mathbb{E}\left(\left(1 - e^{-qT_{\{2^{-n}\}}}\right) 1_{\{O_{2^{-n}}=0,T^+_{2^{-n}}<\infty \text{ or } T^+_{2^{-n}}=\infty\}}\right)\\
&= 2^n \mathbb{E}\left(\left(1 - e^{-qT^+_{2^{-n}}}\right) 1_{\{O_{2^{-n}}=0,T^+_{2^{-n}}<\infty \text{ or } T^+_{2^{-n}}=\infty\}}\right)
\end{aligned}$$

and so, we obtain the required formula from (6) noting $\eta = 0$ in the unbounded variation case

$$\begin{aligned}
&x^{-1}\mathbb{E}\left(1 - e^{-qT_x^+}\right) - x^{-1}\mathbb{E}\left(\left(1 - e^{-qT_x^+}\right) 1_{\{O_x>0,T_x^+<\infty\}}\right)\\
&\quad\longrightarrow \kappa + \int_{(t,h)\in(0,\infty)\times[0,\infty)} (1 - e^{-qt})\Lambda(dt,dh) - \int_{(t,h)\in(0,\infty)^2} (1 - e^{-qt})\Lambda(dt,dh).
\end{aligned}$$

□

Although this is not necessary for our proof of Lemma 6, we would like to mention that we can calculate explicitly $\widetilde{z}_n$ or, as Andreas Kyprianou pointed out to us, we can extend (11):

Proposition 7. *In the relevant case $\delta = 1$ of the setting of (11), cf. (6), we also have*

$$\mathbb{P}(T_x^+ \in dt, O_x = 0)dx = \mathbb{P}(T_x^+ = \overline{G}_{T_x^+-} \in dt, O_x = 0, x - \overline{X}_{T_x^+-} = 0; T_x^+ < \infty)dx = \mathscr{U}(dt,dx),$$

Proof. Indeed, note that for $H_x^{-1} = \inf\{s \geq 0 : H_s > x\}$ we have $O_x = 0$ iff $\Delta H_{H_x^{-1}} = 0$ and $T_x^+ = \tau_{H_x^{-1}}$ for a.e. $x \geq 0$ a.s., so that as H has unit drift coefficient $\delta = 1$,

$$\begin{aligned}
&\int_{x\in(0,\infty)} \int_{t\in(0,\infty)} e^{-\alpha t - \beta x}\mathbb{P}(T_x^+ \in dt, O_x = 0)dx = \mathbb{E}\left(\int_0^\infty e^{-\alpha\tau_{H_x^{-1}} - \beta H_{H_x^{-1}}} 1_{\{\Delta H_{H_x^{-1}}=0\}} dx\right)\\
&= \mathbb{E}\left(\int_0^\infty e^{-\alpha\tau_{H_x^{-1}} - \beta H_{H_x^{-1}}} dH_x^{-1}\right) = \mathbb{E}\left(\int_0^\infty e^{-\alpha\tau_s - \beta H_s} ds\right) = \int_{(x,t)\in(0,\infty)^2} e^{-\alpha t - \beta x}\mathscr{U}(dt,dx).
\end{aligned}$$

□

Proof of Theorem 3. Let X be a Lévy process that possesses a partial right inverse. If X is of unbounded variation, we have $\lim_{n\to\infty} z_n = \rho(q)$ by (13); and (14) together with Lemma 5 and Lemma 6 proves the claim.

When X is of bounded variation we refer to [8, Section 5.3] which discusses in this case the Laplace exponent $\rho(q)$ of the partial right inverse and how the right inverse relates to ladder processes.
This establishes (9). The characteristics can now be read off by inverting the Laplace transform $u^q(-y)/u^q(0) = \mathbb{E}(e^{-qT_{\{-y\}}}; T_{\{-y\}} < \infty)$ for $y \geq 0$, where we recall $u^q(0) = u^q(0+)$, which entails $\mathbb{P}(T_{\{0\}} = 0) = 1$. □

Let us briefly explore the context of the last part of this proof. For $y = 0$, note that $\mathbb{P}(T_{\{0\}} = 0) = 1$ is a trivial consequence of the definition $T_{\{0\}} = \inf\{t \geq 0 : X_t = 0\}$. Indeed, this is the appropriate notion to use in the light of Theorem 1, where excursions below the supremum ending at zero before passing positive do *not* get marked by a further return time $T^{>}_{\{0\}} = \inf\{t > 0 : X_t = 0\}$, which in the bounded variation case would have Laplace transform $\mathbb{E}(e^{-qT^{>}_{\{0\}}}; T^{>}_{\{0\}} < \infty) = u^q(0-)/u^q(0+) < 1$, see e.g. [6, Theorem 43.21].

