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SLE AND α-SLE DRIVEN BY LÉVY PROCESSES[1]

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Stochastic Loewner evolutions (SLE) with a multiple $\sqrt{\kappa}B$ of Brownian motion B as driving process are random planar curves (if $\kappa \le 4$) or growing compact sets generated by a curve (if $\kappa > 4$). We consider here more general Lévy processes as driving processes and obtain evolutions expected to look like random trees or compact sets generated by trees, respectively. We show that when the driving force is of the form $\sqrt{\kappa}B + \theta^{1/\alpha}S$ for a symmetric α-stable Lévy process S, the cluster has zero or positive Lebesgue measure according to whether $\kappa \le 4$ or $\kappa > 4$. We also give mathematical evidence that a further phase transition at $\alpha = 1$ is attributable to the recurrence/transience dichotomy of the driving Lévy process. We introduce a new class of evolutions that we call α-SLE. They have α-self-similarity properties for α-stable Lévy driving processes. We show the phase transition at a critical coefficient $\theta = \theta_0(\alpha)$ analogous to the $\kappa = 4$ phase transition.

1. Introduction. *Loewner evolutions* are certain processes $(K_t)_{t\ge 0}$ taking values in the space of closed bounded subsets of the complex upper half plane $\mathbb{H}$ (or other simply connected domains), driven by a càdlàg function $U : [0, \infty) \to \mathbb{R}$. They are best described via ordinary differential equations

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U(t)}, \qquad g_0(z) = z, \tag{1.1}$$
$$z \in \overline{\mathbb{H}} = \{x + iy \in \mathbb{C} : y \ge 0\},$$

as follows. ∂_t is the right derivative as U is right-continuous. For each $z \in \overline{\mathbb{H}}$, the solution of (1.1) is well defined on a time interval $[0, \zeta(z))$. Then the process $K_t := \{z \in \overline{\mathbb{H}} : \zeta(z) \le t\}$, $t \ge 0$, is a strictly increasing family of compact subsets of $\overline{\mathbb{H}}$. We refer to K_t as the *cluster*.

Loewner [15] introduced these in the 1920s in a complex function-theoretic framework of conformal mappings [the solutions $g_t : \mathbb{H} \setminus K_t \to \mathbb{H}$ of (1.1) are conformal mappings]. In the late 1990s, Schramm [20] noticed that $U(t) = \sqrt{\kappa}B_t$ for a standard Brownian motion B leads to an interesting class of *stochastic* Loewner

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evolutions SLE_κ, some of which he conjectured to be scaling limits of important lattice models in statistical physics, subsequently proved in collaboration with Lawler and Werner [13, 14] and by Smirnov [21]. Some introductory texts [11, 24] are now available. Cardy [6] gives a recent review of mathematical progress and further physical conjectures.

Brownian motion is a suitable driving process since its independent identically distributed (i.i.d.) increments translate into a composition of i.i.d. conformal mappings that describe, in a sense, independent growth increments. Furthermore, Loewner evolutions transform well under Brownian scaling, making SLE_κ conformally invariant: that is, on the one hand, the distribution of $(K_t)_{t\geq 0}$ is invariant under homotheties (the only conformal automorphisms of $\mathbb{H}$ leaving start and end points 0 and ∞ fixed), up to a linear time change; on the other hand, we can naturally consider SLE_κ in other simply connected domains by application of a conformal mapping.

In this paper we discard the Brownian scaling property and consider the larger class of processes with stationary independent increments (Lévy processes) as driving processes. Such processes are necessarily discontinuous (except for Brownian motion, with drift). Whereas SLE_κ is either a simple curve ($\kappa \leq 4$) or generated by a curve ($\kappa > 4$) [18, 20], here, roughly, each discontinuity corresponds to a jump of the growth point on the boundary of the growing compact set. This leads to tree-like structures. Beliaev and Smirnov [2] briefly mention such models in a complex analysis context as examples of fractal domains with high multifractal spectrum.

These models were recently introduced in the physics literature by Rushkin et al. [19] who study driving processes of the form $U(t) = \sqrt{\kappa} B_t + \theta^{1/\alpha} S_t$ for a standard Brownian motion B and an independent symmetric α-stable Lévy process S. They observe two phase transitions:

1. The Brownian phase transition of SLE_κ at $\kappa = 4$ is not affected by the additional driving force $\theta^{1/\alpha} S$. It can be expressed in terms of $p(x) = \mathbb{P}(\zeta(x) < \infty)$ as $p(x) = 0$ for all $x \in \mathbb{R} \setminus \{0\}$ for $\kappa \leq 4$ versus $p(x) > 0$ for all $x \in \mathbb{R} \setminus \{0\}$ for $\kappa > 4$. Due to the jumps, simulations look like trees and bushes, respectively.
2. There is another phase transition at $\alpha = 1$, which in the simulations yields "isolated trees/bushes" for $0 < \alpha < 1$ and "forests of trees/bushes" for $1 \leq \alpha < 2$.

We strengthen their results from $x \in \mathbb{R}$ to $z \in \overline{\mathbb{H}}$ and rigorously establish the following theorem.

THEOREM 1.1. *Let $(K_t)_{t\geq 0}$ be an SLE driven by $U_t = \sqrt{\kappa} B_t + \theta^{1/\alpha} S_t$ for a Brownian motion B and an independent symmetric α-stable process S, with $\zeta(z) = \inf\{t \geq 0 : z \in K_t\}$. Then*:

(i) *if $0 \leq \kappa \leq 4$ and $U \not\equiv 0$, then for all $z \in \overline{\mathbb{H}} \setminus \{0\}$, we have $\mathbb{P}(\zeta(z) = \infty) = 1$*;
(ii) *if $\kappa > 4$ and $1 \leq \alpha < 2$, then for all $z \in \overline{\mathbb{H}} \setminus \{0\}$, we have $\mathbb{P}(\zeta(z) < \infty) = 1$*;

(iii) *if* $\kappa > 4$ *and* $0 < \alpha < 1$, *then for all* $z \in \overline{\mathbb{H}} \setminus \{0\}$, *we have* $0 < \mathbb{P}(\zeta(z) < \infty) < 1$ *and* $\lim_{z \to 0, z \in \overline{\mathbb{H}} \setminus \{0\}} \mathbb{P}(\zeta(z) < \infty) = 1$.

Our methods combined with some probabilistic reasoning allow us to deduce the following corollary. Recall that Lévy processes C_t that are just the sums of finite numbers of jumps ΔC_s in any bounded interval $s \in [0, t]$ are called compound Poisson processes. A Lévy process U is called recurrent (transient) if for all $a < 0 < b$ we have $\int_0^\infty 1_{\{a < U_t < b\}}\, dt = \infty$ (resp. $< \infty$) a.s.

COROLLARY 1.2. *Suppose that in the notation of the theorem, the driving process is changed as follows, in terms of* $S_t^c = S_t - \sum_{s \le t} \Delta S_s 1_{\{|\Delta S_s| > c\}}$, *that is, S without its big jumps, for some* $c > 0$, *and independent compound Poisson processes R and T, recurrent and transient, respectively*:

(i) *if* $U_t = \sqrt{\kappa} B_t + \theta^{1/\alpha} S_t^c + R_t$ *or* $U_t = \sqrt{\kappa} B_t + \theta^{1/\alpha} S_t^c + T_t$, *and* $0 \le \kappa \le 4$, *but* $\kappa > 0$ *or* $\theta > 0$ *to avoid trivialities, then for all* $z \in \overline{\mathbb{H}} \setminus \{0\}$, *we have* $\mathbb{P}(\zeta(z) = \infty) = 1$;

(ii) *if* $U_t = \sqrt{\kappa} B_t + \theta^{1/\alpha} S_t^c + R_t$ *and* $\kappa > 4$ *and* $0 < \alpha < 2$, *then for all* $z \in \overline{\mathbb{H}} \setminus \{0\}$, *we have* $\mathbb{P}(\zeta(z) < \infty) = 1$;

(iii) *if* $U_t = \sqrt{\kappa} B_t + \theta^{1/\alpha} S_t^c + T_t$, *and* $\kappa > 4$ *and* $0 < \alpha < 2$, *then for all* $z \in \overline{\mathbb{H}} \setminus \{0\}$, *we have* $0 < \mathbb{P}(\zeta(z) < \infty) < 1$ *and* $\lim_{z \to 0, z \in \overline{\mathbb{H}} \setminus \{0\}} \mathbb{P}(\zeta(z) < \infty) = 1$.

This is strong evidence that the phase transition "at $\alpha = 1$" is attributable to the recurrence/transience dichotomy of Lévy processes. Under suitable regularity conditions on $\mathbb{P}(|U_t| > x) \approx x^{-\alpha}$ as $x \to \infty$, such as regular variation, this is, of course, equivalent to $1 \le \alpha \le \infty$ versus $0 < \alpha \le 1$, where a finer distinction is well known at the critical value $\alpha = 1$.

Since recurrence and transience are governed only by rare big jumps, we *expect* that in the $\kappa \le 4$ case the phase transition is not reflected in the local geometry of the cluster. Heuristically, in both cases pockets in the clusters will stabilize and remain unchanged after a while; in the transient case even the big trees themselves will remain unchanged eventually, whereas in the recurrent case bigger and bigger trees, possibly from the far left and the far right, will almost meet above these unchanged pockets, and this is reflected in the conformal mappings g_t in that a whole pocket is mapped onto a very small portion of the upper half plane that "disappears in the limit" as $t \to \infty$; for $\kappa > 4$ bigger bushes actually meet above pockets, thereby incorporating the pockets in the cluster. We *show* in Proposition 3.5 that the phase transition is reflected in the large-time asymptotics of the g_t, thereby making rigorous another observation in [19].

We leave the geometry of the cluster for further research, but establish the following result.

THEOREM 1.3. *In the situation of Theorem* 1.1, *denote Lebesgue measure on* $\mathbb{H}$ *by m and* $B(0, r) = \{z \in \mathbb{H} : |z| \le r\}$ *for* $r > 0$. *Then*:

(i) *if* $0 \le \kappa \le 4$, *then* $m(\bigcup_{t\ge 0} K_t) = 0$ *a.s.*;
(ii) *if* $\kappa > 4$ *and* $1 \le \alpha < 2$, *then* $m(\mathbb{H} \setminus \bigcup_{t\ge 0} K_t) = 0$ *a.s.*;
(iii) *if* $\kappa > 4$ *and* $0 < \alpha < 1$, *then*

$$\lim_{r\downarrow 0} \frac{m(\bigcup_{t\ge 0} K_t \cap B(0,r))}{m(B(0,r))} = 1 \quad \textit{and} \quad \lim_{r\uparrow\infty} \frac{m(\bigcup_{t\ge 0} K_t \cap B(0,r))}{m(B(0,r))} = 0 \qquad \textit{a.s.}$$

We actually believe that (ii) can be strengthened to $\bigcup_{t\ge 0} K_t = \overline{\mathbb{H}}$ a.s. The other extreme is when the driving process is a compound Poisson process $U(t) = C_t$ with successive jump times J_n, $n \ge 1$, and jump heights X_n, $n \ge 1$. C is piecewise constant and hence the evolution can be decomposed and expressed as

$$g_{J_n+t} = \vartheta_{-X_1-\cdots-X_n} \circ g_t^0 \circ (\vartheta_{X_n} \circ g^0_{J_n-J_{n-1}}) \circ \cdots \circ (\vartheta_{X_1} \circ g^0_{J_1}),$$
$$0 \le t < J_{n+1} - J_n, n \ge 0,$$

a composition of independent and identically distributed conformal mappings $\vartheta_{X_j} \circ g^0_{J_j-J_{j-1}}$, $j \ge 1$, where $g_t^0(z) = \sqrt{z^2+4t}$ is the conformal mapping from $\mathbb{H} \setminus [0, 2\sqrt{t}i]$ to $\mathbb{H}$ that is associated with a driving function $U^0 \equiv 0$ and $\vartheta_x(z) = z - x$ is a translation by $x \in \mathbb{R}$. The flow $(\vartheta_{U_t} \circ g_t)_{t\ge 0}$ is similar to flows of bridges (on [0, 1] instead of $\mathbb{H}$) studied by Bertoin and Le Gall [4].

Clearly, $(K_t)_{t\ge 0}$ is here a forest of trees growing from $\mathbb{R}$, with $g^0_{J_j-J_{j-1}}$ creating branches and ϑ_{X_j} moving the growth point on the boundary. Specifically, $K_t \cup \mathbb{R}$ is path connected and, more precisely, has the *tree property* that for all $y, z \in K_t \cup \mathbb{R}$ there is a simple path $\rho : [0, 1] \to \overline{\mathbb{H}}$, unique up to time parameterization, from $\rho(0) = y$ to $\rho(1) = z$ with $\rho(s) \in K_t \cup \mathbb{R}$ for all $s \in [0, 1]$. If U is not a compound Poisson process, for example, an α-stable Lévy process, we have been unable to show that $K_t \cup \mathbb{R}$ is path connected, but we believe that the following holds.

CONJECTURE 1. *If U_t is a Lévy process with diffusion component $\sqrt{\kappa}B_t$ for some $\kappa \ge 0$, then*:

(i) *if* $0 \le \kappa \le 4$, *then* $K_t \cup \mathbb{R}$ *has the tree property for all* $t \ge 0$. *There is a simple left-continuous function* $\gamma : (0,\infty) \to \mathbb{H}$ *such that* $K_t \cap \mathbb{H} = \{\gamma(s) : 0 < s \le t\}$, *for all* $t \ge 0$;
(ii) *if* $\kappa > 4$, *then* $K_t \cup \mathbb{R}$ *is generated by a left-continuous function* $\gamma : (0,\infty) \to \overline{\mathbb{H}}$ *in that* $\mathbb{H} \setminus K_t$ *is the unbounded connected component of* $\mathbb{H} \setminus \{\gamma(s) : 0 < s \le t\}$, *for all* $t \ge 0$.

This conjecture is a theorem for Brownian SLE_κ (see Rohde and Schramm [18] and Lawler et al. [13]) when γ is indeed continuous. In the setting of Theorem 1.1, the difficult part is to show path connectedness of $\mathbb{R} \cup K_t$, which is not obvious as the logarithmic spiral (see Marshal and Rohde [16]) exemplifies. Heuristically, the $\kappa = 4$ phase transition is not affected by the small jumps since locally, the

Brownian fluctuations dominate jump fluctuations as is expressed, for example, in $(U_{at}/\sqrt{a})_{t\geq 0} \to \sqrt{\kappa}B$ in distribution as $a \downarrow 0$, in the setting of the conjecture.

As a consequence of the scaling properties of (1.1) and Brownian motion of the same index 2, for $\theta = 0$, any $\kappa \geq 0$ and $a > 0$, the process $(\sqrt{a}K_t)_{t\geq 0}$, where $\sqrt{a}K_t = \{\sqrt{a}z : z \in K_t\}$, has the same distribution as $(K_{at})_{t\geq 0}$. The analogous statement for a pure α-stable driving process, that is, $\kappa = 0$ and $\theta > 0$, is not true: the distributions of $(a^{1/\alpha}K_t)_{t\geq 0}$ and $(K_{at})_{t\geq 0}$ are different. Scaling of index 2 is intrinsic to (1.1).

However, we can construct clusters $(K_t)_{t\geq 0}$ such that $(a^{1/\alpha}K_t)_{t\geq 0}$ and $(K_{at})_{t\geq 0}$ have the same distribution by modifying (1.1) to

$$\begin{aligned} \partial_t g_t(z) &= \frac{2|g_t(z) - U(t)|^{2-\alpha}}{g_t(z) - U(t)}, \qquad g_0(z) = z, \\ z &\in \overline{\mathbb{H}} = \{x + iy \in \mathbb{C} : y \geq 0\}, \end{aligned} \tag{1.2}$$

for some $1 < \alpha \leq 2$. This equation still defines a process $(K_t)_{t\geq 0}$ of growing compact subsets of $\overline{\mathbb{H}}$, for a given càdlàg driving process U, and has intrinsic scaling properties of index α. We call this equation the α-Loewner equation. The most interesting driving processes are α-stable processes, that is, $\kappa = 0$ in our setting. We then derive the following phase transition.

THEOREM 1.4. *Let $1 < \alpha < 2$. If $(K_t)_{t\geq 0}$ is the α-SLE driven by $U_t = \theta^{1/\alpha}S_t$ for a symmetric α-stable process S, then there exists $\theta_0(\alpha) > 0$ such that*:

(i) *if $0 < \theta < \theta_0(\alpha)$, then for all $z \in \overline{\mathbb{H}} \setminus \{0\}$, we have $\mathbb{P}(\zeta(z) = \infty) = 1$*;
(ii) *if $\theta > \theta_0(\alpha)$, then for all $z \in \overline{\mathbb{H}} \setminus \{0\}$, we have $\mathbb{P}(\zeta(z) < \infty) = 1$.*

Note that all driving processes are recurrent here, so the analogue to case (iii) in the previous results does not arise. One could, however, for example, add a transient compound Poisson process to the driving process and obtain the analogue to case (iii). We will also deduce the analogue of Theorem 1.3.

COROLLARY 1.5. *In the situation of Theorem 1.4, we have*:

(i) *if $0 \leq \theta < \theta_0(\alpha)$, then $m(\bigcup_{t\geq 0} K_t) = 0$ a.s.*;
(ii) *if $\theta > \theta_0(\alpha)$, then $m(\mathbb{H} \setminus \bigcup_{t\geq 0} K_t) = 0$ a.s.*

This class of growth processes $(K_t)_{t\geq 0}$ seems new and interesting. Theorem 1.4 and the discussion before describe some parallels to the class SLE_κ, $\kappa \geq 0$. Our methods are strong enough to prove these analogous results, even though the functions g_t that solve (1.2) are not conformal mappings. The canonical driving processes are now jump processes, so we expect the self-similar clusters to be trees or structures generated by trees. Again, such structures are easily rigorously established for piecewise constant (e.g., compound Poisson) driving functions, but remain conjectural for stable processes. It would be interesting to know if α-SLE driven by α-stable driving processes are scaling limits of natural lattice models.

The structure of this paper is as follows. In Section 2, we recall and extend some preliminary results on fractional Laplacians, harmonic functions and hitting time distributions; we also give an introduction to Loewner evolutions and provide further and more detailed motivation for our class of driving functions. Sections 3 and 4 study the stochastic differential equation of Bessel type that is associated with (1.1) for stochastic driving functions U and deal with the proof of Theorem 1.1 in the cases $z = x \in \mathbb{R}$ and $z \in \mathbb{H}$, respectively. In Section 5 we study the increasing cluster K_t and prove Theorem 1.3. Section 6 is devoted to properties of α-SLE and the proof of Theorem 1.4.

2. Preliminaries.

2.1. *Symmetric α-stable processes and the fractional Laplacian.* Symmetric α-stable Lévy processes are Markov processes $(S_t)_{t\ge 0}$ starting from $S_0 = 0$, with *stationary independent increments* and càdlàg sample paths, whose distribution is given by

$$\mathbb{E}(e^{i\lambda S_t}) = e^{-t\psi(\lambda)},$$
$$\psi(\lambda) = |\lambda|^\alpha = \int_{\mathbb{R}\setminus\{0\}} \big(1 - e^{i\lambda x} + i\lambda x 1_{\{|x|\le 1\}}\big)|x|^{-\alpha-1}\,dx$$

for some $0 < \alpha < 2$. We use Chapter VIII of Bertoin [3] as our main reference. We can include $\alpha = 2$, where $S_t = \sqrt{2}B_t$ is a Brownian motion B_t, and S has as generator the Laplacian $\Delta_x = \partial_x^2$ on $\mathbb{R}$. Brownian motion has the scaling property of index 2, called *Brownian scaling property* that $(\sqrt{\kappa}B_t)_{t\ge 0}$ has the same distribution as $(B_{\kappa t})_{t\ge 0}$. For $0 < \alpha < 2$, the process S has the *scaling property of index α* that $(\theta^{1/\alpha}S_t)_{t\ge 0}$ has the same distribution as $(S_{\theta t})_{t\ge 0}$. The infinitesimal generator of S is the *fractional Laplacian on* $\mathbb{R}$, defined by the formula

$$\Delta_x^{\alpha/2} w(x) = \lim_{\varepsilon\downarrow 0} \mathcal{A}(1,-\alpha) \int_{\{x'\in\mathbb{R}:|x'-x|>\varepsilon\}} \frac{w(x') - w(x)}{|x - x'|^{1+\alpha}}\,dx', \tag{2.1}$$

where w is a function on $\mathbb{R}$ such that the limit exists for all $x \in \mathbb{R}$, and $\mathcal{A}(1,-\alpha)$ is the constant $\alpha 2^{\alpha-1}\pi^{-1/2}\Gamma((1+\alpha)/2)/\Gamma(1-\alpha/2)$. We refer to Stein [22] for an introduction and properties of the fractional Laplacian. We recall here that the domain of $\Delta_x^{\alpha/2}$ includes the Schwarz space of rapidly decreasing functions. It will be important in the sequel to apply (2.1) as a *formal generator* to functions where the limit does not exist for all $x \in \mathbb{R}$, such as power functions with a singularity at zero.

