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Burgers turbulence initialized by a regenerative impulse *

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Summary. In this article we look at a one-dimensional infinitesimal particle system governed by the completely inelastic collision rule. Considering uniformly spread mass, we feed the system with initial velocities, so that when time evolves the corresponding velocity field fulfils the inviscid Burgers equation. More precisely, we suppose here that the initial velocities are zero, except for particles located on a stationary regenerative set for which the velocity is some given constant number. We give results of a large deviation type. First, we estimate the probability that a typical particle is located at time 1 at distance at least D from its initial position, when D tends to infinity. Its behaviour is related to the left tail of the gap measure of the regenerative set. We also show the same asymptotics for the tail of the shock interval length distribution. Second, we analyse the event that a given particle stands still at time T as T tends to infinity. The data to which we relate its behaviour are the right tail of the gap measure of the regenerative set. We conclude with some results on the shock structure.

Keywords. Inviscid Burgers equation, random initial velocity, regenerative sets, subordinators, large deviations

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1 Introduction

Assume a one-dimensional infinitesimal particle system whose initial mass is uniformly spread. An impulse is given to this system by inducing initial velocities to the particles. Let the evolution of the system obey to the principles of the completely inelastic collision rule, i.e. whenever two clumps of particles meet they build a larger particle clump. Their new common velocity is given by the momentum preservation rule; note that there is a loss of energy.

For a mathematical treatment of this particle system (see E, Rykov and Sinai [9]) it appears most convenient to model the velocity $u(x,t)$ of a particle that is at time $t \geq 0$ in location $x \in \mathbb{R}$. The initial configuration at time $t = 0$ then consists of a specification of the initial velocity $u(x,0)$, $x \in \mathbb{R}$. This model is commonly referred to as the (inviscid) Burgers turbulence as it was introduced by Burgers in the 1920-30s as a simplifying model to describe the turbulence of fluid particles, cf. Burgers [7], Woyczyński [23] and Leonenko [15] for overviews.

Burgers [7] in the 1970s was one of the first to look at this model initialized by random data, specifically a white noise. The interest increased considerably in the early 1990s when larger classes of Gaussian processes were considered as initial velocities or initial potentials (cf. [20], [21], [1], [2], [18], [19], [15]). Their results include properties of the shock structure and the behaviour of the velocities at positive times. We refer to Woyczyński [23] and Leonenko [15] for many more important references on works in this area.

We consider here a quite different class of processes which in general do not lead to nice marginal distributions or satisfy any scaling property, but have another set of properties that can be exploited in order to estimate statistics of the solution process to some extent. We consider a regenerative impulse, i.e. on a stationary regenerative set particles are assigned a positive constant initial velocity, whereas particles outside the regenerative set have zero initial velocity. In particular, all movements go from the left to the right. Unlike most of the initial data considered by other authors, our system is not Gaussian and has a memory, i.e. initial velocities in different locations are not independent. These initial potentials arise naturally as limits when looking at functions of stationary Markov processes as initial velocities (cf. Remark 1 in subsection 2.3).

We give here large deviation estimates for a few fundamental events. First, we investigate into the event that the location $x(a,1)$ at time 1 of a particle is at least at distance D from its original position $a \in \mathbb{R}$ as D tends to infinity. Thanks to stationarity the marginal behaviour is the same in each fixed location a, so we focus on the origin. The basic data describing a regenerative set are the thickness parameter and the gap measure. The interesting case is when the thickness parameter is zero. Then the important information is the left tail of the gap measure (at zero), which can be motivated by intuition, namely that a high velocity (distance travelled in unit time) is strongly connected with a great lot of small gaps (hence many impulse locations) close to the initial particle. Under certain regularity conditions, the probability of this event decreases faster than a quadratic negative exponential. We determine the exponential power. We establish the same asymptotic behaviour also for the event that at time 1 the zero particle is situated within a particle clump of a size at least s as s tends to infinity. Second, we look at the event that a particular particle stands still at time T as T tends to infinity. The probability of this event behaves roughly like the right tail of the gap measure, which is a plausible result as it appears natural that being situated in a large gap favours a late hit and vice versa. Our results cover both a polynomial and a linearly exponential decay. Furthermore, we carry out an analysis into the shock structure. A fundamental characteristic is that at any fixed time there exist intervals of non-moving particles. In the important case of a vanishing thickness parameter, we carry the analysis further by showing that the shock behaviour between two such intervals is discrete, provided that the gap measure satisfies a mild regularity condition at the origin.

The structure of the paper is as follows. A preliminary section introduces the particle system in question including the initial data considered here. For paedagogical purposes, this includes an introduction to a discrete mass analogue to the Burgers model which elucidates the behaviour of the latter; in particular, we develop the Hopf-Cole formulas by elementary methods. The three main sections state and discuss the precise results and present their proofs. On this way a number of lemmas providing auxiliary results on subordinators are established that may be of independent interest. Subordinators arise naturally in our setting as regenerative sets are known to correspond to the ranges of subordinators (cf. Fristedt [11], Bertoin [5]).

2 Preliminaries on the dynamic of inelastic collisions

2.1 Evolution of a discrete particle system with inelastic collisions

Throughout this subsection, let the initial mass of our particle system consist of unit point masses on the integers. To each particle $n \in \mathbb{Z}$ we assign an initial velocity u_n. Then the behaviour of the particle system consists of movements and collisions of clumps having positive integer mass, which we assume to take place inelastically. This is a discrete mass analogue to the (inviscid) Burgers turbulence.

It is convenient to introduce an initial potential ψ given by

$$\psi_n = \sum_{j=1}^{n} u_j, \qquad \psi_{-n} = -\sum_{j=0}^{n-1} u_{-j}, \quad n \geq 0, \qquad \Rightarrow \quad u_m = \psi_m - \psi_{m-1}, \quad m \in \mathbb{Z}$$

right-continuously continued by $\psi(x,0) := \psi_m$ for $x \in [m, m+1)$, $m \in \mathbb{Z}$.

To analyse this model to some extent, we focus on the important special case of non-negative velocities (i.e. increasing potentials). We shall only use the most basic physical formula expressing the momentum needed to transfer a number of unit mass objects within time t a certain distance in a one-dimensional space.

We note that in order for a particle at $a \in \mathbb{Z}$ to pass the location $x \in \mathbb{R}$, $x \geq a$, by time $t > 0$, a certain amount of momentum is necessary. For some $b \leq a$, the momentum available must allow all particles $b, \ldots, [x]$ by time t to get to $[x]$ and further to x yielding

$$\psi_{[x]} - \psi_{b-1} \geq \frac{1}{t} \sum_{i=0}^{[x]-b} i + \frac{1}{t}([x] - b + 1)(x - [x]) = \frac{([x] - b + 1)(2x - [x] - b)}{2t}.$$

This leads to the necessary condition for the particle initially at $a \in \mathbb{Z}$ to pass the location $x \geq a$ by time $t > 0$

$$\psi_{[x]} \geq \inf_{b \leq a} \left\{ \psi_{b-1} + \frac{([x]-b+1)(2x-[x]-b)}{2t} \right\}.$$

Clearly, this is not a sufficient condition as it is possible that the momentum is badly spread in the sense that, given a b where the infimum is attained, there is a $c \in \mathbb{Z}$, $c \geq b$, such that the particles $c, \ldots, [x]$ pass x fairly quickly leaving too little momentum to $b, \ldots, c-1$, i.e. their momentum is insufficient to get all these particles to c and further beyond x:

$$\psi_{c-1} - \psi_{b-1} \leq \frac{1}{t} \sum_{i=1}^{c-b} i + \frac{1}{t}(c-b)(x-c) = \frac{(c-b)(2x-c-b+1)}{2t}$$

which is easily seen to be equivalent to

$$\psi_{c-1} + \frac{([x]-c+1)(2x-[x]-c)}{2t} \leq \psi_{b-1} + \frac{([x]-b+1)(2x-[x]-b)}{2t}.$$

Putting the two results together, we conclude that the left-most location $b \in \mathbb{Z}$ at which the infimum

$$\inf_{b \leq [x]+1} \left\{ \psi_{b-1} + \frac{([x]-b+1)(2x-[x]-b)}{2t} \right\}$$

is achieved presents the left-most location to reach or pass x by time t. $b = [x]+1$ corresponds to the case where no particle to the left of x has passed or reached x by time t. We exclude the case $b = -\infty$ by imposing the general assumption $\psi(x,0) = o(x^2)$ which makes the quadratic eventually dominate ψ. If the infimum is attained more than once, we see that the right-most, c say, of these is the left-most to pass. All particles $b, \ldots, c-1$ just reach x at time t. We can then calculate the speed of the particle clump situated at x at time t to be

$$u(x,t) = \frac{\psi_{c-1} - \psi_{b-1}}{c-b} = \frac{2x-c-b+1}{2t} = \frac{1}{2}\frac{x-b}{t} + \frac{1}{2}\frac{x-(c-1)}{t}.$$

This determines $u(x,t)$ at all essential locations, i.e. almost everywhere with respect to the mass distribution at time t. We use the standard notation $a(x,t)$ for the right-most value $c \in \mathbb{Z}$ attaining the infimum. As is clear from the infimum expression as a function of x, multiple attaining of the infimum at one location x always implies uniqueness at $x+$ and $x-$. The values are the right-most and left-most attaining at time x, respectively. Therefore $a(\cdot, t)$ is right-continuous. $a(x-,t), \ldots, a(x,t)-1$ are the initial locations whose particles are situated in x at time t.