Proof of Corollary 4. Suppose that X has a partial right inverse. In terms of the characteristics $(\kappa_K, \eta_K, \Lambda_K)$ of the minimal partial right inverse K, this is indeed a full right inverse if $\rho(0+) = \kappa_K = 0$. But if X is recurrent (and has a partial right inverse), then X does not drift to $-\infty$, so $\kappa = 0$, and $\mathbb{P}(T_{\{-y\}} = \infty) = 0$ for all $y \in \mathbb{R}$, so indeed

$$\kappa_K = \kappa + \int_{[0,\infty)^2} \mathbb{P}(T_{\{-y\}} = \infty)\Lambda(ds, dy) = 0.$$ □

Similarly, it is also straightforward to show that in the transient case X possesses a full right inverse if and only if X drifts to $+\infty$ and has no positive jumps in that $\Pi((0,\infty)) = 0$.

3 The excursion measure away from K; proof of Theorem 1

Although Theorem 1 is more refined than Theorem 3, it is now a straightforward consequence.

Proof of Theorem 1 and Corollary 2. The proof relies crucially on [8, Theorem 2]. First recall that the semigroup within excursions away from the right inverse is $(P^{\dagger}_t(y, dz), t \geq 0)$. With this, the excursion measure n^Z is uniquely determined by its entrance laws. Let us first check that the entrance laws given in (8) satisfy (3), which is (vi) of [8, Theorem 2]. A standard excursion measure computation similar to the proof of that theorem together with the Wiener-Hopf factorization gives directly

$$\int_0^\infty e^{-qt} \int_{\mathbb{R}} e^{i\lambda y} \widetilde{n}^R_t(dy)dt = \frac{k(q,0)}{q} \int_{\mathbb{R}\setminus\{0\}} e^{i\lambda x} \mathbb{P}(R_{\gamma(q)} \in dx)$$
$$= \frac{k(q,0)}{q} \left(\mathbb{E}(e^{i\lambda R_{\gamma(q)}}) - \mathbb{P}(R_{\gamma(q)} = 0) \right) = \frac{k(q,0)\widehat{k}(q,0)}{q\widehat{k}(q,i\lambda)} - \eta = \frac{k(q,-i\lambda)}{q + \psi(\lambda)} - \eta, \qquad (24)$$

where $\widehat{k}(\alpha, \beta)$ is the Laplace exponent of the bivariate descending ladder process, which is the ascending ladder process of $-X$, and where $\gamma(q)$ is an independent exponential random variable with rate parameter q; we recall $R_{\gamma(q)} \stackrel{d}{=} \underline{X}_{\gamma(q)} = \inf\{X_s, 0 \leq s \leq \gamma(q)\}$; the Wiener-Hopf identity

$$\frac{k(q,0)}{k(q,-i\lambda)} \frac{\widehat{k}(q,0)}{\widehat{k}(q,i\lambda)} = \frac{q}{q + \psi(\lambda)}$$

can be found in [2, Formulas (4.3.4) and (4.3.7)].
Next we compute the joint transform of the remaining part of the RHS of (8).