LEMMA 2.1. *For $p \in \mathbb{R}$, define a function $w_p : \mathbb{R} \to \mathbb{R}$ by $w_p(0) = 0$ and*

$$w_p(x) = |x|^{p-1}, \qquad x \in \mathbb{R}\setminus\{0\}, p \ne 1; \qquad w_1(x) = \ln|x|, \qquad x \in \mathbb{R}\setminus\{0\}.$$

Then,

$$\Delta_x^{\alpha/2} w_p(x) = \mathcal{A}(1,-\alpha)\gamma(\alpha,p)|x|^{p-\alpha-1} \qquad \text{for all } x \in \mathbb{R}\setminus\{0\}, \text{ and } p \in (0,\alpha+1), \tag{2.2}$$

where $\gamma(\alpha,p) = \alpha^{-1}(p-1)\int_0^\infty v^{p-2}(|v-1|^{\alpha-p} - (v+1)^{\alpha-p})\,dv$ *for* $p \neq 1$ *and* $\gamma(\alpha,1) = \alpha^{-1}\int_0^\infty v^{-1}(|v-1|^{\alpha-1} - (v+1)^{\alpha-1})\,dv$.

PROOF. We assume without loss of generality that $x > 0$. By definition (2.1) we have for $p \neq 1$

$$\begin{aligned}
&\Delta_x^{\alpha/2} w_p(x)\\
&\quad = \lim_{\varepsilon\downarrow 0} \mathcal{A}(1,-\alpha)\int_{\{x':|x'-x|>\varepsilon\}} \frac{|x'|^{p-1} - x^{p-1}}{|x-x'|^{1+\alpha}}\,dx'\\
&\quad = \lim_{\varepsilon\downarrow 0} \mathcal{A}(1,-\alpha)x^{p-\alpha-1}\int_{\{x':|x'-1|>\varepsilon\}} \frac{|x'|^{p-1} - 1}{|x'-1|^{1+\alpha}}\,dx'\\
&\quad = \lim_{\varepsilon\downarrow 0} \mathcal{A}(1,-\alpha)x^{p-\alpha-1}\int_{\{x':|x'|>\varepsilon\}} \frac{|x'+1|^{p-1} - 1}{|x'|^{1+\alpha}}\,dx'\\
&\quad = \lim_{\varepsilon\downarrow 0} \mathcal{A}(1,-\alpha)x^{p-\alpha-1}\int_{\varepsilon}^{\infty} \frac{|x'+1|^{p-1} + |x'-1|^{p-1} - 2}{|x'|^{1+\alpha}}\,dx'\\
&\quad = \mathcal{A}(1,-\alpha)\frac{(p-1)x^{p-\alpha-1}}{\alpha}\\
&\qquad \times \int_{\{x':x'>0\}} \frac{(x'+1)^{p-2} + (x'-1)^{p-2}I_{\{x'>1\}} - (1-x')^{p-2}I_{\{0<x'\le 1\}}}{|x'|^{\alpha}}\,dx'\\
&\quad = \mathcal{A}(1,-\alpha)\frac{(p-1)x^{p-\alpha-1}}{\alpha}\int_0^\infty v^{p-2}\big(|v-1|^{\alpha-p} - (v+1)^{\alpha-p}\big)\,dv.
\end{aligned} \tag{2.3}$$

We use the transformation $(x'+1)/x' = v$ and $(x'-1)/x' = v$ in the last step of (2.3). The case $p = 1$ can be proved in the same way. $\square$

REMARK 2.1. By Lemma 2.1, it is easy to check that w_α is a harmonic function on $\mathbb{R}\setminus\{0\}$ for the symmetric α-stable process. When $\alpha > 1$, w_δ is subharmonic and superharmonic on $\mathbb{R}\setminus\{0\}$ when $\delta \in (\alpha,\alpha+1)\cup(0,1)$ and $\delta \in [1,\alpha)$, respectively. When $0 < \alpha < 1$, w_δ is subharmonic and superharmonic on $\mathbb{R}\setminus\{0\}$ when $\delta \in [1,\alpha+1)\cup(0,\alpha)$ and $\delta \in (\alpha,1)$, respectively. When $\alpha = 1$, w_δ is a subharmonic function on $\mathbb{R}\setminus\{0\}$ when $\delta \in (0,1)\cup(1,\alpha+1)$.

By Lemma 4.2 in [7], we can alternatively express the coefficients in Lemma 2.1 as $\gamma(\alpha,p) = \int_0^1((u^{p-1}-1)(1-u^{\alpha-p})(1-u)^{-1-\alpha} + (u^{p-1}-1)(1-u^{\alpha-p})(1+$

$u)^{-1-\alpha})\,du$ for $p \neq 1$ and $\gamma(\alpha, 1) = \int_0^1((1-u^{\alpha-1})\ln(u)(1-u)^{-1-\alpha} + (1-u^{\alpha-1})\ln(u)(1+u)^{-1-\alpha})\,du$. See also [5], Lemma 5.1, [9], Appendix, [19], Appendix for other expressions of these or closely related results.

2.2. *Bessel-type processes and exit times.* Let $(B_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ be standard Brownian motion and an independent symmetric α-stable process with generator $\Delta_x^{\alpha/2}$, on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Define $U_t = \sqrt{\kappa}B_t + \theta^{1/\alpha}S_t$ and the conformal mappings $(g_t)_{t\geq 0}$ of SLE driven by U_t via (1.1). Let $h_t = g_t - U_t$; then we have the Bessel-type stochastic differential equation

$$dh_t(z) = \frac{2\,dt}{h_t(z)} - dU_t, \qquad h_0(z) = z, \qquad z \in \overline{\mathbb{H}} \setminus \{0\}. \tag{2.4}$$

$h_t(z) = h_{1,t}(z) + ih_{2,t}(z)$, $t \geq 0$, is an $\overline{\mathbb{H}}$-valued Markov process, well defined until hitting zero, for every $z \in \overline{\mathbb{H}} \setminus \{0\}$ starting from $z = z_1 + iz_2$. The formal generator of the process h is

$$Af(z) = \frac{-2z_2}{z_1^2 + z_2^2}\partial_{z_2} f(z) + \frac{2z_1}{z_1^2 + z_2^2}\partial_{z_1} f(z) + \frac{\kappa}{2}\partial_{z_1}^2 f(z) + \theta\Delta_{z_1}^{\alpha/2} f(z). \tag{2.5}$$

It will be convenient to adopt a Markov process setup $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (h_t)_{t\geq 0}, (P_z)_{z\in\overline{\mathbb{H}}\setminus\{0\}})$, slightly abusing notation, where h_t under P_z has the same distribution as $h_t(z)$ under $\mathbb{P}$. In this vein, $\zeta = \inf\{t \geq 0 : h_{t-} = 0 \text{ or } h_{t-} = U_t - U_{t-}\}$. We make a convention that $h_t = \Upsilon$, a cemetery point $\Upsilon \notin \overline{\mathbb{H}}$, for $t \geq \zeta$ and $f(\Upsilon) = 0$ for any function f. For a Borel set $D \subset \mathbb{H}$, denote $G_D(z, dz') = \int_0^\infty P_t^D(z, dz')\,dt$, where $(P_t^D(z, dz'))_{t\geq 0}$ is the transition kernel for the process $(h_t)_{t\geq 0}$ killed when leaving D.

LEMMA 2.2. *Let D be an open subset of $\mathbb{H}$ bounded away from 0, that is, such that $B(0, r) \subseteq D^c$ for some $r > 0$. Let $\tau = \inf\{t \geq 0 : h_t \notin D\}$ be the exit time from D, where h_t is as in (2.4). Then for every Borel set $B \subseteq \overline{D}^c$ and every $z \in D$,*

$$P_z\{h_\tau \in B\} = \int_D G_D(z, dz') \int_{\{z_1''\in\mathbb{R}: z_1''+iz_2'\in B\}} \frac{\theta\mathcal{A}(1, -\alpha)}{|z_1'' - z_1'|^{1+\alpha}}\,dz_1'', \tag{2.6}$$

where $z' = z_1' + iz_2'$.

PROOF. We only need to prove that

$$E_z f(h_\tau) = \theta\mathcal{A}(1, -\alpha)\int_D G_D(z, dz') \int_{-\infty}^{\infty} \frac{f(z_1'' + iz_2')}{|z_1'' - z_1'|^{1+\alpha}}\,dz_1'', \tag{2.7}$$

for each C^2 function f on $\overline{\mathbb{H}}$ with compact support satisfying $\mathit{supp}\, f \subseteq \overline{D}^c$. In fact, by Dynkin's formula (see, e.g., Itô [10]), we have for all $z \in D$

$$\begin{aligned} E_z f(h_\tau) &= E_z \int_0^\tau Af(h_t)\,dt = E_z \int_0^\tau \theta \Delta_{z_1}^{\alpha/2} f(h_t)\,dt \\ &= \int_0^\infty \int_D P_t^D(z, dz')\theta \Delta_{z_1'}^{\alpha/2} f(z')\,dt \\ &= \theta \mathcal{A}(1, -\alpha) \int_D G_D(z, dz') \int_{-\infty}^\infty \frac{f(z_1'' + iz_2')}{|z_1'' - z_1'|^{1+\alpha}}\,dz_1'', \end{aligned}$$

which is (2.7). □

Let $b > a > 0$ and define "inner" and "outer" exit times of $h_{1,t}$ from $\{x \in \mathbb{R} : a < |x| < b\}$ as

$$\begin{aligned} \tau_{a,b} &= \inf\{t \ge 0 : |h_{1,t}| \le a;\, |h_{1,s}| < b, \forall s \le t\}, \\ \tau_{b,a} &= \inf\{t \ge 0 : |h_{1,t}| \ge b;\, |h_{1,s}| > a, \forall s \le t\}, \end{aligned} \tag{2.8}$$

where $\inf \varnothing = +\infty$. Let $\mu_{a,b}(z, dx')$ and $\mu_{b,a}(z, dx')$ be the conditional probability distributions under P_z of $h_{1,\tau_{a,b}}$ and $h_{1,\tau_{b,a}}$ on events $\{\tau_{a,b} < \infty\}$ and $\{\tau_{b,a} < \infty\}$, respectively. Set $U_{a,b} = \{z \in \overline{\mathbb{H}} : a < \|z\| < b\}$, where $\|z\| = \|z_1 + iz_2\| = \max\{|z_1|, |z_2|\}$. Denote similarly

$$\begin{aligned} \overline{\tau}_{a,b} &= \inf\{t \ge 0 : \|h_t\| \le a, \|h_s\| < b, \forall s \le t\}, \\ \overline{\tau}_{b,a} &= \inf\{t \ge 0 : \|h_t\| \ge b, \|h_s\| > a, \forall s \le t\}, \end{aligned} \tag{2.9}$$

and let $\overline{\mu}_{a,b}(z, dx')$ and $\overline{\mu}_{b,a}(z, dx')$ be the conditional probability distributions of $h_{1,\overline{\tau}_{a,b}}$ and $h_{1,\overline{\tau}_{b,a}}$ on events $\{\overline{\tau}_{a,b} < \infty, h_{2,\overline{\tau}_{a,b}} \ne a\}$ and $\{\overline{\tau}_{b,a} < \infty\}$, respectively.

LEMMA 2.3. *Let $b > a > 0$, then the following assertions are true*:

(1) *Let $z \in \overline{\mathbb{H}}$ such that $a < |z_1| < b$. Then $\mu_{a,b}(z, dx)$ is absolutely continuous on $\{x : |x| < a\}$ with density function $x \mapsto \varphi_{a,b}(z, x)$; $\mu_{b,a}(z, dx)$ is absolutely continuous on $\{x : |x| > b\}$ with density function $x \mapsto \varphi_{b,a}(z, x)$ such that for all $|x| < a/3$, respectively $|x| > 2b$,*

$$\varphi_{a,b}(z, x) < \frac{3 \cdot 2^{3+4\alpha}}{a}, \qquad \varphi_{b,a}(z, x) < 2^{3+4\alpha} \frac{(2b)^\alpha \alpha}{|x|^{1+\alpha}}. \tag{2.10}$$

(2) *Let $z \in U_{a,b} \subset \overline{\mathbb{H}}$. Then $\overline{\mu}_{a,b}(z, dx)$ is absolutely continuous on $\{x : |x| < a\}$ with density function $x \mapsto \overline{\varphi}_{a,b}(z, x)$; $\overline{\mu}_{b,a}(z, dx)$ is absolutely continuous on $\{x : |x| > b\}$ with density function $x \mapsto \overline{\varphi}_{b,a}(z, x)$ such that the same upper bounds as in (2.10) hold.*

PROOF. We only prove (2) as the proof of (1) is similar. Let $|x| \geq |x'| \geq 2b$. Then for any $|u| < b$, we have

$$2^{-2-2\alpha} \frac{|x'|^{1+\alpha}}{|x|^{1+\alpha}} \leq \frac{|x'-u|^{1+\alpha}}{|x-u|^{1+\alpha}} \leq 2^{2+2\alpha} \frac{|x'|^{1+\alpha}}{|x|^{1+\alpha}}. \tag{2.11}$$

Let $z \in \overline{\mathbb{H}}$ such that $z \in U_{a,b}$. For $|x| > b$, denote

$$f(x) = \frac{1}{P_z\{\overline{\tau}_{a,b} > \overline{\tau}_{b,a}\}} \int_{U_{a,b}} \frac{\theta \mathcal{A}(1,-\alpha)}{|x - z_1'|^{1+\alpha}} G_{U_{a,b}}(z, dz'). \tag{2.12}$$

By Lemma 2.2, we know that f is the density of $\overline{\mu}_{b,a}$ on $\{x : |x| > b\}$. By (2.11) and (2.12), we see that for $|x| > x' = 2b$

$$2^{-2-2\alpha} \frac{(2b)^{1+\alpha}}{|x|^{1+\alpha}} f(2b) \leq f(x) \leq 2^{2+2\alpha} \frac{(2b)^{1+\alpha}}{|x|^{1+\alpha}} f(2b). \tag{2.13}$$

Hence we have

$$2\int_{2b}^{\infty} 2^{-2-2\alpha} \frac{(2b)^{1+\alpha}}{|x|^{1+\alpha}} f(2b)\,dx \leq \int_{-\infty}^{-2b} f(x)\,dx + \int_{2b}^{\infty} f(x)\,dx \leq 1,$$

which leads to $f(2b) \leq b^{-1}\alpha 2^{2\alpha}$. Thus the assertion concerning $\overline{\mu}_{b,a}$ follows from (2.13).

Now let $|x| \leq |x'| \leq a/3$. Then for any $|u| > a$ we have

$$2^{-2-2\alpha} \leq \frac{|u - x'|^{1+\alpha}}{|u-x|^{1+\alpha}} \leq 2^{2+2\alpha}. \tag{2.14}$$

Denote

$$\begin{aligned} f(x) = {} & \frac{1}{P_z\{\overline{\tau}_{a,b} < \overline{\tau}_{b,a}, h_{2,\overline{\tau}_{a,b}} \neq a\}} \\ & \times \int_{U_{a,b}} \frac{\theta \mathcal{A}(1,-\alpha)}{|z_1' - x|^{1+\alpha}} G_{U_{a,b}}(z, dz'), \qquad |x| < a. \end{aligned} \tag{2.15}$$

By definition of $\overline{\mu}_{a,b}$ and Lemma 2.2, we know that f is the density of $\overline{\mu}_{a,b}$ on $\{x : |x| < a\}$. By (2.16) and (2.17), we see that for $|x| < x' = a/3$

$$2^{-2-2\alpha} f(a/3) \leq f(x) \leq 2^{2+2\alpha} f(a/3). \tag{2.16}$$

Hence we have

$$\int_{-a/3}^{a/3} 2^{-2-2\alpha} f(a/3)\,dx \leq \int_{-a/3}^{a/3} f(x)\,dx \leq 1,$$

which leads to $f(a/3) \leq 3a^{-1}2^{1+2\alpha}$. Thus the assertion concerning $\overline{\mu}_{a,b}$ follows from (2.16). $\square$

REMARK 2.2. Let $g(x) = \ln|x|$ or $g(x) = |x|^{p-1}$ for $x \neq 0$ and $0 < p < \alpha + 1$. By Lemma 2.3, we see that $\int g\mu_{a,b}$, $\int g\mu_{b,a}$, $\int g\overline{\mu}_{a,b}$ and $\int g\overline{\mu}_{b,a}$ are all finite.

Whether conditional distributions such as $\mu_{a,b}$ have atoms at a and $-a$ depends on the so-called creeping properties of Lévy processes (and how they are affected by a drift); see Millar [17] and Vigon [23]. Specifically, there will be atoms if $\kappa > 0$. The measure $\mu_{b,a}$ will have atoms at b and $-b$ if $\kappa > 0$, or if $\kappa = 0$ and $\alpha < 1$.

2.3. *Growing clusters, Loewner evolutions and independent increments*. The Riemann mapping theorem implies that for a compact set $K \subset \overline{\mathbb{H}}$ such that $\mathbb{H} \setminus K$ is simply connected, the family of conformal mappings $k : \mathbb{H} \setminus K \to \mathbb{H}$ is a set of three real dimensions. Since $\infty \notin K$, it is natural to choose $k(\infty) = \infty$, the only point one can consistently fix for all compact sets K, with compositions of such conformal mappings in mind. The expansion at infinity then takes the form

$$k(z) = a\left(z + b + \frac{\operatorname{hcap}(K)}{z}\right) + O\left(\frac{1}{z^2}\right)$$

for remaining parameters $a > 0$ and $b \in \mathbb{R}$,

where $\operatorname{hcap}(K)$ is called the half-plane capacity (see Lawler [11], Section 3.4). It measures the size of K. Any increasing process $(K_t)_{t\geq 0}$ of compact sets with continuously increasing capacities can be (time-)parameterized such that $\operatorname{hcap}(K_t) = 2t$. Choosing $a = 1$ is natural; $b = b_g := 0$ is one choice specifying a family of conformal mappings $(g_t)_{t\geq 0}$. Under the local growth condition

$$\bigcap_{\varepsilon>0} \overline{\{g_t(z) : z \in K_{t+\varepsilon} \setminus K_t\}} = \{\text{single point}\} =: \{U(t)\} \qquad \text{for all } t \geq 0, \tag{2.17}$$

where $\overline{C}$ denotes the closure of a Borel set $C \subset \overline{\mathbb{H}}$; this growth point $b = b_h(t) := -U(t)$ is another choice for the parameter b specifying another family of conformal mappings $(h_t)_{t\geq 0}$. It can be checked that $(K_t)_{t\geq 0}$ is then the Loewner evolution driven by $(U(t))_{t\geq 0}$, the family $(g_t)_{t\geq 0}$ solves Loewner's differential equation (1.1) (see Lawler [11], Section 4.1), and $h_t(z) = g_t(z) - U(t)$ solves the Bessel equation (2.4) when integrating suitable test functions. In general, $(U(t))_{t\geq 0}$ may be just measurable. However, we will assume in the sequel that $(U(t))_{t\geq 0}$ is càdlàg. The local growth condition, even with a càdlàg function $(U(t))_{t\geq 0}$, is strictly weaker than the condition

$$\begin{aligned} g_t^{-1}(\{U(t)\}) &:= \bigcap_{\varepsilon>0} \overline{g_t^{-1}(B(U(t), \varepsilon))} \\ &= \bigcap_{\varepsilon>0} \overline{K_{t+\varepsilon} \setminus K_t} = \{\text{single point}\} =: \{\gamma(t)\}, \end{aligned} \tag{2.18}$$

for a càdlàg function $\gamma:(0,\infty)\to\overline{\mathbb{H}}$, where $B(x,\varepsilon)=\{z\in\mathbb{H}:|z-x|\le\varepsilon\}$. In general, even under the local growth condition, equality may fail. If equality holds, one can ask whether $(K_t)_{t\ge0}$ is *generated* by a function γ in a suitable class of functions, that is, $\mathbb{H}\setminus K_t$ is the unbounded connected component of $\mathbb{H}\setminus\overline{\{\gamma(s),0<s\le t\}}$, or even whether $\mathbb{H}\cap K_t=\{\gamma(s-),0<s\le t\}$, that is,

$$\Big\{z\in\mathbb{H}\setminus K_{t-}:\lim_{\varepsilon\downarrow0}g_{t-\varepsilon}(z)=U(t-)\Big\}=\mathbb{H}\cap K_t\setminus K_{t-}=\{\gamma(t-)\}. \tag{2.19}$$

In fact, SLE_κ for $4<\kappa<8$ are examples where (2.18) holds but (2.19) fails—further points in the left-hand member of (2.19) are called "swallowed points." The logarithmic spiral of Marshal and Rohde [16] is an example where (2.18) fails—here the otherwise well-defined and continuous function γ has neither left nor right limits at the time of the singularity, even though the driving function $(U(t))_{t\ge0}$ is continuous. Werner [24] remarks that one can build examples with a dense set of such singularities at different scales. In a rather more regular setting, it is shown in [16] that 1/2-Hölder continuity of $(U(t))_{t\ge0}$ with small norm is sufficient for the existence and continuity of a simple curve γ.