2.2 Evolution of a continuous particle system with inelastic collisions

Now consider the continuous analogue of the discrete system - let the initial mass of our particle system be uniformly spread on $\mathbb{R}$, i.e. according to Lebesgue measure. Assigning an initial velocity to every particle now means the specification of a function $u(\cdot,0)$, an initial potential is then simply any integral function $\psi(\cdot,0)$ of $u(\cdot,0)$. Collisions are again to take place completely inelastically.

For convenience, let again the initial potential be increasing and satisfy $\psi(x,0) = o(x^2)$. W.l.o.g. we assume $\psi(0,0) = 0$. The analogous considerations to the above, basically replacing sums by integrals, lead to the definition of $a(x,t)$ as the right-most location attaining the infimum in

$$\inf_{a\le x}\left\{\psi(a,0)+\frac{(x-a)^2}{2t}\right\} \tag{1}$$

$x \mapsto a(x,t)$ is the *inverse Lagrangian* (at time t). As it is increasing, we define its right-continuous inverse

$$x(a,t) := \inf\{x\in\mathbb{R} : a(x,t) > a\}, \qquad a\in\mathbb{R},$$

which is called the *Lagrangian function* and which gives the position at time t of the particle initially started in a. In particular, the jump locations x of $a(\cdot,t)$ are the locations of particle clumps at time t, formed by the particles initially in $[a(x-,t),a(x,t))$. Consequently, $S_L(a) := [a(x(a,t)-,t),a(x(a,t),t))$ represents the so-called *Lagrangian shock interval* containing the particle initially located at $a\in\mathbb{R}$. The velocity of the clump located at $x = x(a,t)\in\mathbb{R}$ at time t is calculated to be

$$u(x,t) = \frac{\psi(a(x,t),0)-\psi(a(x-,t),0)}{a(x,t)-a(x-,t)} = \frac{1}{2}\frac{x-a(x-,t)}{t} + \frac{1}{2}\frac{x-a(x,t)}{t}.$$

However, in general these *Eulerian shock points* are not the only locations for which a velocity has to be assigned. As the mass was initially continuously distributed, there may well remain continuously spread parts, so-called *Eulerian regular points* (their initial positions are called *Lagrangian regular points*). However, it is clear that these cannot have participated in any shocks and have hence kept their initial speed. Specifically, denoting the union of the (closures of the) Lagrangian shock intervals by $\mathcal{S}_t$, for all $a\notin\mathcal{S}_t$ the velocity does not change up to time t. This yields for these locations

$$x(a,t) = a + tu(a,0) =: x,\ a(x,t) = a,\ u(x,t) = u(a,0) = u(a(x,t),0) = \frac{x-a(x,t)}{t}.$$

Now, this latter formula could be performed as a definition for all $x\in\mathbb{R}$ and amended (affecting only the jump locations) by averaging

$$\tilde{u}(x,t) := \frac{x-a(x,t)}{t}, \qquad u(x,t) = \frac{1}{2}\tilde{u}(x-,t) + \frac{1}{2}\tilde{u}(x,t) \tag{2}$$

to sum up the above considerations into a unifying form. We refer to u defined in this way as what it is, the *velocity field of the particle model*. Note that we also assign non-zero values where there is no particle mass. We refer to these locations as rarefaction intervals and denote by R_x the (possibly empty) rarefaction interval including $x \in \mathbb{R}$.

This model is known as the *zero viscosity Burgers turbulence model*, an equivalence that can be made precise in various ways (cf. Hopf [13], Cole [8], Lax [14], E et al. [9]) - for essentially arbitrary initial velocities. The so-called *Hopf-Cole limit solution to the inviscid Burgers equation* is usually chosen to be right-continuous and coincides with $\tilde{u}$ rather than u. We refer to the definitions of $a(x,t)$ and $\tilde{u}(x,t)$ via (1) and (2) respectively as the *Hopf-Cole formulas.*

This argument can be generalised to include non-monotonic initial potentials. Furthermore, we can consider initial data $\psi(\cdot,0)$ to which we can associate initial velocities $u(\cdot,0) = \psi_x(\cdot,0)$ in the distributional sense. These are highly irregular generalised functions. However, from our representation of the solution, it is clear that $x \mapsto u(x,t)$ for every $t > 0$ is a well-defined function in the ordinary sense, i.e. singularities at time $t = 0$ disappear immediately.

Under the condition of an increasing initial potential, a representation of the event that the zero particle stands still at time T is immediate from the above formulae. We can write it as $\{u(0,T) = 0\} = \{x(0,T) = 0\} = \{a(0,T) = 0\}$ and hence by the condition

$$-\psi(a,0) \le \frac{a^2}{2T} \qquad \textit{for all } a \le 0. \tag{3}$$

Also, the length of the Lagrangian shock interval $S_L(0)$ can be expressed. Namely, $|S_L(0)|$ exceeds s if and only if there exists a minorising parabola

$$q(a) = \eta - \frac{1}{2}(\xi - a)^2, \qquad \xi, \eta \in \mathbb{R},$$

and touch locations $a_1 \le 0 \le a_2$ satisfying $a_2 - a_1 > s$ such that

$$q \le \psi(\cdot,0), \quad q(a_1) = \psi(a_1,0) \quad \textit{and} \quad q(a_2) = \psi(a_2,0). \tag{4}$$

Similarly, the event that the particle started in $-D$ has travelled distance D by time 1, is $\{x(-D,1) \ge 0\} = \{a(0-,1) \le -D\}$ which can be expressed by

$$\psi(a,0) + \frac{1}{2}a^2 \le \inf_{-D \le x \le 0} \left\{ \psi(x,0) + \frac{1}{2}x^2 \right\} \quad \textit{for some } a \le -D$$

In this work we shall meet stationarity conditions that allow translating the events; and particle clumps will a.s. not occur at fixed positions which then yields $\{x(-D,1) \ge 0\} = \{a(0,1) \le -D\} = \{u(0,1) \ge D\}$ a.s. Finally, we give a domination for the distance event that shall be useful in the sequel

$$\{x(-D,1) \ge 0\} \subset \{\textit{there is } a \le -D \textit{ such that } \psi(a,0) \le -a^2/2\}. \tag{5}$$

2.3 Regenerative sets as initial impulse

We recall first the notion of stationary regenerative sets in $\mathbb{R}$. As standard reference we mention Fristedt [11] who also gives credits for early treatments of regenerative sets under different names.

The prototype of such a random set is the zero-set of a stationary Markov processes X:

$$\mathcal{R}_0 = \{y \in \mathbb{R} : X_y = 0\}.$$

However, the connection of regenerative sets and Markov processes is not intrinsic as very different Markov processes may have the same zero-set. It is therefore natural to look for a description of regenerative sets that does not refer to an underlying Markov process.

Essentially, a *stationary regenerative set* is a random closed subset of the real line $\mathcal{R} \subset \mathbb{R}$ that has two properties. First, we assume $\mathcal{R}$ to be *stationary*, i.e. the distribution of $\mathcal{R} - s$ does not depend on $s \in \mathbb{R}$. Second, if we denote by g_0 the first point of $\mathcal{R}$ to the left of the origin, we require $(\mathcal{R} - g_0) \cap (-\infty, 0]$ and $(\mathcal{R} - g_0) \cap [0, \infty)$ be independent; this is what we call the *regeneration property*. We refer to g_0 as the first nonpositive regeneration point. We also denote by d_0 the first nonnegative point of $\mathcal{R}$. Then the analogous regeneration property is true for d_0.