$$\begin{aligned}
&\int_0^\infty e^{-qt}\int_{[0,t]\times[0,\infty)}\int_{\mathbb{R}} e^{i\lambda z}P^\dagger_{t-s}(y,dz)\Lambda(ds,dy)dt\\
&\quad=\int_{[0,\infty)\times(0,\infty)} e^{-qs}\int_s^\infty e^{-q(t-s)}\mathbb{E}_y(e^{i\lambda X_{t-s}};T_{\{0\}}>t-s)dt\Lambda(ds,dy)\\
&\quad=\frac{1}{q}\int_{[0,\infty)\times(0,\infty)} e^{-qs}\mathbb{E}_y(e^{i\lambda X_{\gamma(q)}};T_{\{0\}}>\gamma(q))\Lambda(ds,dy),
\end{aligned}$$

where $\gamma(q)$ is an independent exponential random variable with rate parameter q. Also recall

$$\mathbb{P}_y(T_{\{0\}}\le\gamma(q))=\mathbb{E}(e^{-qT_{\{-y\}}})=\frac{u^q(-y)}{u^q(0)},\qquad \text{where } u^q(0)=u^q(0+),$$

so that

$$\begin{aligned}
&\frac{qe^{i\lambda y}}{q+\psi(\lambda)}=\mathbb{E}_y(e^{i\lambda X_{\gamma(q)}})=\mathbb{E}_y(e^{i\lambda X_{\gamma(q)}};T_{\{0\}}>\gamma(q))+\mathbb{E}_y(e^{i\lambda X_{\gamma(q)}};T_{\{0\}}\le\gamma(q))\\
&\quad=\mathbb{E}_y(e^{i\lambda X_{\gamma(q)}};T_{\{0\}}>\gamma(q))+\mathbb{P}_y(T_{\{0\}}\le\gamma(q))\mathbb{E}(e^{i\lambda X_{\gamma(q)}})\\
&\quad=\mathbb{E}_y(e^{i\lambda X_{\gamma(q)}};T_{\{0\}}>\gamma(q))+\frac{u^q(-y)}{u^q(0)}\frac{q}{q+\psi(\lambda)}.
\end{aligned}$$

Thus we have computed $\mathbb{E}_y(e^{i\lambda X_{\gamma(q)}};T_{\{0\}}>\gamma(q))$. Hence

$$\begin{aligned}
&\int_0^\infty e^{-qt}\int_{[0,t]\times[0,\infty)}\int_{\mathbb{R}} e^{i\lambda z}P^\dagger_{t-s}(y,dz)\Lambda(ds,dy)dt\\
&\quad=\frac{1}{q+\psi(\lambda)}\int_{[0,\infty)\times(0,\infty)} e^{-qs}\left(e^{i\lambda y}-\frac{u^q(-y)}{u^q(0)}\right)\Lambda(ds,dy).
\end{aligned}$$

Next observe that by using (6) and adding the numerator of the first term in (24) we get

$$\begin{aligned}
&k(q,-i\lambda)+\int_{[0,\infty)\times(0,\infty)} e^{-qs}\left(e^{i\lambda y}-\frac{u^q(-y)}{u^q(0)}\right)\Lambda(ds,dy)\\
&\quad=\kappa+\eta q-i\lambda+\int_{[0,\infty)^2}\left(1-e^{-qs+i\lambda y}+e^{-qs+i\lambda y}-e^{-qs}\frac{u^q(-y)}{u^q(0)}\right)\Lambda(ds,dy)=\rho(q)-i\lambda,
\end{aligned}$$

where the last equality holds by Theorem 3. Together with (24), this proves (8) since (3) holds for the measure on the RHS of (8).
To finish the proof of Theorem 1, we note that the RHS of (7) is Markovian with semi-group $(P^\dagger_t(y,dz),t\ge0)$ and check that the RHS of (7) has as entrance laws the RHS of (8). Specifically,

$$\begin{aligned}
&(\widetilde{n}^R\oplus\mathbb{K})(\{\omega\in D:\omega(t)\in dz;\zeta(\omega)>t\})\\
&\quad=\widetilde{n}^R(\{\omega_1\in D:\omega_1(t)\in dz;\zeta^+(\omega_1)>t\})\\
&\qquad+\int_{\{\omega_1\in D:\zeta^+(\omega_1)\le t\}}\mathbb{K}(\omega_1;\{\omega_2\in D:\omega_2(t-\zeta^+(\omega_1))\in dz;\zeta(\omega_2)>t-\zeta^+(\omega_1)\})\widetilde{n}^R(d\omega_1)\\
&\quad=\widetilde{n}^R_t(dz)+\int_{[0,t]\times[0,\infty)} P^\dagger_{t-s}(y,dz)\Lambda(ds,dy)
\end{aligned}$$

since $\widetilde{n}^R(\{\omega_1 \in D : \zeta^+(\omega_1) \in ds, \omega_1(\zeta^+(\omega_1)) \in dy\}) = \Lambda(ds, dy)$. $\square$