Let us discuss further the geometric reasons for the choice of parameters, as they provide further motivation for stochastic driving functions that are linear combinations of stable processes with stationary independent increments. The first was $\infty\mapsto\infty$. Alternatively, one could fix $x\mapsto x$ for any specific $x\in\mathbb{R}$, the boundary of $\overline{\mathbb{H}}$, provided $x\notin K$ but K need not be compact. This is related to Loewner evolutions "from 0 to x," rather than "from 0 to ∞."

Now let $(K_t)_{t\ge0}$ be a Loewner evolution driven by any measurable function $(U(t))_{t\ge0}$, growing "from 0 to ∞"; denote the associated solution to Loewner's equation by $(g_t)_{t\ge0}$. The only conformal coordinate changes that leave zero and infinity fixed are homotheties $z\mapsto cz$ inviting us to investigate $\widetilde{k}_t(z)=cg_t(z/c)$, $t\ge0$. Clearly, these conformal mappings grow $(cK_t)_{t\ge0}$, where $\operatorname{hcap}(cK_t)=c^2\operatorname{hcap}(K_t)$, so that we reparameterize $k_t=\widetilde{k}_{c^{-2}t}$ and obtain

$$\partial_t k_t(z)=\frac{2}{k_t(z)-cU_{c^{-2}t}},\qquad k_0(z)=z,\qquad z\in\overline{\mathbb{H}}, \tag{2.20}$$

so that $(cK_{c^{-2}t})_{t\ge0}$ is a Loewner evolution driven by $(cU_{c^{-2}t})_{t\ge0}$. This is the scaling property of index 2 that is therefore intrinsic to Loewner's equation.

PROPOSITION 2.4 ([12, 18] for SLE_κ). (a) *A family $(K_t)_{t\ge0}$ of random compact sets is generated by a flow $h_t:\mathbb{H}\setminus K_t\to\mathbb{H}$ with stationary independent "increments" $h_{s,t}=h_t\circ h_s^{-1}$, $s\le t$, if and only if the driving function $(U(t))_{t\ge0}$ has the finite-dimensional distributions of a Lévy process.*

(b) *If $(U(t))_{t\ge0}$ is a Lévy process, then the distribution of $(\sqrt{a}K_{a^{-1}t})_{t\ge0}$ is the same as that of $(K_t)_{t\ge0}$ if and only if $(U(t))_{t\ge0}$ is a multiple of Brownian motion.*

(c) *If $U=\sqrt{\kappa}B+\theta^{1/\alpha}S$ for a Brownian motion B and an independent symmetric stable process of index $\alpha\in(0,2)$, then $(\sqrt{a}K_{a^{-1}t})_{t\geq 0}$ has the same distribution as a Loewner evolution driven by $\widetilde{U}=\sqrt{\kappa}B+\widetilde{\theta}^{1/\alpha}S$, where $\widetilde{\theta}=a^{\alpha/2-1}\theta$.*

PROOF. For (a) just note that for fixed $s\geq 0$ and $h_t^{(s)}=h_{s+t}\circ h_s^{-1}$, we have by (2.4)

$$dh_t^{(s)}(z)=dh_{s+t}(h_s^{-1}(z))=\frac{2\,dt}{h_{s+t}(h_s^{-1}(z))}-dU_{s+t}=\frac{2\,dt}{h_t^{(s)}(z)}-dU_t^{(s)},$$
$$h_0^{(s)}(z)=z, z\in\overline{\mathbb{H}}\setminus\{0\},$$

where $U_t^{(s)}=U_{s+t}-U_s$, and this easily yields the result. (b) and (c) are simple consequences of the scaling properties of Loewner's equation, (2.20), and of B and S (see Section 2.1). □

The property in (b) is called conformal invariance. For any simply connected domain $D\subset\mathbb{C}$, $D\neq\mathbb{C}$, one can now uniquely define SLE_κ from one boundary point α to another boundary point β by conformal mappings $f:\mathbb{H}\to D$ with $f(0)=\alpha$ and $f(\infty)=\beta$, up to a linear time change. For any other Lévy process, the definition is not unique. However, note that for the driving processes in (c), the properties of SLE studied in this paper do not depend on θ.

3. $\mathbb{R}$-valued Bessel-type processes driven by $U=\sqrt{\kappa}B+\theta^{1/\alpha}S$. By (2.4), it is easy to see that $(h_t(x))_{0\leq t<\zeta(x)}$ is $\mathbb{R}$-valued for all $x\in\mathbb{R}\setminus\{0\}$. In this case their formal generator A reduces to

$$Af(x)=\frac{2}{x}\partial_x f(x)+\frac{\kappa}{2}\partial_x^2 f(x)+\theta\Delta_x^{\alpha/2}f(x)\qquad\text{for all } x\in\mathbb{R}\setminus\{0\}.$$

PROPOSITION 3.1. *When $0\leq\kappa\leq 4$ and $0<\alpha<2$, we have $\zeta(x)=\infty$ a.s. for all $x\in\mathbb{R}\setminus\{0\}$.*

PROOF. We will use the same notation as in Lemma 2.1 and always assume that $\kappa>0$. The case $\kappa=0$ can be proved similarly.

Case 1. $0<\alpha\leq 1$. By Lemma 2.1, we have for $y\in\mathbb{R}\setminus\{0\}$

$$\begin{aligned}Aw_1(y)&=\frac{2}{y}\partial_y w_1(y)+\frac{\kappa}{2}\partial_y^2 w_1(y)+\theta\Delta_y^{\alpha/2}w_1(y)\\&\geq\theta\Delta_y^{\alpha/2}w_1(y)=\theta\mathcal{A}(1,-\alpha)\gamma(\alpha,1)|y|^{-\alpha}\geq 0.\end{aligned}$$

For $0<a<b$, let $\tau_{a,b}$ and $\tau_{b,a}$ be the inner and outer exit times defined in (2.8). Let $\mu_{a,b}$ and $\mu_{b,a}$ be the corresponding conditional probability distribution. By

Dynkin's formula we have

$$\ln|x| \le P_x\{\tau_{a,b} < \tau_{b,a}\} \int_{\{|y|\le a\}} \ln|y| \mu_{a,b}(x,dy) + P_x\{\tau_{a,b} > \tau_{b,a}\} \int_{\{|y|\ge b\}} \ln|y| \mu_{b,a}(x,dy).$$

Therefore

$$P_x\{\tau_{a,b} < \tau_{b,a}\} \le \frac{\ln|x| - \int_{\{|y|\ge b\}} \ln|y| \mu_{b,a}(x,dy)}{\int_{\{|y|\le a\}} \ln|y| \mu_{a,b}(x,dy) - \int_{\{|y|\ge b\}} \ln|y| \mu_{b,a}(x,dy)}. \tag{3.1}$$

By Lemma 2.3 we know that $\int_{\{|y|\ge b\}} \ln|y|\mu_{b,a}(x,dy)$ is bounded for fixed b uniformly in $a < b$. Letting $a \downarrow 0$ in (3.1), we get $\zeta = \infty$, P_x-a.s.

Case 2. $0 < \kappa < 4$, $1 < \alpha < 2$. Let $f_1 = w_{3/2-2/\kappa}$. First we prove the case $\kappa \ge 2$. By Lemma 2.1 we have for $y \ne 0$

$$\begin{aligned} Af_1(y) &= \left(\frac{2}{y}\partial_y + \frac{\kappa}{2}\partial_y^2\right) w_{3/2-2/\kappa}(y) + \theta \Delta_y^{\alpha/2} w_{3/2-2/\kappa}(y) \\ &= \left(\frac{1}{2} - \frac{2}{\kappa}\right)\left(1 - \frac{\kappa}{4}\right)|y|^{-3/2-2/\kappa} \\ &\quad + \theta \mathcal{A}(1,-\alpha)\gamma\left(\alpha, \frac{3}{2} - \frac{2}{\kappa}\right)|y|^{1/2-2/\kappa-\alpha}. \end{aligned} \tag{3.2}$$

Noticing that $(\frac{1}{2} - \frac{2}{\kappa})(1 - \frac{\kappa}{4}) < 0$, we can find a constant c such that $Af_1(y) - cf_1(y) < 0$ for all $y \ne 0$. Again by Dynkin's formula we obtain

$$f_1(x) \ge E_x[e^{-c\tau_{a,b}} f_1(h_{\tau_{a,b}})] + E_x[e^{-c\tau_{b,a}} f_1(h_{\tau_{b,a}})]. \tag{3.3}$$

If $P_x\{\zeta < \infty\} > 0$, we can choose $b, T \in \mathbb{R}$ big enough such that $P_x\{\lim_{a\downarrow 0} \tau_{a,b} < T\} > 0$. Hence by (3.3), we get $f_1(x) \ge e^{-cT} P_x\{\lim_{a\downarrow 0} \tau_{a,b} < T\} a^{1/2-2/\kappa} + E_x[e^{-c\tau_{b,a}} f_1(h_{\tau_{b,a}})]$, which is impossible when taking $a \downarrow 0$. When $0 < \kappa < 2$, we can take $f_1 = w_{1/2}$ and use the same method.

Case 3. $\kappa = 4$, $1 < \alpha < 2$. By Lemma 2.1 we have $(\frac{2}{y}\partial_y + 2\partial_y^2) w_1(y) = 0$. Therefore for $y \ne 0$ and $c > 0$ we have

$$\begin{aligned} &A(w_1 + cw_{3-\alpha})(y) \\ &\quad = c\left(\frac{2}{y}\partial_y + 2\partial_y^2\right) w_{3-\alpha}(y) + \theta\Delta_y^{\alpha/2} w_1(y) + c\theta\Delta_y^{\alpha/2} w_{3-\alpha}(y) \\ &\quad = 2c(2-\alpha)^2 |y|^{-\alpha} + \theta\mathcal{A}(1,-\alpha)\gamma(\alpha,1)|y|^{-\alpha} \\ &\qquad + c\theta\mathcal{A}(1,-\alpha)\gamma(\alpha, 3-\alpha)|y|^{2-2\alpha}. \end{aligned} \tag{3.4}$$

By (3.4) and noticing that $-\alpha < 2 - 2\alpha$, we can find c large enough and $r > 0$ small enough such that $Af_2(y) > 0$ for $|y| < r$, $y \ne 0$. Then following the same method as in case 1, we can prove $P_x\{\tau_{0,r} < \tau_{r,0}\} = 0$, which leads to the conclusion. $\square$

PROPOSITION 3.2. *When* $4 < \kappa$ *and* $1 \le \alpha < 2$, *we have* $\zeta(x) < \infty$ *a.s. for all* $x \in \mathbb{R} \setminus \{0\}$.

PROOF. We will use the same notation as in Lemmas 2.1 and 2.3. Without loss of generality we assume $x > 0$.

Case 1. $2 - 4/\kappa \le \alpha < 2$. In this case $\gamma(\alpha, 2-4/\kappa) \le 0$. We get by Lemma 2.1 that $Aw_{2-4/\kappa} \le 0$. By Dynkin's formula we have

$$
\begin{aligned}
&P_x\{\tau_{a,b} < \tau_{b,a}\} \\
&\quad \ge \frac{\int_{\{|y|\ge b\}} |y|^{1-4/\kappa} \mu_{b,a}(x, dy) - |x|^{1-4/\kappa}}{\int_{\{|y|\ge b\}} |y|^{1-4/\kappa} \mu_{b,a}(x, dy) - \int_{\{|y|\le a\}} |y|^{1-4/\kappa} \mu_{a,b}(x, dy)}.
\end{aligned}
\tag{3.5}
$$

By Lemma 2.3, letting $a \downarrow 0$ and then $b \uparrow \infty$, we get the conclusion.

Case 2. $1 < \alpha < 2 - 4/\kappa$. By Lemma 2.1, we can check $Aw_\alpha < 0$. Hence we can get the same conclusion by the method above.

Case 3. $\alpha = 1$. By Lemma 2.1, we can check that there exists a number $c > 0$ satisfying $Aw_{3/2-2/\kappa}(y) < 0$ for $0 < |y| < c$. Hence we obtain $\lim_{y\downarrow 0} P_y\{\tau_{0,c} < \tau_{c,0}\} = 1$ by Dynkin's formula. Now, by the Markov property, we only need to prove that $P_x\{\tau_{a,\infty} < \infty\} = 1$ for all $a > 0$ and $x \ne 0$. Here $\tau_{a,\infty} = \inf_{b>a} \tau_{a,b}$.

By Lemma 2.1, we have $Aw_1(y) < 0$ for $y \ne 0$. Hence we have by Dynkin's formula

$$
P_x\{\tau_{a,b} < \tau_{b,a}\} \ge \frac{\ln|x| - \int_{\{|y|\ge b\}} \ln|y| \mu_{b,a}(x, dy)}{\int_{\{|y|\le a\}} \ln|y| \mu_{a,b}(x, dy) - \int_{\{|y|\ge b\}} \ln|y| \mu_{b,a}(x, dy)}.
$$

By Lemma 2.3, letting $b \uparrow \infty$, we have $P_x\{\tau_{a,\infty} < \infty\} = 1$. $\square$

LEMMA 3.3. *Let* $4 < \kappa$ *and* $0 < \alpha < 1$. *There exist constants* $k_1, k_2 > 0$ *depending on* κ, α, θ *such that*:

$$
P_x\{\zeta = \infty\} > k_2 \qquad \text{for all } x \ge k_1. \tag{3.6}
$$

PROOF. By Lemma 2.1, we can choose c large enough such that $Aw_{\alpha/2+1/2}(y) < 0$ for $|y| > c/2$. Hence we have

$$
\begin{aligned}
&P_x\{\tau_{c/2,b} > \tau_{b,c/2}\} \\
&\quad \ge \frac{\int_{\{|y|\le c/2\}} |y|^{\alpha-1} \mu_{b,a}(x, dy) - c^{\alpha-1}}{\int_{\{|y|\le c/2\}} |y|^{\alpha-1} \mu_{a,b}(x, dy) - \int_{\{|y|\ge b\}} |y|^{\alpha-1} \mu_{b,a}(x, dy)}, \qquad c < x < b.
\end{aligned}
$$

By Lemma 2.3, letting $b \uparrow \infty$, we get the conclusion. $\square$

PROPOSITION 3.4. *Let* $4 < \kappa$ *and* $0 < \alpha < 1$. *There exists constant* $c > 0$ *such that*:

(a) $\frac{1}{c}|x|^{1-4/\kappa} < P_x\{\zeta = \infty\} < c|x|^{1-4/\kappa}, \qquad 0 < |x| \le 1;$

(b) $\frac{1}{c}|x|^{\alpha-1} < P_x\{\zeta < \infty\} < c|x|^{\alpha-1}, \qquad |x| > 1.$

PROOF. First we prove the upper bound in (a). Define functions $u_1(y) = |y|^{1-2/\kappa} \wedge 2$ and $u_2(y) = |y|^{1-4/\kappa} \wedge 2$. Now we suppose $1 - 2/\kappa < \alpha$. By Lemma 2.1 and direct calculation we have

$$
\begin{aligned}
&\frac{1}{c_1} < \lim_{|y|\downarrow 0} |\Delta_y^{\alpha/2} u_1(y)|/|y|^{1-2/\kappa-\alpha} < c_1;\\
&\frac{1}{c_2} < \lim_{|y|\downarrow 0} |\Delta_y^{\alpha/2} u_2(y)|/|y|^{1-4/\kappa-\alpha} < c_2,
\end{aligned}
\tag{3.7}
$$

for some positive constants c_1 and c_2. Choose a small positive real number c_3 such that $u_2(y) - c_3 u_1(y) > 0$ for $y \neq 0$. We have, for $0 < |y| < 1$,

$$
A(u_2 - c_3 u_1)(y) = -c_3(1 - 2/\kappa)|y|^{-1-2/\kappa} + \theta \Delta_y^{\alpha/2}(u_2 - c_3 u_1)(y). \tag{3.8}
$$

Let $f_1 = u_2 - c_3 u_1$. By (3.7) and (3.8), we can find a positive real number c_4 such that $Af_1(y) < 0$ for $y \neq 0$ and $|y| < c_4$. Applying the same notation as in Proposition 3.1, we have for $0 < a < c_4$

$$
P_x\{\tau_{a,c_4} > \tau_{c_4,a}\} \le \frac{f_1(x)}{\int_{|y|\ge c_4} f_1(y)\mu_{c_4,a}(x,dy) - \int_{|y|\le a} f_1(y)\mu_{a,c_4}(x,dy)}.
$$

By Lemma 2.3, letting $a \downarrow 0$ in the equality above, we have

$$
P_x\{\zeta = \infty\} \le P_x\{\tau_{0,c_4} > \tau_{c_4,0}\} \le \frac{x^{1-4/\kappa}}{\lim_{a\downarrow 0} \int_{|y|\ge c_4} f_1(y)\mu_{c_4,a}(x,dy)},
$$

which gives the second inequality in (a). When $1 - 2/\kappa \ge \alpha$, we can prove the upper bound in the same way as above by noticing that

$$
\begin{aligned}
&\frac{1}{c} < \lim_{|y|\downarrow 0} |\Delta_y^{\alpha/2} u(y)|/\ln|y| < c \qquad &&\text{when } \beta = \alpha;\\
&|\Delta_y^{\alpha/2} u(y)| < c, \qquad y \in (-1,1) &&\text{when } \beta > \alpha,
\end{aligned}
\tag{3.9}
$$

for some constant c depending on β and α, where $u(y) = |y|^\beta \wedge 2$. This can be checked directly; see also Proposition 2.3 in [8] and Proposition 2.5 in [7].

Next we prove the lower bound in (a). We use the notation k_1 and k_2 as in Lemma 3.3. Let $u_3(y) = |y|^{1-4/\kappa} \wedge M$ for some $M > 0$. Choose M big enough such that $Au_3(y) > 0$ for $0 < |y| < k_1$. By this fact and applying the same method as above, we can prove that for some constant c_5

$$
P_x\{\tau_{k_1,0} < \tau_{0,k_1}\} \ge c_5|x|^{1-4/\kappa}, \qquad 0 < x < k_1.
$$

Hence by the Markov property and Lemma 3.3 we get $P_x\{\zeta = \infty\} \ge k_2 c_5 |x|^{1-4/\kappa}$ and complete the proof of (a). We omit the proof of (b) as it can be proved by similar discussions. $\square$

To end this section that dealt with

$$\begin{aligned} p(x) &= P_x(\zeta < \infty) \\ &= P_x(h_{t-} = 0 \text{ or } h_{t-} = U_t - U_{t-} \text{ for some } t \geq 0), \qquad x \in \mathbb{R} \setminus \{0\}, \end{aligned}$$

we briefly turn to a related quantity studied by [19], namely,

$$\widetilde{p}(x) = P_x\Big(\liminf_{t\to\infty} |h_t| = 0\Big) \qquad \text{with the convention } |h_t| = |\Upsilon| = 0 \text{ for } t \geq \zeta.$$

While $p(x)$ has a geometric meaning, $\widetilde{p}(x)$ does not, so it is of limited interest for the study of the growing clusters. However, it is of some interest in the study of recurrence and transience of Bessel-type processes and it exhibits the phase transition at $\alpha = 1$ observed in [19]. Our methods allow us to rigorously establish their result.

PROPOSITION 3.5. *In the situation of Theorem* 1.1:

(a) *if* $1 \leq \alpha < 2$, *then* $\widetilde{p}(x) = 1$ *for all* $x \in \mathbb{R} \setminus \{0\}$;
(b) *if* $0 < \alpha < 1$ *and* $\kappa > 4$, *then* $\widetilde{p}(x) = p(x) \in (0, 1)$ *for all* $x \in \mathbb{R} \setminus \{0\}$;
(c) *if* $0 < \alpha < 1$ *and* $0 \leq \kappa \leq 4$, *then* $\widetilde{p}(x) = 0$ *for all* $x \in \mathbb{R} \setminus \{0\}$.