In the Markov process context the regeneration property is simply the strong Markov property of the reversed process $\hat{X}_y = X_{-y}$ at its first passage time of zero. Like the Markov property, the regeneration property is in fact applicable at much more general random times. Namely, if we introduce the forward and backward natural filtrations $\mathcal{F}^{\rightarrow}$ and $\mathcal{F}^{\leftarrow}$ of $\mathcal{R}$ (completed in the usual way to include all sets of zero probability) by

$$\mathcal{F}^{\rightarrow}_y = \sigma(\mathcal{R} \cap (-\infty, y]), \qquad \mathcal{F}^{\leftarrow}_y = \sigma(\mathcal{R} \cap [y, \infty)), \qquad y \in \mathbb{R},$$

the regeneration property is valid for any stopping time τ (w.r.t. $\mathcal{F}^{\rightarrow}$ or $\mathcal{F}^{\leftarrow}$) that satisfies $\tau \in \mathcal{R} \cup \{\pm\infty\}$ a.s., i.e. conditionally under $\{-\infty < \tau < \infty\}$, $(\mathcal{R} - \tau) \cap (-\infty, 0]$ and $(\mathcal{R} - \tau) \cap [0, \infty)$ are independent.

Regenerative sets can also be characterised by a *thickness parameter* $\theta \in [0, 1]$ and a *gap measure* ν on $(0, \infty)$ satisfying

$$\mu := \theta + \int_{(0,\infty)} x\nu(dx) < \infty \quad \textit{and} \quad \theta + \int_{(0,\infty]} (1 \wedge x)\nu(dx) = 1.$$

The distribution of $\mathcal{R}$ is specified as follows: the two random variables $g_0 \le 0 \le d_0$ have joint distribution

$$P(d_0 - g_0 \in dz, g_0 \in dy) = \frac{\theta}{\mu}\delta_{(0,0)}(dz \times dy) + \frac{1}{\mu}1_{\{z \ge -y \ge 0\}} dy\nu(dz) \qquad (6)$$

Furthermore, independently, $(\mathcal{R} - d_0) \cap [0, \infty)$ and $(g_0 - \mathcal{R}) \cap [0, \infty)$ are the ranges of two independent subordinators (i.e. increasing processes with stationary independent increments) $\sigma^{\pm}$ with *drift coefficient* θ and *Lévy measure* ν, i.e. for $t \ge s \ge 0$

$$E(exp\{-q(\sigma^{\pm}_t - \sigma^{\pm}_s)\}) = \exp\{-(t - s)\Phi(q)\}$$

where

$$\Phi(q) = \theta q + \int_{(0,\infty)} (1 - e^{-qx})\nu(dx), \quad q \geq 0,$$

is called the *Laplace exponent* of $\sigma^{\pm}$. Note that $E(\sigma_1^{\pm}) = \mu$.

In particular, integration yields from (6)

$$P(|g_0| > x) = (1/\mu) \int_x^{\infty} \bar{\nu}(y)dy, \quad x > 0, \tag{7}$$

where $\bar{\nu}(y) = \nu([y,\infty))$. We focus on g_0 rather than d_0 since the left hand half of $\mathcal{R}$ will be the important one in our applications.

Example 1 If the gap measure ν is finite, σ is a compound Poisson process with a drift θs added. In this case we can write $\nu = cF$ for a constant $c \geq 0$ and a probability measure F. $\mathcal{R} \cap (-\infty, 0]$ has the following law: there is a family of independent random variables $X_{2n-1} \sim Exp(\theta)$, $X_{2n} \sim F$, $n \geq 1$. Define $S_n = |g_0| + X_1 + \ldots + X_n$. Then we have $s \in \mathcal{R} \cap (-\infty, 0]$ if and only if $S_{2n} \leq -s \leq S_{2n+1}$ for some $n \geq 0$. Note that $\theta = 0$ corresponds to $S_{2n} = S_{2n+1}$, i.e. $\mathcal{R}$ is a.s. a collection of isolated points whereas $\theta > 0$ leads to a collection of intervals of an exponentially distributed length.

As already apparent in the special setting of the example, the parameter θ is an important indicator of the size of the regenerative set. In fact, it is true in full generality that $\theta > 0$ corresponds to a *heavy* regenerative set, entailing $\lambda\!\!\lambda(\mathcal{R}) > 0$ a.s. where $\lambda\!\!\lambda$ denotes Lebesgue measure, whereas $\theta = 0$ corresponds to $\lambda\!\!\lambda(\mathcal{R}) = 0$ a.s., called a *light* regenerative set.

In the heavy case an initial velocity $u = k1_{\mathcal{R}}$, $k > 0$ constant, to the particle system yields an increasing initial potential ψ which can be identified with the continuous inverse function of $\sigma_{\cdot/k}$ where $\sigma_s = d_0 + \sigma_s^+$, $\sigma_{-s} = g_0 - \sigma_s^-$ is the so-called associated *two-sided subordinator* whose range is $\mathcal{R}$. In the light case, $u = k1_{\mathcal{R}}$ yields a constant potential, i.e. there is no energy in the system. The light case can be treated differently, as a limit case of the heavy case, and can still be represented by the inverse of the associated two-sided subordinator as an initial potential. This is a function that increases precisely on $\mathcal{R}$ but it is not differentiable on $\mathcal{R}$. Its derivative can be defined in a distributional sense, corresponding to a measure concentrated on $\mathcal{R}$ that (for infinite ν) can be identified as some Hausdorff measure restricted to $\mathcal{R}$. As such, the initial potential ψ is the integral of $1_{\mathcal{R}}$ w.r.t. this Hausdorff measure. Cf. Fristedt and Pruitt [11]. Loosely speaking, an infinite velocity is assigned on $\mathcal{R}$.

When entering the particle system, the case of a finite ν is of minor importance. In order not to exclude it categorically, we stick to the definition of an initial potential as the inverse of the associated two-sided subordinator. In the light case, this corresponds to independent exponential jumps of the potential which is again a stationary behaviour but when integrating $1_{\mathcal{R}}$ w.r.t. the counting measure (which is the adequate Hausdorff measure) these weights have to be taken into account.

Since the range of a process is invariant under time changes, any subordinator having deterministic drift $c = \theta/k$ and Lévy measure $\Pi = \nu/k$ for some intensity $k \in (0, \infty)$ has the same range as with intensity $k = 1$. Given a subordinator σ with unit intensity and $\psi(x, 0) = \sup\{t \in \mathbb{R} : \sigma_t \leq x\}$, we can pass to a subordinator with intensity k, $t \mapsto \sigma_{t/k}$, which corresponds to the potential $x \mapsto k\psi(x, 0)$. The mean of the time-changed subordinator is now $m = \mu/k$. When entering the particle system in the way described above, a high intensity corresponds to higher initial velocities on the same regenerative set. We refer to k as the *intensity parameter* of a *regenerative impulse*. Note that the distribution of the regenerative impulse is completely determined by Π (or its tail $\bar{\Pi}(t) := \Pi(t, \infty)$, $t > 0$) and m. For technical reasons we shall often use these as a parametrization. In fact, k will not influence our results.

Remark 1 *In the setting of a stationary Markov process (call the stationary distribution μ), for suitable approximations of the Dirac distribution*

$$f_n \mu \to \delta_0 \quad \Rightarrow \quad \psi_n(x, 0) := \int_0^x f_n(X_y) dy \to \psi(x, 0)$$

is well-known and means here that initial velocities $f_n(X_y)$, $y \in \mathbb{R}$, in the limit lead to the local time $\psi(x, 0)$ of X at zero as initial potential.