A Alternative proof of Theorem 3

Suppose that X possesses right inverses. We introduce an alternative construction that we first use to sketch a heuristic proof of Theorem 3. Informally, for all $n \geq 0$ we approximate K by the ascending ladder time process τ, but when an excursion away from the supremum with $e^R_x(\Delta\tau_x) = \Delta H_x > 2^{-n}$ appears, our approximation $\widetilde{K}(n)$ of K makes a jump whose size is the length of this excursion plus the time needed by X to return to the starting height $H_{x-} = \overline{X}_{\tau_{x-}} = X_{\tau_{x-}}$ of the excursion, after which we iterate the procedure. Then $X_{\widetilde{K}_x(n)}$ evolves like an ascending ladder height process H with jumps of sizes exceeding 2^{-n} removed.
Let us formalize this. Fix $n \geq 0$. Consider the bivariate subordinator (τ, H). To begin an inductive definition, let

$$S_1(n) = \inf\{x \geq 0 : \Delta H_x > 2^{-n}\}$$

$$\widetilde{K}_x(n) = \tau_x, \quad 0 \leq x < S_1(n)$$

$$\widetilde{K}_{S_1(n)}(n) = \inf\left\{t \geq \tau_{S_1(n)} : X_t = X_{\tau_{S_1(n)-}}\right\}, \quad \text{if } S_1(n) < \infty.$$

Given $(\widetilde{K}_x(n),\, 0 \leq x \leq S_m(n))$ and $T_m(n) = \widetilde{K}_{S_m(n)}(n) < \infty$, let $X^{(m)}_t(n) = X_{T_m(n)+t} - X_{T_m(n)}$, $t \geq 0$. With $(\tau^{(m)}(n), H^{(m)}(n))$ as the bivariate ladder subordinator of $X^{(m)}(n)$, define

$$S_{m+1}(n) = S_m(n) + \inf\left\{x \geq 0 : \Delta H^{(m)}_x(n) > 2^{-n}\right\}$$

$$\widetilde{K}_{S_m(n)+x}(n) = T_m(n) + \tau^{(m)}_x(n), \quad 0 \leq x < S_{m+1}(n) - S_m(n)$$

$$\widetilde{K}_{S_{m+1}(n)}(n) = T_m(n) + \inf\left\{t \geq \tau^{(m)}_{S_{m+1}(n)-S_m(n)}(n) : X^{(m)}_t(n) = X^{(m)}_{\tau^{(m)}_{S_{m+1}(n)-S_m(n)}(n)-}(n)\right\}.$$

Thus we have defined $\widetilde{K}_x(n)$ for all $x \geq 0$ and $n \geq 0$ a.s. Now it *must be expected that*

$$K_x = \inf_{y>x} \sup_{n\geq 0} \widetilde{K}_{L_n(y)}(n) = \lim_{n\to\infty} \widetilde{K}_x(n), \quad \text{where } L_n(y) = \inf\{x \geq 0 : X_{\widetilde{K}_x(n)} \geq y - 2^{-n}\}. \tag{25}$$

To derive formula (10) of Theorem 3, consider the Poisson point process $(\Delta\widetilde{K}_x(n), x \geq 0)$ whose intensity measure we can calculate from $((\Delta\tau_x, \Delta H_x), x \geq 0)$ using standard thinning (keep if $\Delta H_x \leq 2^{-n}$, modify if $\Delta H_x > 2^{-n}$), marking by independent $T_{\{-\Delta H_x\}}$ if $\Delta H_x > 2^{-n}$ and mapping $(\Delta\tau_x, \Delta H_x, T_{\{-\Delta H_x\}}) \mapsto \Delta\tau_x + T_{\{-\Delta H_x\}} = \Delta\widetilde{K}_x(n)$ of Poisson point processes, as

$$\Lambda_{\widetilde{K}(n)}(dt) = \int_{y\in[0,2^{-n}]} \Lambda(dt, dy) + \int_{(s,y)\in[0,\infty)\times(2^{-n},\infty)} \mathbb{P}\left(s + T_{\{-y\}} \in dt\right) \Lambda(ds, dy),$$