PROOF. First let $\kappa > 4$. Clearly $\widetilde{p}(x) \geq p(x)$, so the case $1 \leq \alpha < 2$ follows from Proposition 3.2. For the upper bound in the case $0 < \alpha < 1$, we only need to prove that $\lim_{|x|\downarrow 0} p(x) = 1$, by the Markov property. This is due to the lower bound of $p(x)$ derived from Proposition 3.4(a).

Now let $0 \leq \kappa \leq 4$. For $1 < \alpha < 2$, using the methods of the previous propositions, it is easy to check that $f(x) = |x|^{(\alpha-1)/2}$ is a superharmonic function for $(h_t)_{t\geq 0}$ on $(-\infty, N) \cup (N, \infty)$ if N is big enough. For $\alpha = 1$ take $f(x) = \log|x| - |x|^{-1/2}$. Hence for $x \in \mathbb{R} \setminus \{0\}$

$$P_x\{\tau_N < \infty\} = 1, \tag{3.10}$$

where $\tau_N = \inf\{t \geq 0 : |h_t| < N\}$. For any $0 < a < N$ we can construct a function that is subharmonic on $(-N, -a) \cup (a, N)$, for example, of the form $f(x) = |x|^{-1} 1_{\{|x|>a/2\}} + M 1_{\{|x|\leq a/2\}}$ for big enough M, to prove that there is a $q > 0$ such that

$$P_x\{\tau_{a,2N} < \infty\} > q, \qquad |x| < N. \tag{3.11}$$

Hence, we get $P_x\{\tau_a < \infty\} = 1$ for all x by (3.10), (3.11) and the Markov property.

For $0 < \alpha < 1$ and $0 \leq \kappa < 4$, we can prove the result using the superharmonic function $f(x) = |x|^\beta$ for $\beta = (1 - 4/\kappa) \vee (\alpha - 1)$. When $\kappa = 4$, set $f(x) = |x|^{(\alpha-1)/2}$. This function is superharmonic on $(-\infty, -N) \cup (N, \infty)$ for N big enough and then we can prove that $\lim_{|x|\uparrow\infty} \widetilde{p}(x) = 0$. Therefore, we can prove the assertion by the Markov property and by the fact that h_t has arbitrarily big jumps. $\square$

REMARK 3.1. A Markov process h_t in $\mathbb{R} \setminus \{0\}$ is called recurrent (transient) if for all nonempty relatively compact open sets $B \subset \mathbb{R} \setminus \{0\}$ we have $\int_0^\infty 1_{\{h_t \in B\}}\, dt = \infty$ (resp. $< \infty$) a.s. In our setting it can be shown that $(h_t)_{t\geq 0}$ is recurrent if $1 \leq \alpha < 2$ and $0 \leq \kappa \leq 4$, and transient otherwise.

4. $\overline{\mathbb{H}}$-valued Bessel-type processes driven by $U = \sqrt{\kappa} B + \theta^{1/\alpha} S$. In this section we consider the problem whether the Bessel-type process on the complex upper half plane, given in (2.4), can hit 0. Denote this process by $h_t(z) = h_{1,t}(z) + i h_{2,t}(z)$ and $z = z_1 + i z_2$. For $z \in \overline{\mathbb{H}}$, we have that

$$\begin{aligned} dh_{1,t}(z) &= \frac{2h_{1,t}(z)\, dt}{h_{1,t}^2(z) + h_{2,t}^2(z)} - dU_t, \qquad & h_{1,0}(z) = z_1, \\ dh_{2,t}(z) &= \frac{-2h_{2,t}(z)\, dt}{h_{1,t}^2(z) + h_{2,t}^2(z)}, \qquad & h_{2,0}(z) = z_2. \end{aligned} \tag{4.1}$$

4.1. *The subcritical phase* $0 < \kappa < 4$. We have to prepare some results to deal with the hitting problem. For $\delta > 0$, denote by $V_\delta = \{z = z_1 + i z_2 : 0 < z_2 \leq \delta |z_1|)\}$ the double wedge of slope δ, and $\tau_\delta = \inf\{t \geq 0 : h_t \in V_\delta\}$ the first entrance time.

LEMMA 4.1. *If* $\kappa > 0$, *then for each* $\delta > 0$ *and* $z \in \mathbb{H}$,

$$P_z\{\tau_\delta < \infty\} = 1. \tag{4.2}$$

PROOF. The proof is in five parts.

1. *We reduce the proof to small* z. We only need to prove (4.2) when $z \notin V_\delta$. Without loss of generality we assume that $\delta < 1$. Let $s > 0$ and denote

$$d_{\delta,s} = \inf\{t \geq 0 : h_t \in V_\delta \text{ or } h_{2,t} \leq s\}.$$

We claim that $d_{\delta,s} < \infty$. This is actually true for every càdlàg driving function. In fact, if $d_{\delta,s} = \infty$, then

$$\lim_{t\to\infty} h_{2,t} = z_2 + \lim_{t\to\infty} \int_0^t \frac{-2h_{2,u}\, du}{h_{1,u}^2 + h_{2,u}^2} \leq z_2 - \lim_{t\to\infty} \int_0^t \frac{2s\, du}{z_2^2/\delta^2 + z_2^2} = -\infty,$$

which is absurd for a process in $\overline{\mathbb{H}}$. Next, by the Markov property,

$$\begin{aligned} P_z\{\tau_\delta < \infty\} &= P_z\{h_{d_{\delta,s}} \in V_\delta\} + P_z\{h_{d_{\delta,s}} \notin V_\delta, \tau_\delta < \infty\} \\ &= P_z\{h_{d_{\delta,s}} \in V_\delta\} + E_z\big[I_{\{h_{d_{\delta,s}} \notin V_\delta\}} P_{h_{d_{\delta,s}}}\{\tau_\delta < \infty\}\big]. \end{aligned} \tag{4.3}$$

Notice that $h_{2,d_{\delta,s}} = s$ on $\{h_{d_{\delta,s}} \notin V_\delta, d_{\delta,s} < \infty\}$, and (4.3) implies that we only need to prove (4.2) when $0 < |z_1| < z_2/\delta$ and z_2 small enough.

2. *Locally, the Brownian fluctuations dominate the stable fluctuations.* As $a^{-1/\alpha}S_{at}$ has the same distribution as S_t for $a > 0$, we have

$$\begin{aligned}
&\mathbb{P}\{\theta^{1/\alpha}|S_t| \le \tfrac{1}{2}\sqrt{2\kappa t \ln\ln(1/t)}\} \\
&\qquad = \mathbb{P}\{|S_1| \le \tfrac{1}{2}\theta^{-1/\alpha}t^{1/2-1/\alpha}\sqrt{2\kappa \ln\ln(1/t)}\} \to 1,
\end{aligned}$$

when $t \downarrow 0$. Hence we can find t_0 such that $\mathbb{P}\{\theta^{1/\alpha}|S_t| \le \frac{1}{2}\sqrt{2\kappa t \ln\ln(1/t)}\} \ge 1/2$ for $0 < t < t_0$. Now let $s > 0$ such that

$$s < t_0 \wedge 2\exp\left\{-\frac{1}{2}\exp\frac{288}{\kappa\delta^2}\right\} =: t_1 \tag{4.4}$$

and let $z \in \mathbb{H}$ such that $0 < |z_1| < s/\delta$ and $z_2 = s$. By (4.4), for $0 < t < s$,

$$\begin{aligned}
&\mathbb{P}\{U_t \ge \sqrt{2\kappa t \ln\ln(1/t)}/2\} \\
&\quad \ge \mathbb{P}\{B_t \ge \sqrt{2t \ln\ln(1/t)}\}\mathbb{P}\{\theta^{1/\alpha}|S_t| \le \sqrt{2\kappa t \ln\ln(1/t)}/2\} \\
&\quad \ge \frac{1}{2}\mathbb{P}\{B_1 \ge \sqrt{2\ln\ln(1/t)}\} \\
&\quad \ge \frac{1}{4\sqrt{2\pi}\,\ln(1/t)\sqrt{2\ln\ln(1/t)}}.
\end{aligned} \tag{4.5}$$

The last inequality of (4.5) follows from $\int_x^\infty e^{-y^2/2}\,dy \ge \frac{1}{2x}e^{-x^2/2}\,dy$ for $x > 1$.

3. *$h_{2,t}$ decreases quickest if $h_{1,t} = 0$, and $h_{1,t}$ reflects high values of U_t.* By (4.1), for each $y > 0$ with $h_{2,0} = y$ we have

$$h_{2,u} > y/2 \qquad \text{when } 0 < u < 3y^2/16. \tag{4.6}$$

Therefore, if $U_{s^2/16} \ge s\sqrt{2\kappa \ln\ln(16/s^2)}/8$, then by (4.4) and (4.6),

$$\begin{aligned}
|h_{1,s^2/16}| &= \left|z_1 + \int_0^{s^2/16} \frac{2h_{1,u}}{h_{1,u}^2 + h_{2,u}^2}\,du - U_{s^2/16}\right| \\
&\ge |U_{s^2/16}| - s/\delta - \int_0^{s^2/16} \frac{2}{s}\,du \\
&\ge s\sqrt{2\kappa \ln\ln(4/s^2)}/8 - 2s/\delta \\
&\ge s/\delta,
\end{aligned}$$

which leads to

$$\{U_{s^2/16} \ge s\sqrt{2\kappa \ln\ln(16/s^2)}/8\} \subseteq \{\tau_\delta \le s^2/16\}. \tag{4.7}$$

By (4.5) and (4.7), we obtain

$$
\begin{aligned}
P_z\{\tau_\delta \le s^2/16\} &\ge \mathbb{P}\bigl\{U_{s^2/16} \ge s\sqrt{2\kappa \ln\ln(16/s^2)}/8\bigr\} \\
&\ge \frac{1}{8\sqrt{2\pi}\ln(4/s)\sqrt{2\ln(2\ln(4/s))}}.
\end{aligned}
\tag{4.8}
$$

4. *Consider a positive starting height $s_0 < t_1$ and levels $s_0/2^n$, $n \ge 1$. We control τ_δ between successive levels.* Define $T_n = \inf\{t \ge 0 : h_{2,t} = s_0/2^n\}, n \ge 1$ and $T_0 = 0$. Let $p_n = P_z\{\tau_\delta \in (T_{n-1}, T_n]\}$. By (4.6) and (4.8) we have

$$
p_1 \ge \frac{1}{8\sqrt{2\pi}\ln(4/s_0)\sqrt{2\ln(2\ln(4/s_0))}}.
$$

By the Markov property, (4.6) and (4.8), we have

$$
\begin{aligned}
p_n &= E_z\bigl[P_z[\tau_\delta \in (T_{n-1}, T_n]|\mathcal{F}_{T_{n-1}}]\bigr] \\
&\ge E_z\biggl[I_{\{\tau_\delta > T_{n-1}\}} P_{h_{T_{n-1}}}\biggl\{|h_{1,T_{n-1}}| < s_0/(2^{n-1}\delta), \tau_\delta \le \biggl(\frac{s_0}{2^{n-1}}\biggr)^2/16\biggr\}\biggr] \\
&\ge \frac{1}{8\sqrt{2\pi}(\ln(4/s_0) + (n-1)\ln 2)\sqrt{2\ln(2\ln(4/s_0) + 2(n-1)\ln 2)}} \\
&\qquad \times P_z\{\tau_\delta > T_{n-1}\} \\
&= \frac{1}{8\sqrt{2\pi}(\ln(4/s_0) + (n-1)\ln 2)\sqrt{2\ln(2\ln(4/s_0) + 2(n-1)\ln 2)}} \\
&\qquad \times \Biggl(1 - \sum_{k=1}^{n-1} p_k\Biggr).
\end{aligned}
$$

5. *We conclude.* Now the proof is complete if we show $\sum_{n\ge 1} p_n = 1$. Otherwise, we would have $\sum_{n\ge 1} p_n < 1$ and

$$
\begin{aligned}
\sum_{n\ge 1} p_n &\ge \sum_{n\ge 1} \frac{1}{8\sqrt{2\pi}(\ln(4/s_0) + (n-1)\ln 2)\sqrt{2\ln(2\ln(4/s_0) + 2(n-1)\ln 2)}} \\
&\qquad \times \Biggl(1 - \sum_{k=1}^{n-1} p_k\Biggr) \\
&\ge \sum_{n\ge 1} \frac{1}{8\sqrt{2\pi}(\ln(4/s_0) + (n-1)\ln 2)\sqrt{2\ln(2\ln(4/s_0) + 2(n-1)\ln 2)}} \\
&\qquad \times \Biggl(1 - \sum_{k\ge 1} p_k\Biggr) \\
&= \infty,
\end{aligned}
$$

which is a contradiction, so we must have $\sum_{n\geq 1} p_n = 1$ as required. $\square$

LEMMA 4.2. *Let $z = z_1 + iz_2 \in \mathbb{H}$ and let $0 < \kappa < 4$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that $P_z\{\zeta < \infty\} < \varepsilon$ for $z \in V_\delta$, the double wedge of slope δ.*

PROOF. For convenience, we will use the notation of Lemmas 2.1 and 2.2. For example, we still use notation $\tau_{a,b}$ and $\tau_{b,a}$ for the inner and outer exit times of $(h_{1,t})_{t\geq 0}$ from $\{x \in \mathbb{R} : a < |x| < b\}$. We also denote the exit time by $\tau = \tau_{a,b} \wedge \tau_{b,a}$. For $c \geq 0$ and a C^2 function f, set

$$A_c f(y) = \frac{2y}{y^2 + c^2}\partial_y f(y) + \frac{\kappa}{2}\partial_y^2 f(y) + \theta\Delta_y^{\alpha/2} f(y) \qquad \text{for } y \neq 0. \tag{4.9}$$

Let $\beta = (2/\kappa - 1/2) \wedge (1 - \alpha)$ if $\alpha < 1$ and $\beta = (2/\kappa - 1/2) \wedge 1/2$ if $1 \leq \alpha < 2$. Then we have $4\kappa^{-1}(1 + \beta)^{-1} - 1 > 0$. Let $0 < k < \varepsilon^{1/\beta} \wedge 1$ and let δ be a positive number such that

$$\delta < k\sqrt{\frac{4}{\kappa(1 + \beta)} - 1}. \tag{4.10}$$

Define $f = w_{1-\beta}$. Noticing that $\Delta^{\alpha/2} w_{1-\beta}(y) \leq 0$, and applying (4.10), we have for any $|y| > kz_1$ and $0 \leq c \leq \delta z_1$

$$\begin{aligned} A_c f(y) &\leq \frac{2y}{y^2 + c^2}\partial_y f(y) + \frac{\kappa}{2}\partial_y^2 f(y) \\ &= \frac{\beta}{|y|^{2+\beta}}\left(\frac{2y^2}{y^2 + c^2} - \frac{\kappa(1 + \beta)}{2}\right) \\ &\leq \frac{-\beta}{|y|^{2+\beta}}\left(\frac{2}{1 + \delta^2/k^2} - \frac{\kappa(1 + \beta)}{2}\right) \\ &\leq 0. \end{aligned} \tag{4.11}$$

Let $\tau = \tau_{a,b} \wedge \tau_{b,a}$ for $kz_1 \leq a < z_1 < b$. By Dynkin's formula,

$$E_z f(h_{1,\tau}) = z_1^{-\beta} + E_z \int_0^\tau A_{h_{2,u}} f(h_{1,u-})\,du. \tag{4.12}$$

Hence by (4.11) and $h_{2,u} \leq \delta z_1$, we obtain $E_z f(h_{1,\tau}) \leq z_1^{-\beta}$. Therefore, by Remark 2.2

$$\begin{aligned} P_z\{\zeta < \infty\} &\leq \lim_{b\uparrow\infty} P_z\{\tau_{kz_1,b} < \tau_{b,kz_1}\} \\ &\leq \lim_{b\uparrow\infty} \frac{z_1^{-\beta} - \int_{\{|y|\geq b\}} |y|^{-\beta}\mu_{b,kz_1}(dy)}{\int_{\{|y|\leq kz_1\}} |y|^{-\beta}\mu_{kz_1,b}(z, dy) - \int_{\{|y|\geq b\}} |y|^{-\beta}\mu_{b,kz_1}(z, dy)} \\ &\leq k^\beta < \varepsilon, \end{aligned}$$

which completes the proof for $0 < \kappa < 4$. $\square$

THEOREM 4.3. *Let $0<\kappa<4$. For any $z\in\overline{\mathbb{H}}\setminus\{0\}$, we have $P_z\{\zeta=\infty\}=1$.*

PROOF. When $z_2=0$, the conclusion follows from Proposition 3.1. When $z_2>0$, the conclusion follows from Lemmas 4.1 and 4.2. □

4.2. *The supercritical phase $\kappa>4$.* We first show that we control the return time to the imaginary axis outside an asymptotically negligible event. This will be useful when we choose regeneration points on the imaginary axis.

LEMMA 4.4. (1) *Let $\kappa>4$, $1\le\alpha<2$ and let $z=z_1+iz_2\in\overline{\mathbb{H}}\setminus\{0\}$. Denote $\widetilde{\tau}=\inf\{t\ge 0: h_{1,t-}=0\}$. Then $\widetilde{\tau}<\infty$ with probability* 1.

(2) *Moreover, for all $\kappa>4$ and $0<\alpha\le 2$, there exist a constant c and an event Θ such that*

$$E_z[I_\Theta\widetilde{\tau}]\le c|z_1|^{1-4/\kappa},\qquad P_z[\Theta^c]<c|z_1|^{1-4/\kappa}\qquad \text{for } 0<|z_1|<1. \tag{4.13}$$

Specifically we can take Θ to be $\{\omega\in\Omega:\tau_{0,2}(\omega)<\tau_{2,0}(\omega)\}$ in (4.13).

PROOF. Define A_c by (4.9). By Lemma 2.1, we have $A_cw_\beta\le 0$ for $\beta=\alpha\wedge(2-4/\kappa)$. Then, applying the same method as in the proof of Proposition 3.2, we can prove (1).

Now let $\alpha\ge 2-4/\kappa$. By the same arguments as in (3.5) we have

$$P_z\{\tau_{0,2}>\tau_{2,0}\}\le\frac{z_1^{1-4/\kappa}}{\int_{\{|y|\ge 2\}}|y|^{1-4/\kappa}\mu_{2,0}(z,dy)}. \tag{4.14}$$

Let $f(x)=x^2\wedge M$ for $x\in\mathbb{R}$ and $M>0$. Choose M big enough such that $\theta\Delta^{\alpha/2}f(y)\ge-\kappa/2$ for $|y|\le 2$. Set $\Theta=\{\tau_{0,2}<\tau_{2,0}\}$. Taking the notation of Lemma 4.2, we have by Dynkin's formula

$$\begin{aligned}E_z[f(h_{1,\tau_{0,2}\wedge\tau_{2,0}})]&\ge z_1^2+E_z\biggl[I_\Theta\int_0^{\widetilde{\tau}}A_{h_{2,u}}f(h_{1,u-})\,du\biggr]\\&\ge z_1^2+E_z\biggl[I_\Theta\int_0^{\widetilde{\tau}}\biggl(\frac{4h_{1,u-}^2}{h_{1,u-}^2+h_{2,u}^2}+\frac{\kappa}{2}\biggr)du\biggr]\\&\ge\frac{\kappa}{2}E_z[I_\Theta\widetilde{\tau}].\end{aligned} \tag{4.15}$$

By (4.14), we have

$$E_z[f(h_{1,\tau_{0,2}\wedge\tau_{2,0}})]\le\frac{Mz_1^{1-4/\kappa}}{\int_{\{|y|\ge 2\}}|y|^{1-4/\kappa}\mu_{2,0}(z,dy)}.$$

Hence (4.13) follows from (4.15).