Assume now a stationary regenerative impulse in the sense indicated above. Due to the stationarity assumption, the position zero behaves like any other location, therefore the event $\{x(a, T) = a\}$ is stationary in a, i.e. behaves stochastically the same for all $a \in \mathbb{R}$. Picking up the representation (3) of the last subsection we simply invert ψ to obtain

$$\begin{aligned} \{\textit{particle in } 0 \textit{ not moving at time } T\} &= \{x(0, T) = 0\} \\ &= \{\sigma_z^- \geq \sqrt{2zT} - |g_0| \textit{ for all } z \geq 0\}. \end{aligned}$$

Also the representation (4) of the length of the Lagrangian shock interval containing zero can be reformulated by inverting ψ. Inverted parabolas take the form

$$f_{\xi,\eta}(z) = \xi - \sqrt{2(\eta - z)}, \qquad z \leq \eta, \qquad \xi, \eta \in \mathbb{R}.$$

For the event we obtain

$$\begin{aligned} \{|S_L(0)| > s\} = \{&\textit{there are } \xi, \eta \in \mathbb{R}, z_1 \leq 0 \leq z_2 \textit{ such that } f_{\xi,\eta} \geq \sigma, \\ &\sigma_{z_1} = f_{\xi,\eta}(z_1), \sigma_{z_2} = f_{\xi,\eta}(z_2), f_{\xi,\eta}(z_2) - f_{\xi,\eta}(z_1) > s\}. \end{aligned}$$

The domination (5) of the distance event translates as well. We can furthermore continue in estimating $g_0 \leq 0$ which yields

$$\begin{aligned} \{x(-D, 1) > 0\} &\subset \{\textit{there is } z < -D^2/2 \textit{ such that } \sigma_z > -\sqrt{2|z|}\} \\ &\subset \{\textit{there is } y > D^2/2 \textit{ such that } \sigma_y^- < \sqrt{2y}\}. \end{aligned}$$

3 The event of getting far away from an initial position

3.1 Formulation and discussion of Theorem 1

Let us consider the event that a typical particle is found at distance greater than D from its initial position within unit time. Closely related, in fact, is the event that the particle clump containing a particular particle exceeds a given strength s. Both events are considered for large values, i.e. when D and s, respectively, tend to infinity. We shall first define the type of decay we encounter:

Definition 1 For $\rho \in (0,\infty)$ we call a function $f:(0,\infty) \to (0,\infty)$ ρ-exponentially decreasing if

$$\liminf_{t\to\infty} -\frac{1}{t^r}\ln(f(t)) = \infty \; for\ all\ r<\rho \quad and \quad \limsup_{t\to\infty} -\frac{1}{t^r}\ln(f(t)) = 0 \; for\ all\ r>\rho.$$

Denoting $I(t) = \int_0^t \bar{\Pi}(x)dx = (1/k)\int_0^t \bar{\nu}(x)dx$, we have

Theorem 1 *Assume a Burgers turbulence model initialized by a stationary regenerative impulse satisfying*

$$\theta = 0 \quad and \quad \lim_{t\downarrow 0} \frac{\ln(I(t))}{\ln(t)} = 1-\alpha \in (0,1]. \tag{8}$$

Then the tails of both the distance law and the distribution of the shock interval length

$$D \mapsto P(x(0,1)>D) \quad and \quad s \mapsto P(|S_L(0)|>s)$$

are $(2-\alpha)/(1-\alpha)$*-exponentially decreasing.*

Remark 2 **1.** Important subordinators like the Gamma subordinator ($\alpha=0$) and the inverse Gaussian subordinator ($\alpha=1/2$) are among the class of subordinators for that the theorem applies. The same is true for the entire class of subordinators introduced by Vershik and Yor, as treated in Appendix A. Within it all parameters $\alpha\in[0,1)$ are represented.

2. The condition (8) can be rewritten in several ways, cf. Proposition III.1 in Bertoin [3]. We mention $\ln(\Phi(\lambda))/\ln(\lambda) \to \alpha \in [0,1)$ as λ tends to infinity, where Φ is the Laplace exponent of the associated subordinator as introduced in subsection 2.3. This is the condition we shall exploit in the sequel. We point out, that due to the smoothing effect of integration, the simpler condition $\ln(\bar{\Pi}(t))/\ln(t) \to -\alpha \in (-1,0]$ as $t\downarrow 0$ is stronger, but indeed unnecessary for our arguments.

3. Due to stationarity each fixed location behaves stochastically the same; that is why we focus on the origin. The shifted events are $\{x(a,1)-a>D\}$, $a\in\mathbb{R}$.

4. The distance events are closely related to the tail of the velocity $u(0,1)$. In fact we have $\{x(-D,1)>0\} = \{a(0,1)<-D\} = \{u(0,1)>D\}$ a.s. by the Hopf-Cole formulas. However, the equality fails where there are particle clumps, due to the averaging adjustment of $u(0,1)$. In particular, $u(0,1)$ is not the typical velocity of a particle as it is merely an instant in a velocity field that, when considering a regenerative impulse, interacts with discretely spread particle clumps, non-moving infinitesimal particles and empty areas; concerning a typical velocity tail, the latter two are not interesting and particle clumps correspond precisely to the locations where the adjustment has to be performed.

5. If the underlying regenerative set is heavy, all velocities are bounded. This is stronger than the limiting case $\alpha = 1$ of Theorem 1 which states a decay faster than ρ-exponential for all $\rho > 0$. Also the particle clump size is bounded at any fixed time.

6. A combination of the distance and the shock interval size parts of Theorem 1 yields a uniformity result to the distance part, extending the result concerning a fixed position, $a = 0$ say, to the left end point of the Lagrangian shock interval $S_L(0) = (l_0, r_0)$ surrounding 0:

Corollary 1 *In the situation of Theorem 1*

$$D \mapsto P(x(l_0+, 1) - l_0 > D)$$

is $(2-\alpha)/(1-\alpha)$*-exponentially decreasing.*

Proof: In order to conclude from Theorem 1 just note that

$$\begin{aligned} P(x(0,1) > D) &\leq P(\sup_{a \in (l_0, r_0)} x(a,1) - a > D) = P(x(l_0+,1) - l_0 > D) \\ &\leq P(l_0 < -D/2) + P(x(l_0+,1) > D/2) \\ &\leq P(|S_L(0)| > D/2) + P(x(0,1) > D/2). \end{aligned}$$

□

The proof of Theorem 1 is given in the following subsections.

3.2 Auxiliary results

This subsection contains two quite general results on (one-sided) subordinators concerning the behaviour of the Laplace exponent Φ and its derivatives and the probabilities $P(c_1\sqrt{t} \leq \sigma_t \leq c_2\sqrt{t})$ respectively.

We start by identifying the asymptotics of related analytic quantities when condition (8) holds.

Lemma 1 *Given a subordinator satisfying condition (8), we have for* $\varphi = \Phi'$

$$\ln(\varphi(\lambda)) \sim (\alpha - 1)\ln(\lambda), \qquad \ln(-\varphi'(\lambda)) \sim (\alpha - 2)\ln(\lambda)$$

as λ *tends to infinity. Furthermore*

$$\ln(\varphi^{-1}(\gamma)) \sim \frac{1}{\alpha - 1}\ln(\gamma)$$

as γ *tends to zero.*

Proof: Let $\varepsilon > 0$. From Remark 2 2. and the wellknown concavity of Φ we conclude

$$\frac{\Phi(\lambda^{1+\varepsilon}) - \Phi(\lambda)}{\lambda^{1+\varepsilon} - \lambda} \leq \varphi(\lambda) \leq \frac{\Phi(\lambda)}{\lambda}$$

which yields

$$\frac{\ln(\Phi(\lambda^{1+\varepsilon}) - \Phi(\lambda))}{\frac{1}{1+\varepsilon}\ln(\lambda^{1+\varepsilon})} - \frac{\ln(\lambda^{1+\varepsilon} - \lambda)}{\ln(\lambda)} \leq \frac{\ln(\varphi(\lambda))}{\ln(\lambda)} \leq \frac{\ln(\Phi(\lambda))}{\ln(\lambda)} - 1$$

which, in the limit, restricts $\ln(\varphi(\lambda))/\ln(\lambda)$ to something in $[\alpha(1+\varepsilon) - (1+\varepsilon), \alpha - 1]$ establishing the first claim.

Second, along the same lines, the convexity of φ entails

$$\frac{\varphi(\lambda) - \varphi(\lambda^{1+\varepsilon})}{\lambda^{1+\varepsilon} - \lambda} \leq -\varphi'(\lambda) \leq \frac{\varphi(\lambda^{1-\varepsilon}) - \varphi(\lambda)}{\lambda - \lambda^{1-\varepsilon}},$$

and

$$\frac{\ln(\varphi(\lambda) - \varphi(\lambda^{1+\varepsilon}))}{\ln(\lambda)} - \frac{\ln(\lambda^{1+\varepsilon} - \lambda)}{\ln(\lambda)} \leq \frac{\ln(-\varphi'(\lambda))}{\ln(\lambda)} \leq \frac{\ln(\varphi(\lambda^{1-\varepsilon}) - \varphi(\lambda))}{\frac{1}{1-\varepsilon}\ln(\lambda^{1-\varepsilon})} - \frac{\ln(\lambda - \lambda^{1-\varepsilon})}{\ln(\lambda)}$$

yields an asymptotic behaviour within $[\alpha - 1 - (1+\varepsilon), (1-\varepsilon)(\alpha-1) - 1]$.