which converges to the claimed expression, as $n \to \infty$. We now make this approach rigorous, the main task being to rigorously establish a variant of (25).
To simplify notation, let us assume in the sequel that X possesses a full right inverse. In the case where only partial right inverses exist we can follow the construction above until the first $m \geq 0$ for which $H^{(m)}(n)$ is killed before its first jump of size exceeding 2^{-n}. If we denote the resulting process by $(\widetilde{K}_x(n), 0 \leq x < \widetilde{\xi}(n))$, we can insert suitable restrictions to events such as $\{\widetilde{\xi}(n) > x\}$ into the following arguments.

Lemma 8. *Let $\widetilde{K}_x(n)$ be as above. Then $((\widetilde{K}_x(n), X_{\widetilde{K}_x(n)}), x \geq 0)$ is a bivariate subordinator with drift coefficient $(\eta, 1)$ and Lévy measure*

$$\widetilde{\Lambda}_n(dt, dz) = \Lambda(dt, dz \cap [0, 2^{-n}]) + \int_{(s,y)\in[0,\infty)\times(2^{-n},\infty)} \mathbb{P}(s + T_{\{-y\}} \in dt, 0 \in dz)\Lambda(ds, dy).$$

Proof. Let $\widetilde{H}_x(n) = X_{\widetilde{K}_x(n)}$. Then $(\widetilde{K}(n), \widetilde{H}(n))$ inherits the drift coefficient $(\eta, 1)$ from (τ, H). By standard thinning properties of Poisson point processes, $((\Delta\widetilde{K}_x(n), \Delta\widetilde{H}_x(n)), 0 \leq x < S_1(n))$ has the distribution of a Poisson point process with intensity measure $\Lambda(dt, dz \cap [0, 2^{-n}])$ run up to an independent exponential time $S_1(n)$ with parameter $\lambda = \Lambda([0, \infty) \times (2^{-n}, \infty))$, and

$$\mathbb{P}(\Delta\widetilde{K}_{S_1(n)} \in dt, \Delta\widetilde{H}_{S_1(n)} \in dz) = \lambda^{-1} \int_{[0,\infty)\times(2^{-n},\infty)} \mathbb{P}(s + T_{\{-y\}} \in dt, 0 \in dz)\Lambda(ds, dy).$$

By the strong Markov property of X at $T_m(n)$, $m \geq 1$, the process $((\Delta\widetilde{K}_x(n), \Delta\widetilde{H}_x(n)), x \geq 0)$ with points at $S_m(n)$, $m \geq 1$, removed, is a Poisson point process with intensity measure $\Lambda(dt, dz \cap [0, 2^{-n}])$ independent of the removed points, which we collect in independent and identically distributed vectors $(S_m(n) - S_{m-1}(n), \Delta\widetilde{K}_{S_m(n)}(n), \Delta\widetilde{H}_{S_m(n)}(n))$, $m \geq 1$. By standard superposition of Poisson point processes, the result follows. □

Lemma 9. *With $\widetilde{K}_x(n)$ as above, we have*

$$K_x = \inf_{y>x} \sup_{n\geq 0} \widetilde{K}_{L_n(y)}(n), \qquad \text{where } L_n(y) = \inf\{x \geq 0 : X_{\widetilde{K}_x(n)} \geq y - 2^{-n}\}.$$

Proof. By construction, the process $(X_{\widetilde{K}_x(n)}, x \geq 0)$ has no jumps of size exceeding 2^{-n}, so that $x - 2^{-n} \leq X_{\widetilde{K}_{L_n(x)}(n)} \leq x$. Note that we have $\widetilde{K}_{L_n(x)}(n) \leq K_x$. Let us define $\widetilde{K}_x$ by

$$\widetilde{K}_x = \liminf_{n\to\infty} \widetilde{K}_{L_n(x)}(n) = \lim_{n\to\infty} \inf_{m\geq n} \widetilde{K}_{L_m(x)}(m) \leq \sup_{n\geq 0} \widetilde{K}_{L_n(x)}(n) \leq K_x,$$

where the limit in the middle member of this sequence of inequalities is an increasing limit of stopping times. Since X is right-continuous and quasi-left-continuous [1, Proposition I.7], we obtain

$$x - 2^{-n} \leq X_{\inf_{m\geq n} \widetilde{K}_{L_m(x)}} \leq x \qquad \Rightarrow \qquad X_{\widetilde{K}_x} = \lim_{n\to\infty} X_{\inf_{m\geq n} \widetilde{K}_{L_m(x)}} = x \qquad \text{a.s.}$$