For the proof of the case $0<\alpha<1$, the argument of Proposition 3.4 is easily adapted and transferred to the upper half plane as above. It also applies to $1\le\alpha<2-4/\kappa$. □

LEMMA 4.5. *Let $\beta > 0$. Let $(a_n)_{n\geq 0}$ be a sequence of positive numbers such that $a_1 < (1+1/\beta)^{-1/\beta}$ and $a_{n+1} \leq a_n - a_n^{1+\beta}/\beta$. Then*

$$a_n \leq (a_1^{-\beta} + n - 1)^{-1/\beta} \qquad \textit{for all } n \geq 1.$$

PROOF. It is easy to see that the assertion is true for $n = 1$. Now suppose that the assertion is true for $n = k$. Notice that $f(x) = x + x^{\beta+1}/\beta$ is a increasing function on $(0, (1+1/\beta)^{-1/\beta})$; we have

$$\begin{aligned} a_{k+1} &\leq a_k - a_k^{\beta+1}/\beta \\ &\leq (a_1^{-\beta} + k - 1)^{-1/\beta} - (a_1^{-\beta} + k - 1)^{-(\beta+1)/\beta}/\beta \leq (a_1^{-\beta} + k)^{-1/\beta}, \end{aligned}$$

which completes the proof. □

THEOREM 4.6. *Let $\kappa > 4$. Then the following assertions are true*:

(1) *When $1 \leq \alpha < 2$, then for any $z \in \overline{\mathbb{H}} \setminus \{0\}$, we have $P_z\{\zeta < \infty\} = 1$.*
(2) *When $0 < \alpha < 1$, then $\lim_{|z|\downarrow 0} P_z\{\zeta < \infty\} = 1$.*

PROOF. (1) When $z_2 = 0$, the conclusion follows from Proposition 3.2. Next, we assume $z_2 > 0$ and, without loss of generality, $z_1 > 0$. By Proposition VIII.4 in [3], there exists a constant positive number k_1 such that

$$\mathbb{P}\{|S_1| > x\} \leq k_1 x^{-\alpha} \qquad \text{for all } x > 0. \tag{4.16}$$

Denote $\beta = 1/4 - 1/\kappa$. Let a_1 be an arbitrary positive number such that

$$a_1 < z_2 \wedge \left(\frac{\beta}{10}\right)^{1/\beta} < \left(1 + \frac{1}{\beta}\right)^{-1/\beta}. \tag{4.17}$$

Denote $\eta_0 = 0$ and $\xi_1 = \inf\{t \geq 0 : h_{2,t} = a_1 - a_1^{1+\beta}/\beta\}$. By (4.1), we can check $\xi_1 < \infty$ a.s. Set

$$\eta_1 = \inf\{t \geq \xi_1 : h_{1,t} = 0\}.$$

By the Markov property and Lemma 4.4 we have $\eta_1 < \infty$ a.s. Define by induction

$$a_{n+1} = h_{2,\eta_n}; \qquad \xi_{n+1} = \eta_n + \frac{5a_{n+1}^{2+\beta}}{4\beta};$$

$$b_{n+1} = a_{n+1} - \frac{a_{n+1}^{1+\beta}}{\beta}; \qquad \eta_{n+1} = \inf\{t \geq \xi_{n+1} : h_{1,t} = 0\}.$$

By the definitions above and Lemma 4.4 we see that $\xi_n \leq \eta_n < \xi_{n+1} \leq \eta_{n+1} < \infty$, and these are sums of decreasing amounts of waiting time and subsequent return times of h_t to the imaginary axis. We will show that for almost all $n \geq 1$, we have good control of real and imaginary parts of h_t so as to deduce that we reach zero in finite time. Specifically, set

$$E_n = \bigcap_{t\in[\eta_{n-1},\xi_n]} \{|h_{1,t}| \le a_n\}; \qquad H_n = \{h_{2,\xi_n} \le b_n\}. \tag{4.18}$$

Next we prove a lemma for preparation. $\square$

LEMMA 4.7. *We have, for all $n \ge 2$,*

$$P_z[E_n^c|\mathcal{F}_{\eta_{n-1}}] \le \sqrt{\frac{160\kappa}{\beta\pi}} a_n^{\beta/2} \exp\left\{-\frac{\beta a_n^{-\beta}}{40\kappa}\right\} + \frac{10k_1\theta}{4^{1-\alpha}\beta} a_n^{2+\beta-\alpha}; \tag{4.19}$$

$$E_n \subseteq H_n. \tag{4.20}$$

PROOF. Denote $\xi_n' = \inf\{t \ge 0 : h_{2,t} = a_n/2\}$. By (4.1), we can prove

$$h_{2,\xi_n} > a_n/2. \tag{4.21}$$

In fact, if $h_{2,\xi_n} \le a_n/2$ we have $\xi_n' < \xi_n$ and hence

$$\begin{aligned} \frac{a_n}{2} = h_{2,\xi_n'} &= a_n + \int_{\eta_{n-1}}^{\xi_n'} \frac{-2h_{2,u}}{h_{1,u}^2 + h_{2,u}^2}\,du \\ &\ge a_n - \int_{\eta_{n-1}}^{\xi_n'} \frac{2}{h_{2,u}}\,du \\ &> a_n - 5a_n^{1+\beta}/\beta. \end{aligned} \tag{4.22}$$

By (4.22), we have $a_n < 10a_n^{1+\beta}/\beta \le 10a_1^{\beta} a_n/\beta$, which contradicts (4.17).

By (4.17), (4.21) and (4.1), for $\eta_{n-1} < t \le \xi_n$, we have

$$\begin{aligned} |h_{1,t}| &= \left| \int_{\eta_{n-1}}^{t} \frac{2h_{1,u}}{h_{1,u}^2 + h_{2,u}^2}\,du + U_t - U_{\eta_{n-1}} \right| \\ &\le |U_t - U_{\eta_{n-1}}| + \int_{\eta_{n-1}}^{\xi_n} \frac{4}{a_n}\,du \\ &= |U_t - U_{\eta_{n-1}}| + 5a_n^{1+\beta}/\beta \\ &\le |U_t - U_{\eta_{n-1}}| + a_n/2. \end{aligned} \tag{4.23}$$

By the reflection principle and (4.16),

$$\begin{aligned} P_z&\left[\sup_{\eta_{n-1}<t\le\xi_n} |U_t - U_{\eta_{n-1}}| > a_n/2 \middle| \eta_{n-1}\right] \\ &\le 2P_z[\sqrt{\kappa}|B_{\xi_n} - B_{\eta_{n-1}}| > a_n/4|\eta_{n-1}] \\ &\quad + 2P_z[\theta^{1/\alpha}|S_{\xi_n} - S_{\eta_{n-1}}| > a_n/4|\eta_{n-1}] \end{aligned} \tag{4.24}$$

$$\leq 2P_z\big[|B_1| > \beta^{1/2}a_n^{-\beta/2}/\sqrt{20\kappa}|\eta_{n-1}\big]$$
$$+ 2P_z\left[|S_1| > \left(\frac{4\beta}{5\theta}\right)^{1/\alpha} a_n^{1-(2+\beta)/\alpha}/4\middle|\eta_{n-1}\right]$$
$$\leq \sqrt{\frac{160\kappa}{\beta\pi}}a_n^{\beta/2}\exp\left\{-\frac{\beta a_n^{-\beta}}{40\kappa}\right\} + \frac{10k_1\theta}{4^{1-\alpha}\beta}a_n^{2+\beta-\alpha}.$$

Combining (4.23) and (4.24), we obtain the first inequality in (4.19).

Now suppose $|h_{1,u}| \leq a_n$ when $\eta_{n-1} \leq u \leq \xi_n$. Then we have

$$\frac{2h_{2,u}}{h_{1,u}^2 + a_n^2/4} > \frac{4}{5a_n}.$$

By (4.21),

$$h_{2,\xi_n} = a_n + \int_{\eta_{n-1}}^{\xi_n} \frac{-2h_{2,u}}{h_{1,u}^2 + h_{2,u}^2}\,du \leq a_n - \int_{\eta_{n-1}}^{\xi_n} \frac{4}{5a_n}\,du = a_n - a_n^{1+\beta}/\beta = b_n,$$

which proves (4.20). □

CONTINUATION OF THE PROOF OF THEOREM 4.6. Denote

$$\begin{aligned} \widetilde{\tau}_{0,n} &= \eta_n \wedge \inf\{t \geq \xi_n : h_{1,t} = 0, |h_{1,u}| < 2 \text{ for } \xi_n < u < t\}; \\ \widetilde{\tau}_{2,n} &= \eta_n \wedge \inf\{t \geq \xi_n : h_{1,t} = 2, |h_{1,u}| > 0 \text{ for } \xi_n < u < t\}. \end{aligned} \tag{4.25}$$

By Lemma 4.4, there exists a constant $k_2 > 0$ such that

$$\begin{aligned} E_z\big[I_{\{\widetilde{\tau}_{0,n} < \widetilde{\tau}_{2,n}\}}(\eta_n - \xi_n)|\mathcal{F}_{\xi_n}\big] &< k_2|h_{1,\xi_n}|^{1/2-2/\kappa}, \\ P_z[\widetilde{\tau}_{0,n} > \widetilde{\tau}_{2,n}|\mathcal{F}_{\xi_n}] &< k_2|h_{1,\xi_n}|^{1/2-2/\kappa}, \end{aligned} \tag{4.26}$$

when $0 < |h_{1,\xi_n}| < 1$. Denote $F_n = \{\widetilde{\tau}_{0,n} < \widetilde{\tau}_{2,n}\} \cap E_n$ and set $F = \bigcap_{n\geq 2} F_n$. By definition of a_2, (4.20) and Lemma 4.5

$$\bigcap_{n=2}^{N-1} E_n \subseteq \bigcap_{n=1}^{N}\{a_n \leq (a_1^{-\beta} + n - 1)^{-1/\beta}\} \qquad \text{for all } N \in \mathbb{N}. \tag{4.27}$$

Write $d_n = a_1^{-\beta} + n - 1$. By (4.17), (4.18), (4.26) and (4.27),

$$\begin{aligned} P_z[F] &= \lim_{N\to\infty} P_z\left[\bigcap_{n=2}^{N} F_n\right] \\ &= \lim_{N\to\infty} E_z\big[I_{\bigcap_{n=2}^{N-1} F_n} I_{E_N} P_z[\widetilde{\tau}_{0,N} > \widetilde{\tau}_{2,N}|\mathcal{F}_{\xi_N}]\big] \\ &\geq \lim_{N\to\infty} E_z\big[I_{\bigcap_{n=2}^{N-1} F_n} I_{E_N}(1 - k_2|h_{1,\xi_N}|^{1/2-2/\kappa})\big] \end{aligned} \tag{4.28}$$

$$
\begin{aligned}
&\geq \lim_{N\to\infty} E_z\big[I_{\bigcap_{n=2}^{N-1} F_n} I_{E_N}(1-k_2|a_N|^{1/2-2/\kappa})\big] \\
&\geq \lim_{N\to\infty} E_z\big[I_{\bigcap_{n=2}^{N-1} F_n} I_{E_N}(1-k_2 d_N^{-2})\big] \\
&= \lim_{N\to\infty} (1-k_2 d_N^{-2}) E_z\big[I_{\bigcap_{n=2}^{N-1} F_n} P_z[E_N|\mathcal{F}_{\eta_{N-1}}]\big] \\
&\geq \lim_{N\to\infty} (1-k_2 d_N^{-2}) \\
&\quad \times E_z\left[I_{\bigcap_{n=2}^{N-1} F_n}\left(1-\sqrt{\frac{160\kappa}{\beta\pi}} a_N^{\beta/2} \exp\left\{-\frac{\beta a_N^{-\beta}}{40\kappa}\right\} - \frac{10k_1\theta}{4^{1-\alpha}\beta} a_N^{2+\beta-\alpha}\right)\right] \\
&\geq \lim_{N\to\infty} (1-k_2 d_N^{-2}) \\
&\quad \times \left(1-\sqrt{\frac{160\kappa}{\beta\pi}} d_N^{-1/2} \exp\left\{-\frac{\beta d_N}{40\kappa}\right\} - \frac{10k_1\theta}{4^{1-\alpha}\beta} d_N^{-1-(2-\alpha)/\beta}\right) \\
&\quad \times P_z\left[\bigcap_{n=2}^{N-1} F_n\right] \\
&\geq \prod_{n=1}^{\infty} (1-k_2 d_n^{-2}) \\
&\quad \times \left(1-\sqrt{\frac{160\kappa}{\beta\pi}} d_n^{-1/2} \exp\left\{-\frac{\beta d_n}{40\kappa}\right\} - \frac{10k_1\theta}{4^{1-\alpha}\beta} d_n^{-1-(2-\alpha)/\beta}\right) \\
&\geq 1-\sum_{n=1}^{\infty}\left(k_2 d_n^{-2} + \sqrt{\frac{160\kappa}{\beta\pi}} d_n^{-1/2} \exp\left\{-\frac{\beta d_n}{40\kappa}\right\} + \frac{10k_1\theta}{4^{1-\alpha}\beta} d_n^{-1-(2-\alpha)/\beta}\right).
\end{aligned}
$$

By the definition of d_n and (4.28), we have

$$\lim_{a_1\downarrow 0} P_z[F] = 1. \tag{4.29}$$

Set $\xi = \lim_{n\to\infty} \xi_n$. By Lebesgue's monotone convergence theorem, (4.17), (4.26) and (4.27),

$$
\begin{aligned}
&E_z[I_F(\xi-\xi_1)] \\
&\quad = \lim_{n\to\infty} E_z[I_F(\xi_n-\xi_1)] \\
&\quad = \lim_{n\to\infty} \sum_{k=2}^{n} E_z[I_F(\xi_k-\eta_{k-1})] + \lim_{n\to\infty} \sum_{k=2}^{n} E_z[I_F(\eta_{k-1}-\xi_{k-1})] \\
&\quad = \lim_{n\to\infty} \sum_{k=2}^{n} E_z\left[I_F \frac{5a_k^{2+\beta}}{4\beta}\right] \qquad (4.30)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \lim_{n\to\infty} \sum_{k=2}^{n} E_z\big[E_z[I_F(\eta_{k-1}-\xi_{k-1})|\mathcal{F}_{\xi_{k-1}}]\big] \\
&\le \sum_{k=2}^{\infty} E_z\bigg[I_F \frac{5d_k^{-1-2/\beta}}{4\beta}\bigg] \\
&\quad + \sum_{k=2}^{\infty} E_z\big[E_z\big[I_{\bigcap_{s=1}^{k-1} E_s} I_{\{\tilde{\tau}_{0,k-1}>\tilde{\tau}_{2,k-1}\}}(\eta_{k-1}-\xi_{k-1})|\mathcal{F}_{\xi_{k-1}}\big]\big] \\
&\le \sum_{k=2}^{\infty} \frac{5d_k^{-1-2/\beta}}{4\beta} \\
&\quad + \sum_{k=2}^{\infty} E_z\big[I_{\bigcap_{s=1}^{k-1} E_s} E_z\big[I_{\{\tilde{\tau}_{0,k-1}>\tilde{\tau}_{2,k-1}\}}(\eta_{k-1}-\xi_{k-1})|\mathcal{F}_{\xi_{k-1}}\big]\big] \\
&\le \sum_{k=2}^{\infty} \frac{5d_k^{-1-2/\beta}}{4\beta} + \sum_{k=2}^{\infty} E_z\big[I_{\bigcap_{s=1}^{k-1} E_s} k_2 |h_{1,\xi_{k-1}}|^{1/2-2/\kappa}\big] \\
&\le \sum_{k=2}^{\infty} \frac{5d_k^{-1-2/\beta}}{4\beta} + \sum_{k=2}^{\infty} k_2 E_z\big[I_{\bigcap_{s=1}^{k-1} E_s} a_{k-1}^{1/2-2/\kappa}\big] \\
&\le \sum_{k=2}^{\infty} \frac{5d_k^{-1-2/\beta}}{4\beta} + \sum_{k=2}^{\infty} k_2 d_{k-1}^{-2} \\
&< \infty.
\end{aligned}
$$

By (4.27), we see that $F \subseteq \{\lim_{n\to\infty} a_n = 0\}$. Hence by the definition of ξ, we see $h_{2,\xi} = 0$ on F. From this fact and Proposition 3.2, we know $\zeta < \infty$ on F. Notice a_1 can be arbitrarily small; we obtain the conclusion by (4.29).

By the same proof as above we see that (2) can also be proved. $\square$

4.3. *Remaining critical and boundary values $\kappa = 4$ and $\kappa = 0$.* For $z = z_1 + iz_2$ with $z_2 \ge 0$, denote

$$
\tilde{w}_p(z) = (z_1^2+z_2^2)^{(p-1)/2}, \qquad p \ne 1; \qquad \tilde{w}_1 = \ln(z_1^2+z_2^2). \tag{4.31}
$$

For function f on the upper half plane, we set

$$
Af(z) = \frac{-2z_2}{z_1^2+z_2^2}\partial_{z_2} f(z) + \frac{2z_1}{z_1^2+z_2^2}\partial_{z_1} f(z) + \frac{\kappa}{2}\partial_{z_1}^2 f(z) + \theta\Delta_{z_1}^{\alpha/2} f(z). \tag{4.32}
$$

LEMMA 4.8. *For $0 < p < \alpha+1$ and $\theta = 0$,*

$$
\begin{aligned}
A\tilde{w}_p &= \frac{p-1}{2}(z_1^2+z_2^2)^{(p-5)/2}\big((\kappa-4)z_2^2 + \big(4+\kappa(p-2)\big)z_1^2\big), \\
A\tilde{w}_1 &= (\kappa-4)(z_1^2+z_2^2)^{-2}(z_2^2-z_1^2).
\end{aligned} \tag{4.33}
$$

PROOF. When $p \neq 1$, we have

$$\begin{aligned} Af(z) &= -2(p-1)(z_1^2+z_2^2)^{(p-5)/2} z_2^2 + 2(p-1)(z_1^2+z_2^2)^{(p-5)/2} z_1^2 \\ &\qquad + \frac{1}{2}\kappa(p-1)(z_1^2+z_2^2)^{(p-3)/2} + \frac{1}{2}\kappa(p-1)(p-3)(z_1^2+z_2^2)^{(p-5)/2} z_1^2 \\ &= (p-1)(z_1^2+z_2^2)^{(p-5)/2}\biggl(-2z_2^2+2z_1^2+\frac{\kappa}{2}(z_1^2+z_2^2)+\frac{\kappa}{2}(p-3)z_1^2\biggr) \\ &= \frac{p-1}{2}(z_1^2+z_2^2)^{(p-5)/2}\bigl((\kappa-4)z_2^2+\bigl(4+\kappa(p-2)\bigr)z_1^2\bigr). \end{aligned}$$

The second equality can also be verified directly. □

REMARK 4.1. By (4.33), when $\theta = 0$ we have

$$A\widetilde{w}_{2-4/\kappa} = \frac{(\kappa-4)^2}{2\kappa}(z_1^2+z_2^2)^{-3/2-2/\kappa} z_2^2, \tag{4.34}$$

and hence $A\widetilde{w}_1 = 0$ for $\kappa = 4$.