Finally, as φ strictly decreases to zero, the substitution $\gamma = \varphi(\lambda)$ yields eventually

$$\frac{1}{\alpha - 1 + \varepsilon} \leq \frac{\ln(\varphi^{-1}(\gamma))}{\ln(\gamma)} \leq \frac{1}{\alpha - 1 - \varepsilon}.$$

□

We now use this analytic lemma for a large deviation result using essentially standard techniques. Here we meet probabilities that decrease exponentially in the way that has been postulated in the theorem.

Lemma 2 *Let σ be a subordinator satisfying (8). Then for all $0 \leq c_1 < c_2 \leq \infty$, $\varepsilon > 0$ there is a $k_0 > 0$ such that for all t large enough*

$$P(c_1\sqrt{t} \leq \sigma_t \leq c_2\sqrt{t}) \geq \exp\left\{-k_0 t^{(2-\alpha)/(2-2\alpha)+\varepsilon}\right\}$$

Proof: W.l.o.g. we can assume $c_2 < \infty$. We change the probability measure. Define $Q_t = \exp\{-\lambda_t\sigma_t + t\Phi(\lambda_t)\}P$ where λ_t is chosen such that $E_{Q_t}(\sigma_t) = t\varphi(\lambda_t) = (c_1+c_2)\sqrt{t}/2$. Thanks to the preceding lemma

$$\ln(\lambda_t) \sim \frac{1}{\alpha - 1}\ln((c_1+c_2)/(2\sqrt{t})) \quad \Rightarrow \quad t^{1/(2-2\alpha)-\varepsilon} \leq \lambda_t \leq t^{1/(2-2\alpha)+\varepsilon}$$

for all $t \geq t_0$. Then we have

$$\begin{aligned} P\left(c_1\sqrt{t} \leq \sigma_t \leq c_2\sqrt{t}\right) &= E_{Q_t}\left(e^{\lambda_t\sigma_t - t\Phi(\lambda_t)} 1_{\{c_1\sqrt{t} \leq \sigma_t \leq c_2\sqrt{t}\}}\right) \\ &\geq e^{-t\Phi(\lambda_t)} Q_t\left(c_1\sqrt{t} \leq \sigma_t \leq c_2\sqrt{t}\right) \\ &\geq 2\exp\left\{-k_0 t^{1+\alpha/(2-2\alpha)+\varepsilon}\right\} Q_t(c_1\sqrt{t} \leq \sigma_t \leq c_2\sqrt{t}) \end{aligned}$$

for any $\varepsilon > 0$ sufficiently small and a suitable $k_0 > 0$. Note that for all $t \geq t_1$

$$Q_t(c_1\sqrt{t} \leq \sigma_t \leq c_2\sqrt{t}) = Q_t\left(|\sigma_t - E_{Q_t}(\sigma_t)| \leq \frac{c_2 - c_1}{2}\sqrt{t}\right) \geq 1 - \frac{4Var_{Q_t}(\sigma_t)}{(c_2 - c_1)^2 t} \geq 1/2$$

since $Var_{Q_t}(\sigma_t)/t = -\varphi'(\lambda_t) \leq \lambda_t^{\alpha-2+\varepsilon} \leq t^{(\alpha-2)/(2-2\alpha)+\varepsilon(2-\alpha+1/(2-2\alpha)-\varepsilon)}$ is asymptotically negligible. □

3.3 Proof of Theorem 1

We treat the four bounds for Theorem 1 separately. In fact, only for two of them some effort is needed.

Lemma 2 is the key to the shock strength lower bound.

Proposition 1 (Lower bound - shock interval length part) *Under the condition (8), for all $\varepsilon > 0$ there is an $M \geq 0$ such that for all $s \geq M$*

$$P(|S_L(0)| > s) \geq \exp\left\{-s^{(2-\alpha)/(1-\alpha)+\varepsilon}\right\}.$$

Proof: We restrict our attention to a subevent that can be analysed more easily. The aim is to look at paths for which there is a parabola $a \mapsto \eta - \frac{1}{2}(\xi - a)^2$ which touches the initial potential $\psi(a, 0)$ once below zero and once beyond $s/2$, say. As $\psi(0,0) = 0$, it has to increase very quickly just before zero, then still quite quickly up to $s/2$, not depassing a quadratic threshold, though, after which it has to increase only little. Here, we translate the three conditions into terms of the subordinators $\sigma^{\pm}$ that constitute the inverse of ψ (cf. subsection 2.3), by exchanging coordinate axes. Of course, we also have to choose suitable coefficients that we explain when they enter calculations:

$$\begin{aligned}
\Gamma_1 &= \{-\sigma^-_{2s^2} \geq (3/2 - \sqrt{33/8})s + 1\} \\
\Gamma_2 &= \{|g_0| \leq 1, d_0 \leq 1\} \\
\Gamma_3 &= \{s/2 \leq \sigma^+_{s^2/16} \leq s - 1\} \\
\Gamma_4 &= \{\sigma^+_{s^2/12} - \sigma^+_{s^2/16} \geq (1 + \sqrt{1/12})s\}
\end{aligned}$$

are four independent events. As Γ_2 does not depend on s, $P(\Gamma_2) > 0$ and the probabilities of the other three events behave correctly by Lemma 2, it suffices that on $\Gamma := \Gamma_1 \cap \Gamma_2 \cap \Gamma_3 \cap \Gamma_4$ the particles initially in $[0, s/2)$ are part of a single particle clump at time 1.

For the precise parabola analysis, we introduce the so-called two-sided subordinator σ by $\sigma_s := d_0 + \sigma_s^+$ and $\sigma_{-s} := g_0 - \sigma_s^-$. Then ψ is the continuous inverse of σ. (Cf. subsection 2.3).

As for the constants, look first at the parabola

$$p(z) = 3s/2 - \sqrt{2(s^2/16 - z)}, \qquad z \leq s^2/16.$$

We have $p(-2s^2) = (3/2 - \sqrt{33/8})s$, hence Γ_1 (and $|g_0| \leq 1$) are chosen so as to ensure that $\sigma_{-2s^2} \geq p(-2s^2)$ which means that in order to find the parabola at height $3s/2$ that touches σ, we must move p to the left. Now note that on $\Gamma_3 \cap \{d_0 \leq 1\}$ we have

$p(z) \geq p(0) = (3/2 - \sqrt{1/8})s > s \geq \sigma_{s^2/16} \geq \sigma_z$ for all $z \in [0, s^2/16]$. So, all translations of p to the left stay above σ on the positive part of their domain. Therefore, on $\Gamma_1 \cap \Gamma_2 \cap \Gamma_3$ we have $a(3s/2, 1) < 0$, i.e. $x(0,1) > 3s/2$. On Γ_4, we have $\sigma_{s^2/12} \geq (1 + \sqrt{1/12})s$.

Let us now look for a function

$$f(z) = \xi - \sqrt{2(\eta - z)}, \qquad \xi, \eta \in \mathbb{R},$$

that remains always above the subordinator σ and hits it in one negative and one positive position (z_1 and z_2, say). As 0 is not a Lagrangian regular point (since $x(0,1) > 3s/2$), elementary geometric considerations show that ξ and η exist and are unique. (There are possibly further hits. Take z_1 and z_2 as close to zero as possible.) Furthermore, we conclude that $[f(z_1), f(z_2))$ is (part of) the shock interval around zero, ξ the position of the corresponding clump at time 1. We want to show $f(z_2) > s/2$ on Γ.

First, we noted above that we have $x(0,1) > 3s/2$, therefore we have $\xi > 3s/2$.

Second, if $\eta \leq s^2/12$ then f dominates $g(z) = 3s/2 - \sqrt{2(s^2/12 - z)}$. Now $f(0) \geq g(0) = (3/2 - \sqrt{1/6})s$ means we have no hit before σ exceeds height $(3/2 - \sqrt{1/6})s$, which does not happen before $s^2/16$ (by Γ_3).

Third, if $\eta > s^2/12$ and f hits σ between zero and $s^2/16$, then f fulfils $f(s^2/12) \geq (1 + \sqrt{1/12})s$ (to stay above σ) and $f(0) \leq s$ (to enable the hit) yields the condition $\eta \leq s^2/8$ by estimating the coefficient of differences against the slope $f'(s^2/12)$. (This justifies the upper bound in Γ_3 and the lower bound in Γ_4.) The condition $f(0) \leq s$ now yields $\xi \leq 3s/2$ which contradicts $\xi > 3s/2$.