Now, it is standard to argue that $X_{\widetilde{K}_q} = q$ holds a.s. simultaneously for all $q \in \mathbb{Q} \cap [0, \infty)$ and, since $x \mapsto \widetilde{K}_x$ is increasing and X right-continuous, $\inf_{y>x} \widetilde{K}_y = \inf_{q\in\mathbb{Q}\cap(x,\infty)} \widetilde{K}_q \leq K_x$ is a right-continuous right inverse. Since K is the minimal right-continuous right inverse, $\inf_{y>x} \widetilde{K}_y = K_x$. □

Lemma 10. *Let $\widetilde{K}_x(n)$ be as above. Then for all $x \geq 0$ there is convergence along a subsequence $(n_k)_{k\geq 0}$ of $\lim_{k\to\infty} \widetilde{K}_x(n_k) = K_x$ a.s.*

Proof. Denote by $\mathscr{L}eb$ Lebesgue measure on $[0, \infty)$. By Lemma 8, $\widetilde{H}(n)$ has drift coefficient 1, so $\widetilde{H}_x(n) \geq x$ and $\mathscr{L}eb(\{\widetilde{H}_z(n), 0 \leq z \leq x\}) = x$ a.s., and the distribution of $\widetilde{H}_x(n)$ is such that

$$\mathbb{E}\left(e^{-\beta\widetilde{H}_x(n)}\right) = \exp\left\{-x\beta - x\int_{[0,2^{-n}]} (1 - e^{-\beta y})\Lambda_H(dy)\right\} \to e^{-x\beta} \qquad \text{for all } \beta \geq 0.$$

Therefore, $\widetilde{H}_x(n) \to x$ in probability, and there is a subsequence $(n_k)_{k\geq 0}$ along which convergence holds almost surely. Now let $\varepsilon > 0$. Then there is (random) $N \geq 0$ such that for all $n_k \geq N$, we have $x \leq \widetilde{H}_x(n_k) \leq x + \varepsilon$. Therefore,

$$\limsup_{k\to\infty} \widetilde{K}_x(n_k) \leq \inf_{\varepsilon>0} K_{x+\varepsilon} = K_x.$$

Since $L_n(x) \leq x$ a.s., the previous lemma implies the claimed convergence. □

Proof of Theorem 3. This proof is for the case where X has a full right inverse and we only prove (9). The general case can be adapted. By Lemma 10, we can approximate $K_x = \lim_{k\to\infty} \widetilde{K}_x(n_k)$. The Laplace exponent of $\widetilde{K}_x(n_k)$ follows from Lemma 8 and this yields

$$\begin{aligned}
\rho(q) &= -\ln\left(\mathbb{E}\left(e^{-qK_1}\right)\right) = -\lim_{k\to\infty} \ln\left(\mathbb{E}\left(e^{-q\widetilde{K}_1(n_k)}\right)\right) \\
&= \eta q + \lim_{k\to\infty} \int_{[0,\infty)\times[0,2^{-n_k}]} (1-e^{-qs})\Lambda(ds,dy) \\
&\quad + \lim_{k\to\infty} \int_{[0,\infty)\times(2^{-n_k},\infty)} \int_{[0,\infty)\times[0,\infty)} (1-e^{-qt})\mathbb{P}(s+T_{\{-y\}} \in dt, 0 \in dz)\Lambda(ds,dy) \\
&= \eta q + \int_{[0,\infty)\times\{0\}} (1-e^{-qs})\Lambda(ds,dy) + \int_{[0,\infty)\times(0,\infty)} \left(1-e^{-qs}\mathbb{E}\left(e^{-qT_{\{-y\}}}\right)\right)\Lambda(ds,dy).
\end{aligned}$$

□

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