LEMMA 4.9. *For each* $0 < p < \alpha + 1$, *there exists a constant* c *such that*

$$|\Delta_{z_1}^{\alpha/2}\widetilde{w}_p(z)| \le c(|z_1|^{p-1-\alpha} \wedge |z_2|^{p-1-\alpha}) \qquad \textit{for } z \neq 0, |z| < 1, z \in \overline{\mathbb{H}}. \tag{4.35}$$

PROOF. First we see the case $p < 1$. We claim that function

$$\varphi(t) := \lim_{\varepsilon \downarrow 0} \int_{\{y:|y|>\varepsilon\}} \frac{((y+1)^2+t^2)^{(p-1)/2} - (1+t^2)^{(p-1)/2}}{|y|^{1+\alpha}}\, dy$$

is bounded for $t \in [-1, 1]$. In fact, we have for $|t| \le 1$

$$\begin{aligned} |\varphi(t)| &= \biggl| \int_{-\infty}^{\infty} I_{\{|y|>1/2\}} \frac{((y+1)^2+t^2)^{(p-1)/2} - (1+t^2)^{(p-1)/2}}{|y|^{1+\alpha}}\, dy \biggr| \\ &\qquad + \biggl| \int_{-1/2}^{1/2} \bigl[\bigl(((y+1)^2+t^2)^{(p-1)/2} \\ &\qquad\qquad - (1+t^2)^{(p-1)/2} - (p-1)(1+t^2)^{(p-3)/2} y\bigr)(|y|^{1+\alpha})^{-1}\bigr]\, dy \biggr| \\ &\le \int_{-\infty}^{\infty} I_{\{|y|>1/2\}} \frac{|y+1|^{p-1}+1}{|y|^{1+\alpha}}\, dy \\ &\qquad + \int_{-1/2}^{1/2} \biggl[\biggl(|p-1|\biggl(\biggl(\frac{1}{2}\biggr)^2+t^2\biggr)^{(p-3)/2}|y|^2 \\ &\qquad\qquad + |(p-1)(p-3)|\biggl(\frac{3}{2}\biggr)^2\biggl(\biggl(\frac{1}{2}\biggr)^2+t^2\biggr)^{(p-5)/2}|y|^2\biggr) \end{aligned}$$

$$\times (|y|^{1+\alpha})^{-1}\Big] dy$$

$$\begin{aligned}
&\le \int_{-\infty}^{\infty} I_{\{|y|>1/2\}} \frac{|y+1|^{p-1}+1}{|y|^{1+\alpha}}\,dy \\
&\quad + \int_{-1/2}^{1/2} \frac{|p-1|2^{3-p}+|(p-1)(p-3)|(3/2)^2 2^{p-5}}{|y|^{\alpha-1}}\,dy \\
&< \infty,
\end{aligned}$$

which gives the bound of φ on $[-1,1]$. We denote this bound by c_1. Hence for $|z_2/z_1| \le 1$, we have

$$\begin{aligned}
&|\Delta_{z_1}^{\alpha/2} \widetilde{w}_p(z)| \\
&\qquad = \left| \lim_{\varepsilon\downarrow 0} \mathcal{A}(1,-\alpha) \int_{\{y:|y-z_1|>\varepsilon\}} \frac{(y^2+z_2^2)^{(p-1)/2}-(z_1^2+z_2^2)^{(p-1)/2}}{|y-z_1|^{1+\alpha}}\,dy \right| \\
&\qquad = \mathcal{A}(1,-\alpha)|z_1|^{p-\alpha-1} \\
&\qquad\quad \times \left| \lim_{\varepsilon\downarrow 0} \int_{\{y:|y-1|>\varepsilon\}} \frac{(y^2+(z_2/z_1)^2)^{(p-1)/2}-(1+(z_2/z_1)^2)^{(p-1)/2}}{|y-1|^{1+\alpha}}\,dy \right| \\
&\qquad \le c_1 \mathcal{A}(1,-\alpha)|z_1|^{p-\alpha-1}.
\end{aligned} \tag{4.36}$$

On the other hand,

$$\begin{aligned}
&|\Delta_{z_1}^{\alpha/2} \widetilde{w}_p(z)| \\
&\quad = \mathcal{A}(1,-\alpha)|z_1|^{p-\alpha-1} \\
&\qquad \times \lim_{\varepsilon\downarrow 0} \left| \int_{\{y:|y|>\varepsilon\}} \frac{((y+1)^2+(z_2/z_1)^2)^{(p-1)/2}-(1+(z_2/z_1)^2)^{(p-1)/2}}{|y|^{1+\alpha}}\,dy \right| \\
&\quad = \mathcal{A}(1,-\alpha)|z_2|^{p-\alpha-1} \\
&\qquad \times \lim_{\varepsilon\downarrow 0} \left| \int_{\{y:|y|>\varepsilon\}} \frac{((y+(z_1/z_2))^2+1)^{(p-1)/2}-(1+(z_1/z_2)^2)^{(p-1)/2}}{|y|^{1+\alpha}}\,dy \right|.
\end{aligned} \tag{4.37}$$

By similar calculations as above, we can also find a positive number c_2 such that

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0} \left| \int_{\{y:|y|>\varepsilon\}} \frac{((y+(z_1/z_2))^2+1)^{(p-1)/2}-(1+(z_1/z_2)^2)^{(p-1)/2}}{|y|^{1+\alpha}}\,dy \right| \\
&\qquad \le c_2
\end{aligned} \tag{4.38}$$

for $|z_1/z_2| < 1$. Combining (4.36), (4.37) and (4.38), we get

$$|\Delta_{z_1}^{\alpha/2} \widetilde{w}_p(z)| \le (c_1+c_2)\mathcal{A}(1,-\alpha)(|z_1|^{p-\alpha-1} \wedge |z_2|^{p-\alpha-1})$$

which completes the proof for $p < 1$. The case $p \ge 1$ can be checked with the same method. $\square$

THEOREM 4.10. *Let $\kappa = 4$. Then for any $z \in \overline{\mathbb{H}} \setminus \{0\}$, we have $P_z\{\zeta = \infty\} = 1$.*

PROOF. As in the case of the real line, we need to construct a continuous function f which is subharmonic with respect to A on a pointed neighborhood of zero and satisfies

$$\lim_{|z|\downarrow 0} f(z) = -\infty; \qquad \lim_{|z|\uparrow\infty} f(z) \geq 0. \tag{4.39}$$

First we see the case $\alpha > 1$. Let f_1 be a continuous function on $\overline{\mathbb{H}}$ such that

$$f_1(z) = -\widetilde{w}_{2-\alpha/2}, \qquad |z| \leq 1, z \in \overline{\mathbb{H}}; \qquad f_1(z) = 0, \qquad |z| > 2, z \in \overline{\mathbb{H}}.$$

By (4.35) we can check that there exists a positive number c_1 such that

$$|\Delta_{z_1}^{\alpha/2} f_1(z)| \leq c_1(|z_1|^{1-3\alpha/2} \wedge |z_2|^{1-3\alpha/2}) \qquad \text{for } |z| < 1/2, z \in \overline{\mathbb{H}}. \tag{4.40}$$

By (4.33) and (4.35), there exist positive numbers c_2 and c_3 such that

$$Af_1(z) \geq c_2(z_1^2 + z_2^2)^{-(\alpha+2)/4} \qquad \text{for } \theta = 0 \text{ and } z \in \overline{\mathbb{H}}, \tag{4.41}$$

and

$$|\Delta_{z_1}^{\alpha/2} \widetilde{w}_1(z)| \leq c_3(|z_1|^{-\alpha} \wedge |z_2|^{-\alpha}), \qquad z \in \overline{\mathbb{H}}. \tag{4.42}$$

Denote $f = f_1 + \widetilde{w}_1$. It is easy to see that f satisfies (4.39). By (4.40), (4.41), (4.42), and noticing that $-(\alpha+2)/2 < -\alpha < 1 - 3\alpha/2$, we get

$$\lim_{|z|\downarrow 0} Af(z) = \infty.$$

Hence by (2) in Lemma 2.3 and Dynkin's formula we finish the proof of $\alpha > 1$. When $0 < \alpha \leq 1$, the proof is still valid provided that we define f_1 by

$$f_1(z) = -\widetilde{w}_{1+\alpha/2}, \qquad |z| \leq 1, z \in \overline{\mathbb{H}}; \qquad f_1(z) = 0, \qquad |z| > 2, z \in \overline{\mathbb{H}}.$$

When $\theta = 0$ we can simply choose $f = \widetilde{w}_1$. $\square$

Next we consider the pure jump case, that is, $\kappa = 0$. The proof for this case is similar to the case of $0 < \kappa < 4$. For $\delta, \gamma > 0$, denote $V_{\gamma,\delta} = \{z = (z_1, z_2) : 0 < z_2 \leq \delta |z_1|^{\gamma/2}\}$ and $\sigma_{\gamma,\delta} = \inf\{t \geq 0 : h_t \in V_{\gamma,\delta}\}$.

LEMMA 4.11. *If $\kappa = 0$ and $0 < \alpha < 2$, then for each $\delta > 0$ and $z \in \mathbb{H}$,*

$$P_z\{\sigma_{\alpha,\delta} < \infty\} = 1. \tag{4.43}$$

PROOF. We only need to prove (4.43) when $z \notin V_{\alpha,\delta}$. Without loss of generality we assume that $\delta < 1$. By arguments similar to the case of $0 < \kappa < 4$, we only need to prove (4.43) when $0 < |z_1|^{\alpha/2} < z_2/\delta$ and z_2 small enough.

Now let $s > 0$ such that

$$s < 4\exp\{-\tfrac{1}{2}\exp\{3(2^{4/\alpha})\delta^{-2/\alpha}\theta^{-1/\alpha}\}\} =: t_1 \tag{4.44}$$

and let $z \in \mathbb{H}$ such that $0 < |z_1|^{\alpha/2} < s/\delta$ and $z_2 = s$. By Proposition VIII.4 in [3], there exists a positive number k_1 such that for $0 < t < s$,

$$\mathbb{P}\{U_t \geq (\theta t)^{1/\alpha} \ln\ln(1/t)\} = \mathbb{P}\{S_1 \geq \ln\ln(1/t)\} \geq k_1(\ln\ln(1/t))^{-\alpha}. \tag{4.45}$$

We claim that if $U_{s^2/16} \geq 2^{-4/\alpha}\theta^{1/\alpha}s^{2/\alpha}\ln\ln(16/s^2)$, then

$$|h_{1,u}| \geq (s/\delta)^{2/\alpha} \qquad \text{for some } u \in (0, s^2/16]. \tag{4.46}$$

If this is not true, by (4.6) and (4.44),

$$\begin{aligned}
|h_{1,s^2/16}| &= \left| z_1 + \int_0^{s^2/16} \frac{2h_{1,u}}{h_{1,u}^2 + h_{2,u}^2}\,du - U_{s^2/16} \right| \\
&\geq |U_{s^2/16}| - (s/\delta)^{2/\alpha} - \int_0^{s^2/16} \frac{8(s/\delta)^{2/\alpha}}{s^2}\,du \\
&\geq 2^{-4/\alpha}\theta^{1/\alpha}s^{2/\alpha}\ln\ln(16/s^2) - 2(s/\delta)^{2/\alpha} \\
&\geq (s/\delta)^{2/\alpha},
\end{aligned}$$

which leads to a contradiction. By (4.46)

$$\{U_{s^2/16} \geq 2^{-4/\alpha}\theta^{1/\alpha}s^{2/\alpha}\ln\ln(16/s^2)\} \subseteq \{\sigma_{\alpha,\delta} \leq s^2/16\}. \tag{4.47}$$

By (4.45) and (4.47), we obtain

$$\begin{aligned}
P_z\{\sigma_{\alpha,\delta} \leq s^2/16\} &\geq \mathbb{P}\{U_{s^2/16} \\
&\qquad \geq 2^{-4/\alpha}\theta^{1/\alpha}s^{2/\alpha}\ln\ln(16/s^2)\} \geq k_1(\ln\ln(16/s^2))^{-\alpha}.
\end{aligned} \tag{4.48}$$

Let s_0 be a positive number such that $s_0 < t_1/4$. Define $T_n = \inf\{t \geq 0 : h_{2,t} = s_0/2^n\}$, $n \geq 1$ and $T_0 = 0$. Let $p_n = P_z\{\sigma_{\alpha,\delta} \in (T_{n-1}, T_n]\}$. By the Markov property, (4.6) and (4.48), we have

$$\begin{aligned}
p_n &= E_z[P_z[\sigma_{\alpha,\delta} \in (T_{n-1}, T_n] | \mathcal{F}_{T_{n-1}}]] \\
&\geq E_z\left[I_{\{\sigma_{\alpha,\delta} > T_{n-1}\}} P_{h_{T_{n-1}}}\left\{ |h_{1,T_{n-1}}|^{\alpha/2} < s_0/(2^{n-1}\delta), \sigma_{\alpha,\delta} \leq \left(\frac{s_0}{2^{n-1}}\right)^2 \Big/ 16 \right\}\right] \\
&\geq k_1\big(\ln(2(n+1))\ln 2 - 2\ln s_0\big)^{-\alpha} P_z\{\sigma_{\alpha,\delta} > T_{n-1}\} \\
&\geq k_1\big(\ln(2(n+1))\ln 2 - 2\ln s_0\big)^{-\alpha} \left(1 - \sum_{k=1}^{n-1} p_k\right).
\end{aligned}$$

Hence we can prove (4.43) by the same method as in the case of $0 < \kappa < 4$. $\square$

Recall that we denote $\tau_{a,b} = \inf\{t > 0 : h_{1,t} \leq a; h_{1,u} < b, \text{ for all } 0 \leq u < t\}$.

LEMMA 4.12. *Let $z = (z_1, z_2) \in \mathbb{H} \setminus \{0\}$, $\kappa = 0$.*

(1) *If* $0 < \alpha \le 1$, *then* $P_z\{\zeta < \infty\} = 0$.

(2) *If* $1 < \alpha < 2$, *for any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $P_z\{0, \tau_{0,c(\theta,\alpha)} < \tau_{c(\theta,\alpha),0}\} < \varepsilon$ *for* z *satisfying* $0 < |z_2|/|z_1|^{\alpha/2} < \delta$ *and* $0 < |z_1| < c(\theta,\alpha) := (2\mathcal{A}(1,-\alpha)\gamma(\alpha,\frac{1}{2})\theta)^{-1/(2-\alpha)}$.

PROOF. For convenience, we will use the notation of Lemma 4.2. Here we set

$$A_c f(y) = \frac{2y}{y^2+c^2}\partial_y f(y) + \theta\Delta_y^{\alpha/2} f(y) \qquad \text{for } y \in \mathbb{R}\setminus\{0\}, \tag{4.49}$$

for any C^2 function f. When $0 < \alpha < 1$, we can check that $A_c w_{(\alpha+1)/2}(y) < 0$ for $y \ne 0$. We can also check that $A_c w_1(y) \ge 0$ for $y \ne 0$. Hence we can prove (1) by Dynkin's formula.

Next we assume $1 < \alpha < 2$. Let $0 < |z_1| < c(\theta,\alpha)$. For any $\varepsilon > 0$, let $0 < k < \varepsilon^2 \wedge 1$ and let δ be a positive number such that

$$\delta < \left(\frac{k^\alpha}{2\mathcal{A}(1,-\alpha)\gamma(\alpha,1/2)\theta}\right)^{1/2}. \tag{4.50}$$

Define $f = w_{1/2}$. We claim that $A_c f < 0$ if

$$k|z_1| < |y| < c(\theta,\alpha), \qquad 0 \le c \le \delta|z_1|^{\alpha/2}. \tag{4.51}$$

In fact, when $k^2|z_1|^2 < |y|^2 < \delta^2|z_1|^\alpha$, by (4.50)

$$\begin{aligned} A_c f(y) &= \frac{-|y|^{1/2}}{y^2+c^2} + \mathcal{A}(1,-\alpha)\gamma\left(\alpha,\frac{1}{2}\right)\theta|y|^{-1/2-\alpha} \\ &\le |y|^{-1/2-\alpha}\left(\frac{-|y|^\alpha}{y^2+\delta^2|z_1|^\alpha} + \mathcal{A}(1,-\alpha)\gamma\left(\alpha,\frac{1}{2}\right)\theta\right) \\ &\le |y|^{-1/2-\alpha}\left(\frac{-k^\alpha}{2\delta^2} + \mathcal{A}(1,-\alpha)\gamma\left(\alpha,\frac{1}{2}\right)\theta\right) \\ &\le 0. \end{aligned} \tag{4.52}$$

Similarly, when $c(\theta,\alpha)^2 > |y|^2 \ge \delta^2|z_1|^\alpha$,

$$A_c f(y) \le |y|^{-1/2-\alpha}\left(\frac{-|y|^\alpha}{2y^2} + \mathcal{A}(1,-\alpha)\gamma\left(\alpha,\frac{1}{2}\right)\theta\right) \le 0. \tag{4.53}$$

Combining (4.52) and (4.53), we get the claim. Thus, applying Dynkin's formula to f, we have

$$\begin{aligned} &P_z\{\tau_{0,c(\theta,\alpha)} < \infty\} \\ &\quad\le P_z\{\tau_{k|z_1|,c(\theta,\alpha)} < \tau_{c(\theta,\alpha),k|z_1|}\} \\ &\quad\le \frac{|z_1|^{-1/2} - \int_{\{|y|\ge c(\theta,\alpha)\}}|y|^{-1/2}\mu_{c(\theta,\alpha),k|z_1|}(z,dy)}{\int_{\{|y|\le k|z_1|\}}|y|^{-1/2}\mu_{k|z_1|,c(\theta,\alpha)}(z,dy) - \int_{\{|y|\ge c(\theta,\alpha)\}}|y|^{-1/2}\mu_{c(\theta,\alpha),k|z_1|}(z,dy)} \\ &\quad\le k^{1/2} < \varepsilon, \end{aligned}$$

which completes the proof. $\square$

THEOREM 4.13. *Let $\kappa = 0$ and $0 < \alpha < 2$. For any $z \in \overline{\mathbb{H}} \setminus \{0\}$, we have* $P_z\{\zeta = \infty\} = 1$.

PROOF. When $z_2 = 0$, the conclusion follows from Lemma 3.1. When $z_2 > 0$ and $0 < \alpha \le 1$, the conclusion follows from Lemmas 4.11 and 4.12.

Next we assume $1 < \alpha < 2$ and $z \in \mathbb{H}$. For any $n \in \mathbb{N}$ and $\varepsilon > 0$, by Lemma 4.12, there exists $\delta_n > 0$ such that $P_z\{\tau_{0,c(\theta,\alpha)} < \tau_{0,c(\theta,\alpha)}\} < \varepsilon/2^n$ for $0 < |z_1| < c(\theta,\alpha)$. For any $z \in \mathbb{H}$, define $\tau_1 = \inf\{t > 0; h_t \in V_{\delta_n,\alpha}\}$ and $\sigma_1 = \inf\{t \ge \tau_1; |h_{1,t}| > c(\theta,\alpha)\}$. Define by induction, $\tau_n = \inf\{t \ge \sigma_{n-1}; h_t \in V_{\delta_n,\alpha}, |h_{1,t}| < c(\theta,\alpha)/2\}$ and $\sigma_n = \inf\{t \ge \tau_n; |h_{1,t}| > c(\theta,\alpha)$ or $h_{t-} = 0\}$ for $n \ge 2$. By Lemmas 4.11 and 4.12 as well as the quasi-left continuity of paths, we have

$$P_z\{\zeta < \infty\} = \sum_{n=1}^{\infty} P_z\{\sigma_n = \zeta < \infty\} + P_z\left[\bigcap_{n=1}^{\infty}\{\sigma_n < \zeta < \infty\}\right] \le \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon,$$

which completes the proof. $\square$

4.4. *Proofs of Theorem 1.1 and Corollary 1.2.* The statement of Theorem 1.1 is contained in Theorems 4.3, 4.6, 4.10 and 4.13. To prove Corollary 1.2, we just note that the generator of the stable process with all jumps of size exceeding c removed has as its generator

$$\Delta_{x|c}^{\alpha/2} w(x) = \lim_{\varepsilon \downarrow 0} \mathcal{A}(1, -\alpha) \int_{\{y:\varepsilon < |y-x| < c\}} \frac{w(y) - w(x)}{|x-y|^{1+\alpha}}\, dy,$$

and a computation as in Lemma 2.1 shows that

$$\Delta_{x|c}^{\alpha/2} w_p(x) = \mathcal{A}(1, -\alpha)|x|^{p-1-\alpha}\left(\gamma(\alpha, p) - \frac{p-1}{\alpha}\int_{1-x/c}^{1+x/c} v^{p-2}|1-v|^{\alpha-p}\, dv\right)$$

and for x small enough, the rightmost factor has the same sign as $\gamma(\alpha, p)$. It can now be checked that all arguments can be adapted.

5. The increasing cluster of SLE driven by $U = \sqrt{\kappa} B + \theta^{1/\alpha} S$.

Denote the lifetime of $(h_t(z))_{t \ge 0}$ starting at $h_0(z) = z \in \overline{\mathbb{H}}$ by $\zeta(z)$ as in Section 2.2 and define

$$K_t = \{z \in \overline{\mathbb{H}}, \zeta(z) \le t\},$$

the associated family of strictly increasing compact sets in $\mathbb{H}$, and $\mathbb{H} \setminus K_t$ the associated simply connected open set. First note that unlike the Brownian case, K_t is not always connected by the following lemma.

PROPOSITION 5.1.

$$\mathbb{P}\{K_t \textit{ is a disconnected set in } \overline{\mathbb{H}}\} > 0 \qquad \textit{for all } t > 0.$$

PROOF. Let $t > 0$. Set $\tau = \inf\{s \geq 0 : |U_s| > 1\}$. By (2.4) we have for $u < \tau$

$$|h_u(z)| = \left|z + \int_0^u \frac{2}{h_s(z)}\,ds - U_u\right| \geq |z| - \int_0^u \frac{2}{|h_s(z)|}\,ds - 1.$$

Hence we can check that

$$K_{\tau-} \subseteq B(0, 2t+2) \qquad \text{for } \tau < t. \tag{5.1}$$

Denote Loewner's conformal mapping associated with K_τ by g_τ, and

$$B = \bigl\{U_\tau - U_{\tau-} > 2\sup\{|g_{1,\tau}(z)| : z \in B(0, 2t+2)\} + (4t+5)\bigr\}.$$

By (5.1), we have

$$B \subseteq \{K_\tau \text{ is a disconnected set}\}. \tag{5.2}$$

Set $B' = \{|U_s - U_\tau| \leq 1, \tau < s < \tau + t\}$. By similar arguments as for (5.1) we have

$$g_\tau\bigl(B(0, 2t+2)\bigr) \cap B(U_\tau, 2t+2) = \varnothing \quad \Longrightarrow \quad \overline{K}_{\tau-} \cap \overline{K_t \setminus K_{\tau-}} = \varnothing. \tag{5.3}$$

As $\mathbb{P}[B \cap B'] = \mathbb{P}[B]\mathbb{P}[B'] > 0$, by (5.1)–(5.3), we get the conclusion. □

PROOF OF THEOREM 1.3. In what follows we denote Lebesgue measure on $\overline{\mathbb{H}}$ by $m(\cdot)$. Recall that Theorem 1.3 claims the following: (1) When $\kappa \leq 4$, we have $m(\bigcup_{t>0} K_t) = 0$, a.s. (2) When $\kappa > 4$ and $1 \leq \alpha < 2$, we have $m(\overline{\mathbb{H}} \setminus \bigcup_{t>0} K_t) = 0$, a.s. (3) When $\kappa > 4$ and $0 < \alpha < 1$, we have $\lim_{r\downarrow 0} m(B(0,r) \cap (\bigcup_{t>0} K_t))/m(B(0,r)) = 1$, a.s. and $\lim_{r\uparrow\infty} m(B(0,r) \cap (\bigcup_{t>0} K_t))/m(B(0,r)) = 0$ a.s.