Therefore f cannot hit σ before $s^2/16$. Now the lower bound in Γ_3 is needed to conclude $f(z_2) > s/2$. This completes the proof. □

The distance part is now an easy consequence:

Corollary 2 (Lower bound - distance part) *Under the condition (8) there is for each $\varepsilon > 0$ a $D_0 \geq 0$ such that for all $D \geq D_0$*

$$P(x(0,1) > D) \geq \exp\left\{-D^{(2-\alpha)/(1-\alpha)+\varepsilon}\right\}.$$

Proof: We conclude from the proposition by noting the obvious inclusion

$$\{s_0 > s\} \subset \{x(-s/2, 1) + s/2 > s/2\} \cup \{x(0,1) > s/2\}$$

which implies by stationarity $P(s_0 > s) \leq 2P(x(0,1) > s/2)$. □

For the upper bounds, a martingale argument applies:

Proposition 2 (Upper bound - distance part) *Let $1/\Phi(\lambda) = O(\lambda^{-\alpha})$ for some $\alpha \in (0,1)$, as λ tends to infinity. Then $D \mapsto P(x(0,1) > D)$ is at least $(2-\alpha)/(1-\alpha)$-exponentially decreasing.*

Proof: Define martingales

$$M_a := \exp\left(-D^\rho \sigma_a^- + a\Phi(D^\rho)\right), \qquad a \geq 0.$$

for a parameter $\rho \in \mathbb{R}$ to be chosen later. With

$$T_D = \inf\left\{ s > \frac{1}{2}D^2 : \sigma_s^- < \sqrt{2s} \right\}$$

the optional stopping theorem yields

$$1 \geq E\left(\exp\left(-D^\rho \sqrt{2T_D} + T_D\Phi(D^\rho)\right), T_D < \infty\right)$$

where we used that $\sigma_{T_D}^- = \sqrt{2T_D}$ on $\{T_D < \infty\}$ since σ^- is increasing and $s \mapsto \sqrt{2s}$ is continuous.

Now we wish to replace T_D by $D^2/2$. In order to be able to do so we choose D_0 such that $\Phi(D^\rho) \geq 4D^{\rho-1}$ for all $D \geq D_0$. This is possible if $\rho < 1/(1-\alpha)$. Then it is easy to see that we have for all $D \geq D_0$ a.s. on $\{T_D < \infty\}$

$$\Phi(D^\rho) \geq \frac{\sqrt{2}D^\rho}{\sqrt{T_D} - \sqrt{\frac{1}{2}D^2}} \quad \Rightarrow \quad (T_D - \frac{1}{2}D^2)\Phi(D^\rho) \geq \left(\sqrt{T_D} - \sqrt{\frac{1}{2}D^2}\right)\sqrt{2}D^\rho.$$

This yields

$$1 \geq E\left(\exp\left(-D^{\rho+1} + \frac{1}{2}D^2\Phi(D^\rho)\right), T_D < \infty\right) \geq E\left(\exp\left(D^{\rho+1}\right), T_D < \infty\right).$$

This establishes $\rho + 1$ as an upper exponent for $P(T_D < \infty)$.

The proof is finished as $\{T_D < \infty\} = \{\textit{there is } s > D^2/2 \textit{ such that } \sigma_s^- < \sqrt{2s}\}$ represents the event in question (cf. subsection 2.3). □

The shock interval length upper bound follows from the distance lower bound in the same way as the distance lower bound was concluded from the shock interval length lower bound.

Corollary 3 (Upper bound - shock interval length part) *Let $1/\Phi(\lambda) = O(\lambda^{-\alpha})$ for some $\alpha \in (0,1)$, as λ tends to infinity. Then $D \mapsto P(x(0,1) > D)$ is at least $(2-\alpha)/(1-\alpha)$-exponentially decreasing.*

4 The event of not being involved into the shocks

4.1 Formulation and discussion of Theorem 2

For the sequel it is worth fixing some terminology.

Definition 2 A function $f : (0, \infty) \to (0, \infty)$ is called *exponentially decreasing* if

$$0 < \mu_\ell := \liminf_{t\to\infty} -\frac{1}{t}\ln(f(t)) \le \limsup_{t\to\infty} -\frac{1}{t}\ln(f(t)) =: \mu_u < \infty.$$

Analogously, it is called *polynomially decreasing* if

$$0 < \alpha_\ell := \liminf_{t\to\infty} -\frac{1}{\ln(t)}\ln(f(t)) \le \limsup_{t\to\infty} -\frac{1}{\ln(t)}\ln(f(t)) =: \alpha_u < \infty.$$

μ_ℓ (or α_ℓ) and μ_u (or α_u) are called *(lower* and *upper) exponents.*

Note that the exponential decay defined here is stronger than the 1-exponential decay of Definition 1 as the latter does not necessarily imply $\mu_\ell > 0$ nor $\mu_u < \infty$. By allowing $\alpha_\ell < \alpha_u$ ($\mu_\ell < \mu_u$ respectively) our definition includes cases of an 'oscillation' between different exponents.

Using this terminology, we can formulate our main result. Recall that $\{x(0, T) = 0\}$ denotes the event that at time T the particle started in 0 is still in 0, i.e. it has not yet been hit. Furthermore, we refer to the preliminaries concerning the basics and our notation of Burgers turbulence and regenerative impulse.

Theorem 2 *Assume a Burgers turbulence model initialized by a stationary regenerative impulse having thickness parameter θ, gap measure ν (and intensity parameter $k > 0$).*

a) If $\bar{\nu}$ is polynomially decreasing *with exponents $1 < \alpha_\ell \le \alpha_u < \infty$ then*

$$T \mapsto P(x(0, T) = 0)$$

is polynomially decreasing with lower exponent greater than or equal to $\alpha_\ell - 1$ and upper exponent less than or equal to $\alpha_u - 1$.

b) If $\bar{\nu}$ is exponentially decreasing *with exponents $0 < \mu_\ell \le \mu_u < \infty$, then $T \mapsto P(x(0, T) = 0)$ is also exponentially decreasing as $T \to \infty$, and its upper exponent is less than or equal to $\mu_u/2m$.*

The proof is given in the following subsections.

Remark 3 **1.** We mention here briefly that the exponential case includes regenerative sets that correspond to important subordinators like the Gamma subordinator and the inverse Gaussian subordinator. We refer to Appendix A for further illustrations and their embedding in a larger class of subordinators first considered by Vershik and Yor [22].

2. In the exponential case no explicit lower exponent has been given. It is however part of the statement that there exists a positive lower exponent. Our argument does in fact provide a lower exponent of the following form: denote $\gamma_1 = \varphi(-\mu_\ell+)$,

$\beta(\gamma) = \Phi(\varphi^{-1}(\gamma)) - \gamma\varphi^{-1}(\gamma)$, where $\Phi(\lambda) = -\ln E(e^{-\lambda\sigma_1})$ is the Laplace exponent of the subordinator σ associated to the model, and $\varphi = \Phi'$. Then the lower exponent is greater than or equal to $\beta(2m - r_0)/2m^2$. Here r_0 is the unique positive location in $(2m - \gamma_1, m)$ such that $\beta(2m - r_0) = \mu_\ell r_0$, if it exists, $r_0 = 2m - \gamma_1$ otherwise.

3. The transfer of exponents shows that in the polynomial case the asymptotic behaviour of $P(x(0,\cdot) = 0)$ is only dependent on the asymptotic behaviour of $\bar{\nu}$. In the exponential case the mean drift m enters our exponents of $P(x(0,\cdot) = 0)$. However, this means that, for a given asymptotic behaviour of $\bar{\nu}$, we can make the upper exponent of $P(x(0,\cdot) = 0)$ arbitrarily small by just adding one sufficiently heavy atom at a sufficiently large location to ν which does clearly not affect the asymptotics of $\bar{\nu}$. In other terms, as m depends on the whole of ν, k and θ, so do the asymptotics of $P(x(0,T) = 0)$ in the exponential case.