First we show that the lifetime function $\zeta(\omega, z)$ is measurable from $(\Omega \times \overline{\mathbb{H}}, \mathcal{F} \otimes \mathcal{B}(\overline{\mathbb{H}}))$ to $([0,\infty], \mathcal{B}([0,\infty]))$. Denote $\tau_a^z = \inf\{t \geq 0 : h_t(z) \in B(0,a)\}$ for $h_0(z) = z$ and $a > 0$. For any $r > 0$, we have

$$\{(\omega, z) : \zeta(\omega, z) \leq r\} = \bigcup_{k=1}^{\infty}\bigcap_{l=1}^{\infty}\{(\omega, z) : z \in \overline{\mathbb{H}}, |z| > 1/k, \tau_{1/l}^z(\omega) \leq r\}.$$

Hence we only need to show that $\{(\omega, z) : z \in \overline{\mathbb{H}}, |z| > a, \tau_b^z(\omega) \leq r\} \in \mathcal{F} \otimes \mathcal{B}(\overline{\mathbb{H}})$ for any $a > b > 0$. As the coefficient function of the stochastic differential equation (2.4) is Lipschitz and satisfies the linear growth condition outside any neighborhood of zero, by Theorem 6.4.3 in [1], we know that $(h_t(z))_{t\geq 0}$, $z \in \overline{\mathbb{H}}$, have the flow property before hitting $B(0,b)$. Therefore we have $\{(\omega, z) : z \in \overline{\mathbb{H}}, |z| > a, \tau_b^z(\omega) < r\} \in \mathcal{F} \otimes \mathcal{B}(\overline{\mathbb{H}})$.

Now let $\kappa \leq 4$. By Theorem 1.1(i), we have

$$\begin{aligned}
\mathbb{E}\bigl[m(\{z : \zeta(z) < \infty\})\bigr] &= \mathbb{E}\left[\int_{\overline{\mathbb{H}}} I_{\{\zeta(z)<\infty\}} m(dz)\right] \\
&= \int_{\overline{\mathbb{H}}} \mathbb{E}[I_{\{\zeta(z)<\infty\}}] m(dz) \\
&= \int_{\overline{\mathbb{H}}} P_z\{\zeta < \infty\} m(dz) = 0,
\end{aligned} \tag{5.4}$$

which leads to (1). Similarly, by Theorem 1.1(ii), when $\kappa > 4$ and $1 \le \alpha < 2$, we have for any $n > 0$

$$\begin{aligned}\mathbb{E}\bigl[m\bigl(\{z:\zeta(z)<\infty\},|z|<n\bigr)\bigr] &= \mathbb{E}\biggl[\int_{\overline{\mathbb{H}}} I_{\{|z|<n\}} I_{\{\zeta(z)<\infty\}} m(dz)\biggr] \\ &= \int_{\overline{\mathbb{H}}} I_{\{|z|<n\}} \mathbb{E}\bigl[I_{\{\zeta(z)<\infty\}}\bigr] m(dz) = m(\{z:|z|<n\}).\end{aligned}$$

Hence, we have $m(\overline{\mathbb{H}} \setminus \bigcup_{t>0} K_t) = 0$, a.s. (3) can be proved by Theorem 1.1(iii) and the same method. $\square$

6. $\boldsymbol{\beta}$-SLE driven by $\boldsymbol{\alpha}$-stable processes. Let $(S_t)_{t\ge 0}$ be the standard symmetric α-stable Lévy process. For simplicity we take $(S_t)_{t\ge 0}$ as the standard Brownian motion when $\alpha = 2$. For $1 < \beta \le 2$ define the following generalized SLE $(g_t)_{t\ge 0}$, which we call β-SLE:

$$\partial_t g(z) = \frac{2|g_t(z) - \theta^{1/\alpha} S_t|^{2-\beta}}{g_t(z) - \theta^{1/\alpha} S_t}, \qquad g_0(z) = z, \qquad z \in \overline{\mathbb{H}} \setminus \{0\},$$
$$1 < \beta \le 2, 0 < \alpha \le 2;$$

where the derivative above is the right derivative as S_t is right-continuous. Let $h_t(z) = g_t(z) - \theta^{1/\alpha} S_t$; then we have

$$dh_t(z) = \frac{2|h_t(z)|^{2-\beta}}{h_t(z)}\,dt - \theta^{1/\alpha}\,dS_t, \qquad h_0(z) = z, \qquad z \in \overline{\mathbb{H}} \setminus \{0\}. \tag{6.1}$$

Here $(h_t(z))_{t\ge 0}$ is again a well-defined stochastic process up to hitting zero. In fact, similarly to the SLE model we could use a much more general driving process in the above stochastic differential equation. In our setting, when $x \in \mathbb{R}$, $(h_t(x))_{t\ge 0}$ is an $\mathbb{R}$-valued Markov process and its generator $A^{\alpha,\beta,\theta}$ acting on C^2 function f is

$$A^{\alpha,\beta,\theta} f(y) = \frac{2|y|^{2-\beta}}{y}\partial_y f(y) + \theta \Delta_y^{\alpha/2} f(y) \qquad \text{for all } y \ne 0, 1 < \beta \le 2. \tag{6.2}$$

We also denote simply $h_t = h_t(x)$, where $h_0 = x$ under P_x. Also the lifetime of h_t is again denoted by ζ.

PROPOSITION 6.1. *Let $\theta > 0$, $1 < \beta < 2$, and $x \in \mathbb{R}$ with $x \ne 0$. The following statements are valid*:

(a) *If $\alpha > \beta$, then* $\limsup_{|x|\downarrow 0} P_x\{\zeta = \infty\}|x|^{-\delta} < \infty$ *and* $\limsup_{|x|\uparrow\infty} P_x\{\zeta < \infty\}|x|^{\delta} < \infty$ *for all* $0 < \delta < \alpha - 1$.

(b) *If $\alpha = \beta$, there is a phase transition at $\theta_0(\alpha) = 2/(\mathcal{A}(1,-\alpha)|\gamma(\alpha,1)|)$ as follows*:

$$P_x(\zeta < \infty) = 1 \qquad \textit{if } \theta > \theta_0(\alpha) \quad \textit{and} \quad P_x(\zeta = \infty) = 1 \qquad \textit{if } 0 < \theta \le \theta_0(\alpha).$$

(c) *If* $\alpha < \beta$, *then* $P_x(\zeta = \infty) = 1$.

PROOF. (a) Let $0 < \delta < \alpha - 1$. By Lemma 2.1 we can find a positive constant c_1 such that $A^{\alpha,\beta,\theta} w_{1+\delta}(y) < 0$ if $0 < |y| < c_1$. Hence for $0 < a < x < c_1$ we have

$$
\begin{aligned}
P_x\{\zeta = \infty\} &\le \lim_{a\downarrow 0} P_x\{\tau_{a,c_1} > \tau_{c_1,a}\} \\
&\le \lim_{a\downarrow 0} \frac{\int_{\{|y|\le a\}} |y|^\delta \mu_{c_1,a}(x,dy) - x^\delta}{\int_{\{|y|\le a\}} |y|^\delta \mu_{a,c_1}(x,dy) - \int_{\{|y|\ge c_1\}} |y|^\delta \mu_{c_1,a}(x,dy)} \qquad (6.3)\\
&= x^\delta \Big/ \lim_{a\downarrow 0} \int_{\{|y|\ge c_1\}} |y|^\delta \mu_{c_1,a}(x,dy),
\end{aligned}
$$

which gives the first conclusion in (a). Again by Lemma 2.1 we can find a positive constant c_2 such that $A^{\alpha,\beta,\theta} w_{1-\delta}(y) < 0$ if $|y| > c_2$. Similarly we have for $0 < c_2 < x < b$

$$
\begin{aligned}
P_x\{\zeta < \infty\} &\le \lim_{b\uparrow\infty} P_x\{\tau_{b,c_2} > \tau_{c_2,b}\} \\
&\le x^{-\delta} \Big/ \lim_{b\uparrow\infty} \int_{|y|\le c_2} |y|^{-\delta} \mu_{c_2,b}(x,dy),
\end{aligned}
\qquad (6.4)
$$

which gives the second conclusion in (a).

(b) Let $\beta = \alpha$. Define the function

$$\varphi(p) = \frac{2(1-p)}{\mathcal{A}(1,-\alpha)\gamma(\alpha,p)}, \qquad p \ne 1$$

and

$$\varphi(1) = \frac{2}{\mathcal{A}(1,-\alpha)|\gamma(\alpha,1)|} = \theta_0(\alpha).$$

By Lemma 2.1, we can check that φ is a strictly increasing continuous function on $(0,\alpha)$ and

$$\varphi(0+) := \lim_{p\downarrow 0} \varphi(p) > 0; \qquad \lim_{p\uparrow\alpha} \varphi(p) = \infty. \qquad (6.5)$$

Denote by φ^{-1} the inverse function of φ on $(\varphi(0+),\infty)$. By Lemma 2.1 and (6.2) we have $A^{\alpha,\beta,\theta} w_{\varphi^{-1}(\theta)} = 0$ for $\theta \in (\varphi(0+),\infty)$. Hence when $\theta \in (\varphi(0+),\infty)$, with the help of harmonic function $w_{\varphi^{-1}(\theta)}$ we can prove the conclusion by the same method as in Section 3. When $\theta \in (0,\varphi(0+)]$ we can check that $A^{\alpha,\beta,\theta} w_1 > 0$, which also leads to our conclusion.

(c) By Lemma 2.1 we can find a positive constant c_3 such that $A^{\alpha,\beta,\theta} w_0 - c_3 w_0 < 0$. We can prove (c) by this fact and the same method as in Case 2 of Proposition 3.1. $\square$

The behavior in (a) is new. It did not occur in the same way for SLE since Brownian forcing is at the same time at the top of the self-similarity range $\alpha \in (0, 2]$ and the critical forcing where the phase transition occurs, in particular, where in the upper phase the force is strong enough to overcome the potential of the singularity of h_t at zero. For β-SLE driven by an α-stable process with $\alpha > \beta$, the forcing is more than just strong enough to overcome the singularity at zero, but on the other hand, the outward drift is stronger and makes h_t transient, so that there is positive probability that h_t does not hit zero. In this, there are similarities with $\kappa > 4$ and transient driving force for SLE.

If $\alpha = 2 > \beta$, this can only happen if $\mathbb{R} \cap \bigcup_{t\geq 0} K_t = [a, b]$ for some $-\infty < a < 0 < b < \infty$. This means that the β-SLE cluster then grows more in the vertical direction, whereas adding a transient driving force to SLE yields clusters that grow more in the horizontal direction (and necessarily by disconnecting jumps).

In what follows we concentrate on the critical and as such most interesting case $\beta = \alpha$. We will show that the phase transition indicated in Proposition 6.1 can be extended from $z = x \in \mathbb{R}$ to $z \in \mathbb{H}$ in strong analogy to the well-known $\kappa = 4$ phase transition. Recall for $\delta > 0$, we denote by $V_\delta = \{z = z_1 + iz_2 : 0 < z_2 \leq \delta|z_1|)\}$ the double wedge of slope δ and by $\tau_\delta = \inf\{t \geq 0 : h_t \in V_\delta\}$ the first entrance time of h.

LEMMA 6.2. *Let $\theta > 0$. Then for each $\delta > 0$ and $z \in \mathbb{H}$,*

$$P_z\{\tau_\delta < \infty\} = 1. \tag{6.6}$$

PROOF. By arguments similar to the case of Lemma 4.1, we only need to prove (4.43) when $0 < |z_1| < z_2/\delta$ and z_2 small enough. By (6.1), for each $y > 0$ with $h_{2,0} = y$ we have

$$h_{2,u} > y/2 \qquad \text{when } 0 < u < y^\alpha/2^{2+\alpha}. \tag{6.7}$$

Now let $s > 0$ such that

$$s < 16^{1/\alpha} \exp\left\{-\frac{1}{\alpha} \exp\{3 \cdot 2^{4/\alpha}\delta^{-1}\theta^{-1/\alpha}\}\right\} =: t_1 \tag{6.8}$$

and let $z \in \mathbb{H}$ such that $0 < |z_1| < s/\delta$ and $z_2 = s$.

We claim that if $S_{s^\alpha/16} \geq 2^{-4/\alpha} s \ln\ln(16/s^\alpha)$, then:

$$|h_{1,u}| \geq s/\delta \qquad \text{for some } u \in (0, s^\alpha/16]. \tag{6.9}$$

If this is not true, by (6.7) and (6.8),

$$\begin{aligned}
|h_{1,s^\alpha/16}| &= \left| z_1 + \int_0^{s^\alpha/16} \frac{2h_{1,u}}{(h_{1,u}^2 + h_{2,u}^2)^{\alpha/2}}\, du - \theta^{1/\alpha} S_{s^\alpha/16} \right| \\
&\geq |\theta^{1/\alpha} S_{s^\alpha/16}| - s/\delta - \int_0^{s^\alpha/16} \frac{2^{1+\alpha}}{s^{\alpha-1}\delta}\, du \\
&\geq 2^{-4/\alpha}\theta^{1/\alpha} s \ln\ln(16/s^\alpha) - 2s/\delta \\
&\geq s/\delta,
\end{aligned}$$

which leads to a contradiction. By (6.9)

$$\{S_{s^\alpha/16} \geq 2^{-4/\alpha} s \ln\ln(16/s^\alpha)\} \subseteq \{\tau_\delta \leq s^\alpha/16\}. \tag{6.10}$$

By (4.45) and (6.10), we obtain

$$\begin{aligned} P_z\{\tau_\delta \leq s^\alpha/16\} &\geq P\{U_{s^\alpha/16} \geq 2^{-4/\alpha}\theta^{1/\alpha} s \ln\ln(16/s^\alpha)\} \\ &\geq k_1(\ln\ln(16/s^\alpha))^{-\alpha}. \end{aligned} \tag{6.11}$$

Let s_0 be a positive number such that $s_0 < t_1$. Define $T_n = \inf\{t \geq 0 : h_{2,t} = s_0/2^n\}$, $n \geq 1$ and $T_0 = 0$. Let $p_n = P_z\{\tau_\delta \in (T_{n-1}, T_n]\}$. By the Markov property, (6.7) and (6.11), we have

$$\begin{aligned} p_n &= E_z\big[P_z[\tau_\delta \in (T_{n-1}, T_n] | \mathcal{F}_{T_{n-1}}]\big] \\ &\geq E_z\left[I_{\{\tau_\delta > T_{n-1}\}} P_{h_{T_{n-1}}}\left\{|h_{1,T_{n-1}}|^{\alpha/2} < s_0/(2^{n-1}\delta),\ \tau_\delta \leq \left(\frac{s_0}{2^{n-1}}\right)^\alpha \Big/ 16\right\}\right] \\ &\geq k_1\big(\ln\big(\alpha(n-1)\ln 2 + 4\ln 2 - \alpha \ln s_0\big)\big)^{-\alpha} P_z\{\tau_\delta > T_{n-1}\} \\ &\geq k_1\big(\ln\big(\alpha(n-1)\ln 2 + 4\ln 2 - \alpha \ln s_0\big)\big)^{-\alpha}\left(1 - \sum_{k=1}^{n-1} p_k\right). \end{aligned}$$

Hence we can complete the proof by the same arguments as in Lemma 4.1. $\square$

PROPOSITION 6.3. *Let $1 < \alpha < 2$ and $0 < \theta < \theta_0(\alpha)$. For any $z \in \overline{\mathbb{H}} \setminus \{0\}$, we have $P_z\{\zeta = \infty\} = 1$.*

PROOF. When $z_2 = 0$, the conclusion follows from Proposition 6.1. When $z_2 > 0$, by Lemma 6.2 we only need to prove that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $P_z\{\zeta < \infty\} < \varepsilon$ for z satisfying $0 < |z_2|/|z_1| < \delta$. For $c \geq 0$ and C^2 function f, set

$$A_c^{\alpha,\theta} f(y) = \frac{2y}{(y^2 + c^2)^{\alpha/2}} \partial_y f(y) + \theta \Delta_y^{\alpha/2} f(y) \qquad \text{for } y \neq 0. \tag{6.12}$$

Let $\theta \in (0, \theta_0(\alpha))$ and define

$$b = \varphi^{-1}\left(\frac{\theta_0(\alpha) + (\theta \vee \varphi(0+))}{2}\right).$$

By the definition of φ, we see that $0 < b < 1$. Set $\theta_1 = \theta/\varphi(b)$. It is easy to see that $\theta_1 < 1$. Let $0 < k < \varepsilon^{1/(1-b)} \wedge 1$ and let δ be a positive number such that

$$\delta < k\sqrt{\theta_1^{-2/\alpha} - 1}. \tag{6.13}$$

Define $f = w_b$ and applying (6.13), we have for any $|y| > k|z_1|$ and $0 \le c \le \delta|z_1|$

$$
\begin{aligned}
A_c^{\alpha,\theta} f(y) &= \frac{2(b-1)|y|^{b-1}}{(y^2+c^2)^{\alpha/2}} + \theta \mathcal{A}(1,-\alpha)\gamma(\alpha,b)|y|^{b-1-\alpha} \\
&\le \frac{b-1}{|y|^{\alpha+1-b}}\left(\frac{2}{(1+\delta^2/k^2)^{\alpha/2}} + \theta\mathcal{A}(1,-\alpha)\gamma(\alpha,b)/(b-1)\right) \\
&= \frac{b-1}{|y|^{\alpha+1-b}}\left(\frac{2}{(1+\delta^2/k^2)^{\alpha/2}} - 2\theta/\varphi(b)\right) \qquad (6.14)\\
&= \frac{b-1}{|y|^{\alpha+1-b}}\left(\frac{2}{(1+\delta^2/k^2)^{\alpha/2}} - 2\theta_1\right) \\
&\le 0.
\end{aligned}
$$

By (6.14) and the same calculation as in Lemma 4.2 we have

$$P_z\{\zeta < \infty\} \le k^{1-b} < \varepsilon,$$

which completes the proof. $\square$

Next we consider the case $\theta > \theta_0(\alpha)$. First we prepare a result corresponding to Lemma 4.4.

LEMMA 6.4. *Let $1 < \alpha < 2$ and $\theta > \theta_0(\alpha)$. Let $z = (z_1, z_2) \in \overline{\mathbb{H}} \setminus \{0\}$. Denote $\widetilde{\tau} = \inf\{t \ge 0 : h_{1,t-} = 0\}$. Then $\widetilde{\tau} < \infty$ with probability 1. Moreover, there exist a constant c and an event Θ such that*

$$E_z[I_\Theta \widetilde{\tau}] < c|z_1|^{\varphi^{-1}(\theta)-1}, \qquad P_z[\Theta^c] < c|z_1|^{\varphi^{-1}(\theta)-1} \qquad \text{for } 0 < |z_1| < 1. \tag{6.15}$$

Specifically we can take Θ to be $\{\tau_{0,2} < \tau_{2,0}\}$ in (6.15).

PROOF. We omit the proof as it is the same as for Lemma 4.4. $\square$

LEMMA 6.5. *Let $1 < \alpha < 2$ and $\theta > \theta_0(\alpha)$. Let $\delta > 0$ be such that $(\varphi^{-1}(\theta) - 1)(1 - \delta/\alpha) - 2\delta =: r > 0$. Then there exists a constant number k_3, depending on α, δ and θ, such that for any $a > 0$ and $z = iz_2$*

$$P_z\{L < a^{\alpha+\delta}/\delta\} \le k_3 a^{2\delta} \qquad \text{where } L = \int_0^{3a^r} I_{\{|h_{1,t}|<a\}}\, dt. \tag{6.16}$$

PROOF. It is obvious that we can also assume a to be small enough such that

$$16a^\delta < \delta, \qquad a^{(\varphi^{-1}(\theta)-1)(1-\delta/\alpha)-2\delta} > a^{\alpha+\delta}/\delta. \tag{6.17}$$

Denote $\tau(s) = \inf\{t : t \geq s, h_{1,t} = 0\} - s$ for $s > 0$. By (6.15), we have

$$\begin{aligned} &P_z\{|h_{1,a^{\alpha+\delta}/\delta}| < a^{1-\delta/\alpha}, \tau(a^{\alpha+\delta}/\delta) \geq a^{(\varphi^{-1}(\theta)-1)(1-\delta/\alpha)-2\delta}\} \\ &\quad \leq ca^{2\delta} + ca^{(\varphi^{-1}(\theta)-1)(1-\delta/\alpha)} \\ &\quad \leq 2ca^{2\delta}. \end{aligned} \tag{6.18}$$

We claim that

$$\Big\{\sup_{0<t\leq a^{\alpha+\delta}/\delta} |h_{1,t}| \geq a\Big\} \subseteq \Big\{\sup_{0<t\leq a^{\alpha+\delta}/\delta} \theta^{1/\alpha}|S_t| \geq a/8\Big\}, \tag{6.19}$$

$$\Big\{\sup_{0<t\leq a^{\alpha+\delta}/\delta} |h_{1,t}| \geq a^{1-\delta/\alpha}\Big\} \subseteq \Big\{\sup_{0<t\leq a^{\alpha+\delta}/\delta} \theta^{1/\alpha}|S_t| \geq a^{1-\delta/\alpha}/8\Big\}. \tag{6.20}$$

Let $t' = \inf\{t : |h_{1,t}| \geq a\}$, $t'' = \sup\{t \leq t' : |h_{1,t}| < a/2\}$ and suppose that ω belongs to the left-hand side of (6.19); then by the first inequality of (6.17)

$$\begin{aligned} a/2 &\leq |h_{1,t'} - h_{1,t''-}| \\ &= \Big| \int_{t''}^{t'} \frac{2h_{1,u}}{(h_{1,u}^2 + h_{2,u}^2)^{\alpha/2}}\, du - \theta^{1/\alpha} S_{t'} + \theta^{1/\alpha} S_{t''-} \Big| \\ &\leq |\theta^{1/\alpha}(S_{t'} - S_{t''-})| + \int_{t''}^{t'} 2h_{1,u}^{1-\alpha}\, du \\ &\leq |\theta^{1/\alpha}(S_{t'} - S_{t''-})| + 4a^{1+\delta}/\delta \\ &\leq |\theta^{1/\alpha}(S_{t'} - S_{t''-})| + a/4, \end{aligned} \tag{6.21}$$

which proves (6.19). We omit the proof of (6.20) as the proof is the same. By the reflection principle we have

$$\begin{aligned} \mathbb{P}\Big\{\sup_{0<t\leq a^{\alpha+\delta}/\delta} \theta^{1/\alpha}|S_t| \geq a/8\Big\} &\leq 2\mathbb{P}\{|S_{a^{\alpha+\delta}/\delta}| \geq \theta^{-1/\alpha} a/8\} \\ &\leq 2\mathbb{P}\{|S_1| \geq \delta^{1/\alpha}\theta^{-1/\alpha} a^{-\delta/\alpha}/8\} \\ &\leq 2^{1+3\alpha} k_1 \theta \delta^{-1} a^{\delta}. \end{aligned} \tag{6.22}$$

Similarly we have

$$\mathbb{P}\Big\{\sup_{0<t\leq a^{\alpha+\delta}/\delta} \theta^{1/\alpha}|S_t| \geq a^{1-\delta/\alpha}/8\Big\} \leq 2^{1+3\alpha} k_1 \theta \delta^{-1} a^{2\delta}. \tag{6.23}$$

By (6.17)–(6.20), (6.22) and (6.23),

$$\begin{aligned} &P_z\{L < a^{\alpha+\delta}/\delta\} \\ &\quad \leq P_z\Big\{a \leq \sup_{0<t\leq a^{\alpha+\delta}/\delta} |h_{1,t}| < a^{1-\delta/\alpha}, L < a^{\alpha+\delta}/\delta\Big\} \end{aligned} \tag{6.24}$$

$$
\begin{aligned}
&\quad + P_z\Big\{\sup_{0<t\le a^{\alpha+\delta}/\delta}|h_{1,t}|\ge a^{1-\delta/\alpha}\Big\}\\
&\le P_z\Big\{a\le \sup_{0<t\le a^{\alpha+\delta}/\delta}|h_{1,t}|<a^{1-\delta/\alpha},\\
&\qquad \tau(a^{\alpha+\delta}/\delta)<a^{(\varphi^{-1}(\theta)-1)(1-\delta/\alpha)-2\delta}, L<a^{\alpha+\delta}/\delta\Big\}\\
&\quad + P_z\Big\{\sup_{0<t\le a^{\alpha+\delta}/\delta}|h_{1,t}|<a^{1-\delta/\alpha},\\
&\qquad \tau(a^{\alpha+\delta}/\delta)\ge a^{(\varphi^{-1}(\theta)-1)(1-\delta/\alpha)-2\delta}\Big\}+2^{1+3\alpha}k_1\theta\delta^{-1}a^{2\delta}\\
&\le P_z\Big\{a\le \sup_{0<t\le a^{\alpha+\delta}/\delta}|h_{1,t}|<a^{1-\delta/\alpha},\\
&\qquad \tau(a^{\alpha+\delta}/\delta)<a^{(\varphi^{-1}(\theta)-1)(1-\delta/\alpha)-2\delta},\\
&\qquad \sup_{\tau(a^{\alpha+\delta}/\delta)\le t\le\tau(a^{\alpha+\delta}/\delta)+a^{\alpha+\delta}/\delta}|h_{1,t}|\ge a\Big\}+2^{1+3\alpha}\delta^{-1}(k_1\theta+c)a^{2\delta}\\
&\le P_z\Big\{\sup_{0<t\le a^{\alpha+\delta}/\delta}|h_{1,t}|\ge a, \sup_{\tau(a^{\alpha+\delta}/\delta)\le t\le\tau(a^{\alpha+\delta}/\delta)+a^{\alpha+\delta}/\delta}|h_{1,t}|\ge a\Big\}\\
&\quad +2^{1+3\alpha}(\delta^{-1}k_1\theta+c)a^{2\delta}\\
&\le 2^{1+3\alpha}\big(\delta^{-1}k_1\theta+c+(k_1\theta\delta^{-1})^2 2^{1+3\alpha}\big)a^{2\delta},
\end{aligned}
$$

which completes the proof. $\square$

PROPOSITION 6.6. *Let $1<\alpha<2$ and $\theta>\theta_0(\alpha)$. Let $z\in\overline{\mathbb{H}}\setminus\{0\}$. Then $P_z\{\zeta<\infty\}=1$.*

PROOF. The proof will follow the arguments for Theorem 4.6 with some technical differences. Fix $z=z_1+iz_2\in\overline{\mathbb{H}}$. When $z_2=0$, the conclusion follows from Proposition 6.1. Next, we assume $z_2>0$ and, without loss of generality, $z_1>0$. Denote $\beta>0$ small enough such that

$$(\varphi^{-1}(\theta)-1)(1-\beta/\alpha)\ge 6\beta, \tag{6.25}$$

$$\frac{(\varphi^{-1}(\theta)-1)(1-\beta/\alpha)-2\beta}{2\alpha}\big(\varphi^{-1}(\theta)-1\big)\ge 2\beta. \tag{6.26}$$

Write $\widetilde{\alpha}=(\varphi^{-1}(\theta)-1)(1-\beta/\alpha)-2\beta$. Let a_1 be an arbitrary positive number such that

$$a_1<z_2\wedge\left(\frac{\beta}{\beta+1}\right)^{1/\beta}\quad\text{and}\quad a_1^{1+\beta}/\beta<a_1/2. \tag{6.27}$$

Denote $\eta_0 = 0$ and $\xi_1 = \inf\{t \geq 0 : h_{2,t} = a_1\}$. Set

$$b_1 = a_1 - \frac{a_1^{1+\beta}}{\beta}; \qquad \eta_1 = \inf\{t \geq \xi_1 : h_{1,t} = 0\}.$$

By Lemma 6.4 we have $\eta_1 < \infty$ a.s. Define by induction

$$a_{n+1} = h_{2,\eta_n}; \qquad \xi_{n+1} = \eta_n + 3a_{n+1}^{\widetilde{\alpha}};$$

$$b_{n+1} = a_{n+1} - \frac{a_{n+1}^{1+\beta}}{\beta}; \qquad \eta_{n+1} = \inf\{t \geq \xi_{n+1} : h_{1,t} = 0\}.$$

Let $L_n = \int_{\eta_{n-1}}^{\xi_n} I_{\{|h_{1,t}|<a_n\}}\,dt$. Define events

$$\begin{aligned} E_n &= \{L_n \geq 2^{\alpha/2} a_n^{\alpha+\beta}/\beta\}; \\ G_n &= \{|h_{1,\xi_n}| > 8a_n^{\widetilde{\alpha}/2\alpha}\}; \qquad (6.28)\\ H_n &= \{h_{2,\xi_n} \leq b_n\}. \end{aligned}$$

Next we prove the following assertions:

$$G_n \subseteq \left\{\theta^{1/\alpha} \sup_{\eta_{n-1}<t<\xi_n} |S_{\xi_n} - S_t| > a_n^{\widetilde{\alpha}/2\alpha}\right\}, \tag{6.29}$$

$$E_n \subseteq H_n, \tag{6.30}$$

$$P_z[E_n^c \cup G_n | \mathcal{F}_{\eta_{n-1}}] \leq \left(6\theta k_1 + 2^{\alpha\beta/(\alpha+\beta)} k_3\right) a_n^{2\beta}. \tag{6.31}$$

Suppose that $\theta^{1/\alpha} \sup_{\eta_{n-1}<t<\xi_n} |S_{\xi_n} - S_t| \leq a_n^{\widetilde{\alpha}/2\alpha}$; we will check (6.29) by proving that $|h_{1,\xi_n}| \leq 8a_n^{\widetilde{\alpha}/2\alpha}$. Otherwise we can find $t' \in (\eta_{n-1}, \xi_n)$ such that $|h_{1,t'-}| \leq a_n^{\widetilde{\alpha}/2\alpha}$ and $|h_{1,t}| \geq a_n^{\widetilde{\alpha}/2\alpha}$ for $t \in (t', \xi_n)$. So we have

$$\begin{aligned} |h_{1,\xi_n}| &= \left| \int_{\eta_{n-1}}^{\xi_n} \frac{2h_{1,u}}{(h_{1,u}^2 + h_{2,u}^2)^{\alpha/2}}\,du - \theta^{1/\alpha} S_{\xi_n} + \theta^{1/\alpha} S_{\eta_{n-1}} \right| \\ &\leq \left| \int_{t'}^{\xi_n} \frac{2h_{1,u}}{(h_{1,u}^2 + h_{2,u}^2)^{\alpha/2}}\,du - \theta^{1/\alpha} S_{\xi_n} + \theta^{1/\alpha} S_{t'-} \right| + |h_{1,t'-}| \\ &\leq \left| \int_{t'}^{\xi_n} 2h_{1,u}^{1-\alpha}\,du \right| + 2a_n^{\widetilde{\alpha}/2\alpha} \qquad (6.32)\\ &\leq 6a_n^{\widetilde{\alpha}(1+\alpha)/2\alpha} + 2a_n^{\widetilde{\alpha}/2\alpha} \\ &\leq 8a_n^{\widetilde{\alpha}/2\alpha}. \end{aligned}$$

Now suppose that $L_n \geq 2^{\alpha/2} a_n^{\alpha+\beta}/\beta$. If $h_{2,\xi_n} < a_n/2$, by the second inequality of (6.27), we see that (6.30) is true. When $h_{2,\xi_n} \geq a_n/2$, we have

$$h_{2,\xi_n} = a_n + \int_{\eta_{n-1}}^{\xi_n} \frac{-2h_{2,u}}{(h_{1,u}^2 + h_{2,u}^2)^{\alpha/2}}\,du$$

$$
\begin{aligned}
&\le a_n - \int_{\eta_{n-1}}^{\xi_n} \frac{a_n}{(h_{1,u}^2 + a_n^2)^{\alpha/2}}\,du \\
&\le a_n - 2^{-\alpha/2} \int_{\eta_{n-1}}^{\xi_n} I_{\{|h_{1,t}|<a_n\}} a_n^{1-\alpha}\,du \\
&\le a_n - a_n^{1+\beta}/\beta = b_n,
\end{aligned}
$$

which completes the proof of (6.30). Equation (6.31) can be proved by Lemma 6.5, (6.29), (6.30) and the following results:

$$
\begin{aligned}
&P_z\Big[\theta^{1/\alpha} \sup_{\eta_{n-1}<t<\xi_n} |S_{\xi_n} - S_t| > a_n^{\widetilde{\alpha}/2\alpha} \Big| \mathcal{F}_{\eta_{n-1}}\Big] \\
&\qquad \le 2P_z[|S_{\xi_n-\eta_{n-1}}| > \theta^{-1/\alpha} a_n^{\widetilde{\alpha}/2\alpha} | \mathcal{F}_{\eta_{n-1}}] \\
&\qquad \le 2P_z[|S_1| > 3^{-1/\alpha}\theta^{-1/\alpha} a_n^{-\widetilde{\alpha}/2\alpha} | \mathcal{F}_{\eta_{n-1}}] \le 6\theta k_1 a_n^{\widetilde{\alpha}/2} \le 6\theta k_1 a_n^{2\beta},
\end{aligned}
\tag{6.33}
$$

where we used (6.25) in the last inequality of (6.33).

As for SLE we denote

$$
\begin{aligned}
\widetilde{\tau}_{0,n} &= \inf\{t \ge \xi_n : h_{1,t} = 0, |h_{1,u}| < 2 \text{ for } \xi_n < u < t\}; \\
\widetilde{\tau}_{2,n} &= \inf\{t \ge \xi_n : h_{1,t} \ge 2, |h_{1,u}| > 0 \text{ for } \xi_n < u < t\}.
\end{aligned}
\tag{6.34}
$$

By Lemma 6.4, there exists a constant $k_4 > 0$ such that

$$
\begin{aligned}
E_z\big[I_{\{\widetilde{\tau}_{0,n}<\widetilde{\tau}_{2,n}\}}(\eta_n - \xi_n)|\mathcal{F}_{\xi_n}\big] &< k_4|h_{1,\xi_n}|^{\varphi^{-1}(\theta)-1}, \\
E_z\big[I_{\{\widetilde{\tau}_{0,n}>\widetilde{\tau}_{2,n}\}}|\mathcal{F}_{\xi_n}\big] &< k_4|h_{1,\xi_n}|^{\varphi^{-1}(\theta)-1},
\end{aligned}
\tag{6.35}
$$

when $0 < |h_{1,\xi_n}| < 1$. Denote $F_n = \{\widetilde{\tau}_{0,n} < \widetilde{\tau}_{2,n}\} \cap (E_n \cap G_n^c)$ and set $F = \bigcap_{n\ge 1} F_n$. By (6.30) and Lemma 4.5

$$
\bigcap_{n=1}^{N-1} (E_n \cap G_n^c) \subseteq \bigcap_{n=1}^{N} \{a_n < (a_1^{-\beta} + n - 1)^{-1/\beta}\} \qquad \forall N \in \mathbb{N}.
\tag{6.36}
$$

Write $d_n = a_1^{-\beta} + n - 1$. By (6.26), (6.31), (6.35) and (6.36),

$$
\begin{aligned}
P_z[F] &= \lim_{N\to\infty} P_z\Big[\bigcap_{n=1}^{N} F_n\Big] \\
&= \lim_{N\to\infty} E_z\big[I_{\bigcap_{n=1}^{N-1} F_n} I_{E_N\cap G_N^c} P_z[\widetilde{\tau}_{0,N} < \widetilde{\tau}_{2,N}|\mathcal{F}_{\xi_N}]\big] \\
&\ge \lim_{N\to\infty} E_z\big[I_{\bigcap_{n=1}^{N-1} F_n} I_{E_N\cap G_N^c} \big(1 - k_4|h_{1,\xi_N}|^{\varphi^{-1}(\theta)-1}\big)\big] \\
&\ge \lim_{N\to\infty} E_z\big[I_{\bigcap_{n=1}^{N-1} F_n} I_{E_N\cap G_N^c} \big(1 - 2^{3(\varphi^{-1}(\theta)-1)} k_4 |a_N|^{(\varphi^{-1}(\theta)-1)\widetilde{\alpha}/2\alpha}\big)\big]
\end{aligned}
\tag{6.37}
$$

$$
\begin{aligned}
&\geq \lim_{N\to\infty} E_z\big[I_{\bigcap_{n=1}^{N-1} F_n} I_{E_N\cap G_N^c}\big(1-2^{3(\varphi^{-1}(\theta)-1)}k_4 d_N^{-2}\big)\big]\\
&= \lim_{N\to\infty}\big(1-2^{3(\varphi^{-1}(\theta)-1)}k_4 d_N^{-2}\big) E_z\big[I_{\bigcap_{n=1}^{N-1} F_n} P_z[E_N\cap G_N^c|\mathcal{F}_{\eta_{N-1}}]\big]\\
&\geq \lim_{N\to\infty}\big(1-2^{3(\varphi^{-1}(\theta)-1)}k_4 d_N^{-2}\big)\big(1-\big(6\theta k_1+2^{2\beta/(\alpha+\beta)}k_3\big)a_N^{2\beta}\big)\\
&\quad\times P_z\Bigg[\bigcap_{n=1}^{N-1} F_n\Bigg]\\
&\geq \prod_{n=1}^{\infty}\big(1-2^{3(\varphi^{-1}(\theta)-1)}k_4 d_n^{-2}\big)\big(1-\big(6\theta k_1+2^{2\beta/(\alpha+\beta)}k_3\big)\,d_n^{-2}\big)\\
&\geq 1-\sum_{n=1}^{\infty}\big(6\theta k_1+2^{2\beta/(\alpha+\beta)}k_3+2^{3(\varphi^{-1}(\theta)-1)}k_4\big)\,d_n^{-2}.
\end{aligned}
$$

By the definition of d_n and (6.37), we have

$$
\lim_{a_1\downarrow 0} P_z[F]=1. \tag{6.38}
$$

By Lebesgue's monotone convergence theorem, (6.26), (6.35) and (6.36),

$$
\begin{aligned}
E_z[I_F\zeta] &= \lim_{n\to\infty} E_z[I_F\xi_n]\\
&= \lim_{n\to\infty}\sum_{k=1}^{n} E_z[I_F(\xi_k-\eta_{k-1})]+\lim_{n\to\infty}\sum_{k=1}^{n} E_z[I_F(\eta_{k-1}-\xi_{k-1})]\\
&\leq \sum_{k=1}^{\infty} 3E_z[I_F d_k^{\widetilde{\alpha}/\beta}]\\
&\quad+\sum_{k=1}^{\infty} E_z\big[E_z\big[I_{\bigcap_{s=1}^{k-1}(E_s\cap G_s^c)} I_{\{\widetilde{\tau}_{0,k-1}>\widetilde{\tau}_{2,k-1}\}}(\eta_{k-1}-\xi_{k-1})|\mathcal{F}_{\xi_{k-1}}\big]\big]\\
&\leq \sum_{k=1}^{\infty} 3d_k^{\widetilde{\alpha}/\beta}+\sum_{k=1}^{\infty} E_z\big[I_{\bigcap_{s=1}^{k-1}(E_s\cap G_s^c)} 2^{3(\varphi^{-1}(\theta)-1)}k_4|h_{1,\xi_{k-1}}|^{(\varphi^{-1}(\theta)-1)\widetilde{\alpha}/2\alpha}\big]\\
&\leq \sum_{k=1}^{\infty} 3d_k^{\widetilde{\alpha}/\beta}+\sum_{k=1}^{\infty} 2^{3(\varphi^{-1}(\theta)-1)}k_4 E_z\big[I_{\bigcap_{s=1}^{k-1}(E_s\cap G_s^c)} a_{k-1}^{(\varphi^{-1}(\theta)-1)\widetilde{\alpha}/2\alpha}\big]\\
&\leq \sum_{k=1}^{\infty} 3d_k^{-4}+\sum_{k=1}^{\infty} 2^{3(\varphi^{-1}(\theta)-1)}k_4 d_{k-1}^{-2}\\
&<\infty,
\end{aligned}
$$

which completes the proof. $\square$

PROOFS OF THEOREM 1.4 AND COROLLARY 1.5. The statement of Theorem 1.4 is contained in Propositions 6.3 and 6.6. The proof of the corollary is the same as for SLE with the help of these propositions. $\square$

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