4. In the special polynomial case $\alpha_\ell = \alpha_u =: \alpha$, we have the existence of the following limits

$$\alpha = \lim_{t\to\infty} -\frac{1}{\ln(t)}\ln(\bar{\nu}(t)) \quad \Rightarrow \quad \lim_{T\to\infty} -\frac{1}{\ln(T)}\ln(P(x(0,T) = 0)) = \alpha - 1.$$

This does not follow in the exponential case because we have not been able to provide sharp enough bounds. However, the simple structure of the event in question leads us to conjecturing the analogous statement for the exponential case:

Conjecture 1 *If $\mu := \lim_{t\to\infty} -\frac{1}{t}\ln(\bar{\nu}(t))$ exists then $-\frac{1}{T}\ln(P(x(0,T) = 0))$ converges as well as T tends to infinity.*

An approach trying to establish subadditivity does not work in a reasonably straightforward way.

4.2 Auxiliary results

We shall here present four lemmas which are central in establishing the lower and upper bounds claimed in the theorem. They all concern one-sided subordinators.

The first elementary lemma transfers the asymptotics of the Lévy tail to the integrated tail.

Lemma 3 *Let Π be the Lévy measure of a subordinator. We introduce its integrated tail $\bar{I}(T) = \int_T^\infty \bar{\Pi}(t)dt$.*

a) If $\bar{\Pi}$ is exponentially decreasing with exponents $0 < \mu_\ell \leq \mu_u < \infty$, then so is $\bar{I}$ with exponents at least μ_ℓ and at most μ_u.

b) If $\bar{\Pi}$ is polynomially decreasing with exponents $1 < \alpha_\ell \leq \alpha_u < \infty$ then so is $\bar{I}$ with exponents at least $\alpha_\ell - 1$ and at most $\alpha_u - 1$.

Proof: a) By assumption, there is for all $0 < \mu_1 < \mu_\ell$, $\mu_u < \mu_2 < \infty$ a $t_0 \geq 0$ such that for all $t \geq t_0$

$$e^{-\mu_2 t} \leq \bar{\Pi}(t) \leq e^{-\mu_1 t}$$

which can be integrated from $T \geq t_0$ to ∞ to yield

$$\frac{1}{\mu_2} e^{-\mu_2 T} \leq \bar{I}(T) \leq \frac{1}{\mu_1} e^{-\mu_1 T}$$

establishing the assertion.

b) The same argument also works here. Just, integrating polynomials changes the exponent by one. □

In the second lemma we relate the behaviour of the Lévy tail to the behaviour of the distributional tail at time one.

Lemma 4 *Let σ be a subordinator with drift coefficient c and a polynomially decreasing Lévy tail $\bar{\Pi}$. Then there exists a constant $h > 0$ and for all $\delta < 1$ an $H_\delta < \infty$ such that for all $s \geq 1$*

$$h\bar{\Pi}(s) \leq P(\sigma_1 > s) \leq H_\delta \bar{\Pi}(s^\delta)$$

In particular the decay of $\bar{\Pi}$ and $P(\sigma_1 > \cdot)$ admits the same upper and lower exponents.

Proof: Assume first $c = 0$. The lower bound for $P(\sigma_1 > s)$ is straightforward, namely

$$P(\sigma_1 > s) \geq 1 - e^{-\bar{\Pi}(s)} \geq h\bar{\Pi}(s)$$

first estimating by the Poisson probability that there has occurred at least one jump of size at least s then choosing $h = (1 - \exp(-\bar{\Pi}(1)))/\bar{\Pi}(1)$.

Now, for the upper bound we first get rid of the small jumps by splitting $\sigma = \rho + \tau$ where ρ consists of all jumps having less than unit size, τ of the rest. An elementary calculation shows that

$$P(\sigma_1 > s) \leq P\left(\rho_1 > \frac{s}{2}\right) + P\left(\tau_1 > \frac{s}{2}\right).$$

To proceed we look at the two right hand probabilities separately. We consider the martingales $Z_t = \exp(\lambda\rho_t - t\Lambda(-\lambda))$ where Λ denotes the Laplace exponent of ρ which is defined on the whole real line since the Lévy measure of ρ has compact support. Clearly for all $\lambda > 0$

$$1 \geq E(Z_1 1_{\{\rho_1 > s\}}) \geq \exp\{\lambda s - \Lambda(-\lambda)\} P(\rho_1 > s) \;\Rightarrow\; P(\rho_1 > s) \leq \exp\{-\lambda s + \Lambda(-\lambda)\}$$

which shows an exponential decay for $P(\rho_1 > s)$, in fact faster than any $e^{-\lambda s}$.

τ is a compound Poisson process with rate $r = \bar{\Pi}(1)$ and jump law $\nu = \frac{1}{r}\Pi(\cdot \cap [1,\infty))$; define its nth convolution power $\nu_n = \nu^{(n)}$. It is immediate that $\bar{\nu_n}(s) \le n\bar{\nu}(\frac{s}{n})$. Now for all $N = N(s) \in \mathbb{N}$

$$\begin{aligned} P(\tau_1 > s) &= \sum_{n=1}^{\infty} \frac{r^n}{n!} e^{-r} \bar{\nu_n}(s) \\ &\le e^{-r} \sum_{n=1}^{\infty} \frac{r^{n-1}}{(n-1)!} r\bar{\nu}\left(\frac{s}{n}\right) \\ &\le \bar{\Pi}\left(\frac{s}{N}\right) + re^{-r} \sum_{n=N}^{\infty} \frac{r^n}{n!} \end{aligned}$$

yields two terms to estimate. The first one favours small values of N in order to yield good estimates, the second one large ones, in particular depending on s. A successful compromise is $N = [s^\varepsilon \vee 2r]$ for any small $\varepsilon > 0$ where $[\cdot]$ denotes the integer part. This choice does arbirarily little harm to the first term, yielding the aimed argument s^δ for $\varepsilon = 1 - \delta$. We can estimate the second term - being the tail of an exponential series - by standard methods

$$\sum_{n=N}^{\infty} \frac{r^n}{n!} \le \frac{2r^N}{N!} \le \frac{2}{\sqrt{2\pi N}} \left(\frac{re}{N}\right)^N \le Ke^{-N}$$

(K sufficiently large) establishing a decay faster than polynomial (in s).

Now we can dominate the nonpolynomial terms eventually by the polynomial term; adding a multiplicative constant the domination holds everywhere as all terms are càdlàg in s.

The case $c > 0$ is easily reduced to the preceding case by noting the trivial identity $P(\sigma_1 > s) = P(\sigma_1 - c > s - c)$ where the latter describes the shifted tail of a subordinator without drift component; clearly the shifting does not influence the asymptotic behaviour. □

Note that the lower bound holds for arbitrary Lévy tails; also, h can be chosen uniformly away from zero.

In the special case of $\bar{\Pi}$ regularly varying, the preceding lemma is a corollary of Theorem 8.2.1 in Bingham [6].

The third and fourth lemma provide standard large deviation estimates of the Cramér type; they can hence be seen as rate results for the law of large numbers for subordinators. We treat first the polynomially decreasing Lévy tail, then the exponential case.

Lemma 5 *Let σ be a subordinator with Lévy tail $\bar{\Pi}$ at least polynomially decreasing with lower exponent $\alpha_\ell > 1$. Denoting $m = E\sigma_1$, then for all $\gamma > m$, $t \mapsto P(\sigma_t > \gamma t)$ is at least polynomially decreasing with lower exponent $\alpha_\ell - 1$.*

Proof: W.l.o.g. assume $m = 1$, otherwise perform a linear rescaling of time.

Fix $1 < \alpha < \alpha_\ell$ arbitrarily close to α_ℓ. Then there is a $T_0 \geq 0$ such that for all $T \geq T_0$

$$\bar{\Pi}(T) \leq T^{-\alpha}$$

By Lemma 4 we can even choose $T_0 \geq 0$ such that the same holds for $P(\sigma_1 > \cdot)$ instead of $\bar{\Pi}$, by introducing a multiplicative constant $K_0 > 0$ we can assume for all $x > 0$ that $P(\sigma_1 > x) \leq K_0 x^{-\alpha}$. Putting $S_n = \sigma_n - n$, the same holds true for the tail of the increment distribution F of this centered random walk, by monotonicity. By introducing a function $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = x^{\alpha_\ell}$, $x \leq 0$, $h(x) = K_0$, $x > 0$, we have indeed

$$1 - F(x) \leq x^{-\alpha} h(x) \qquad \textit{for all } x \in \mathbb{R}.$$

In this situation we may apply Theorem 2 of Nagaev [17] to infer that there exists a $K_1 > 0$ such that for all $n \geq 2$ and for all $x \geq n(\gamma - 1)/2$

$$P(S_n \geq x) \leq K_1 n x^{-\alpha} h(x) = K_0 K_1 n x^{-\alpha} \leq K_2 (n-1)(x+1)^{-\alpha}$$

where the restriction $n \geq 2$ is just to be able to find a $K_2 := 2K_0K_1(\gamma/(\gamma-1))^\alpha$ to perform the last inequality (as is easily checked).

Now for all $(\gamma+1)/(\gamma-1) \leq n \leq t \leq n+1$ and $x \geq t(\gamma - 1)$ (which implies $x - 1 \geq (n+1)(\gamma-1)/2$) we conclude from this

$$P(\sigma_t - t \geq x) \leq P(S_{n+1} \geq x - 1) \leq K_2 t x^{-\alpha}$$

which for $x = t(\gamma - 1)$ establishes the assertion when putting $t_0 = 2\gamma/(\gamma-1)$ and $M = K_2(\gamma-1)^{-\alpha}$. Then we can increase t_0 and decrease M to choose $M = 1$. □

The notation fixed in Remark 3 3. is in fact reasonable in a more general subordinator setting with Lévy tail $\bar{\Pi}$ exponentially decreasing having a lower exponent μ_ℓ.

Lemma 6 *Let σ be a subordinator having a Lévy tail at least exponentially decreasing. Then for all $\gamma \in (m, \gamma_1]$ and all $t \geq 0$ also $t \mapsto P(\sigma_t \geq \gamma t)$ is at least exponentially decreasing with lower exponent $\beta(\gamma) = \Phi(\varphi^{-1}(\gamma)) - \gamma\varphi^{-1}(\gamma) > 0$ where φ^{-1} denotes the inverse function of the strictly decreasing, hence invertible function φ. Furthermore, apart from the deterministic subordinator $\sigma_t = mt$, we have $\gamma_1 > m$.*

Proof: First let us note that $\mathcal{D}(\Phi) = \{\lambda \in \mathbb{R} : \Phi(\lambda) < \infty\}$ does include negative values. This is a simple application of the Lévy-Khintchine formula for subordinators which identifies Φ to be essentially an exponential transform of $\bar{\Pi}$, cf. Bertoin [3].

Clearly, the process $Z_t^{(a)} = \exp\{-a\sigma_t + \Phi(a)t\}$ is a martingale for all $a \in \mathcal{D}(\Phi)$. If furthermore $a \leq 0$, we can use the martingale property to conclude for all $\gamma \geq 0$

$$1 \geq E(Z_t^{(a)} 1_{\{\sigma_t \geq \gamma t\}}) \geq \exp\{-t(a\gamma - \Phi(a))\} P(\sigma_t \geq \gamma t)$$

which establishes for every nonpositive $a \in \mathcal{D}(\Phi)$ an exponential bound for the probability of interest. Optimisation by calculus methods yields for $\gamma \in [m, \gamma_1]$ the asserted $a = \varphi^{-1}(\gamma)$.

We note that the well-known properties of Φ being concave and analytic (when extended to $\mathcal{D}(\Phi) + i\mathbb{R}$) imply even strict concavity (except for the trivial cases $\sigma_t = mt$ which precisely correspond to linear Laplace exponents). This means that φ is strictly decreasing. To ease notation we introduce $\delta(a) = \beta(\varphi(a)) = \Phi(a) - a\varphi(a)$. Excluding again $\sigma_t = mt$, $\varphi'(0) = -Var(\sigma_1) < 0$ yields for all $a \in (-\varepsilon, 0)$, $\varepsilon > 0$ sufficiently small $\delta'(a) = a\varphi'(a) > 0$ implying $\beta(\gamma) > 0$ for all $\gamma \in (m, \gamma_1)$ as $\beta(m) = 0$ and using $\delta'(a) = a\varphi'(a) \geq 0$, $\beta = \delta \circ \varphi^{-1}$ is seen to be at least weakly increasing on the whole of (m, γ_1). φ' being positive around the origin also implies the last statement. □

4.3 Proof of Theorem 2

We shall establish the theorem in two steps, first the two upper bounds, second the two lower bounds. In the sequel we stick to terms of subordinators as their occurrence dominates the arguments. The translation into the language of the theorem then relies basically on the representation of the event $\{x(0,T) = 0\}$ in terms of a subordinator, cf. subsection 2.3. Note also that the gap measure ν of a regenerative set is a multiple of the Lévy measure Π of any associated subordinator, hence the asymptotic behaviour of their tails is the same. Recall our notation $\bar{I}(t) = \int_t^\infty \bar{\Pi}(s)ds$

Proposition 3 (Upper bounds) *Let σ be a subordinator with Lévy measure Π satisfying $m = E(\sigma_1) < \infty$, g_0 a (negative) random variable independent of σ and satisfying $P(|g_0| > t) = \frac{\bar{I}(t)}{m}$.*

a) If $\bar{\Pi}$ is at most exponentially decreasing with upper exponent μ_u, then we can find for all $\mu_u < \mu < \infty$ a $T_0 \geq 0$ such that

$$P(\sigma_z \geq \sqrt{2zT} - |g_0| \ for\ all\ z \geq 0) \geq e^{-\mu T/2m} \qquad for\ all\ T \geq T_0.$$

b) If $\bar{\Pi}$ is at most polynomially decreasing with upper exponent $\alpha_u > 1$, then we can find for all $\alpha_u < \alpha < \infty$ a $T_0 \geq 0$ such that

$$P(\sigma_z \geq \sqrt{2zT} - |g_0| \ for\ all\ z \geq 0) \geq T^{-(\alpha-1)} \qquad for\ all\ T \geq T_0.$$

Proof: Assume first $m = 1$. We fix $\varepsilon > 0$ and restict ourselves to the event

$$\left\{|g_0| > \frac{T}{2}(1+\varepsilon)\right\} = \left\{\frac{g_0^2}{2T} > \frac{T(1+\varepsilon)^2}{8}\right\}$$

Since a subordinator is always positive, only $z > \frac{T(1+\varepsilon)^2}{8}$ need to be considered. In what follows we shall establish a deterministic and uniform (in T) upper bound for the random square root function $\sqrt{2zT} - |g_0|$ which as a first step is clearly dominated by $\sqrt{2zT} - \frac{T}{2}(1+\varepsilon)$. In fact, we shall now define the smallest such function and calculate it explicitly:

$$g(z) := \max_{0 \leq T \leq \frac{8z}{(1+\varepsilon)^2}} \left\{\sqrt{2zT} - \frac{T}{2}(1+\varepsilon)\right\} = \frac{z}{1+\varepsilon}$$

where we use standard calculus methods to justify the second equality. Now we can proceed estimating the probability of interest

$$\begin{aligned} &P(\sigma_z \geq \sqrt{2zT} - |g_0| \; for \; all \; z \geq 0) \\ &\quad \geq \; P\left(|g_0| \geq \frac{T(1+\varepsilon)}{2}, \sigma_z \geq \sqrt{2zT} - |g_0| \; for \; all \; z \geq \frac{T(1+\varepsilon)^2}{8}\right) \\ &\quad \geq \; P\left(|g_0| \geq \frac{T(1+\varepsilon)}{2}, \sigma_z \geq \frac{z}{1+\varepsilon} \; for \; all \; z \geq \frac{T(1+\varepsilon)^2}{8}\right) \\ &\quad = \; P\left(|g_0| \geq \frac{T(1+\varepsilon)}{2}\right) P\left(\frac{\sigma_z}{z} \geq \frac{1}{1+\varepsilon} \; for \; all \; z \geq \frac{T(1+\varepsilon)^2}{8}\right) \end{aligned}$$

since g_0 and σ are independent.

As $\frac{\sigma_z}{z} \to E(\sigma_1) > \frac{1}{1+\varepsilon}$ $P - a.s.$, the right hand probability tends to 1 as T tends to infinity. We proceed now for the assertions a) and b) separately.

a) Continuing the above calculation, we choose T_0 so large as to make the right hand probability larger than $\frac{1}{2}$ for $T \geq T_0$. By possibly increasing T_0 we can get an exponential estimate for the left hand probability according Lemma 3 since $P(|g_0| \geq T) = \bar{I}(T)$. More precisely, we conclude

$$P(\sigma_z \geq \sqrt{2zT} - |g_0| \; for \; all \; z \geq 0) \geq \frac{1}{2} \exp\left(-\mu \frac{T(1+\varepsilon)}{2}\right)$$

If $m \neq 1$, we define $\tau_z = \sigma_{z/m}$ which satisfies $E(\tau_1) = 1$ and has Lévy measure Π/m. In particular g_0 has the same distribution when assigned to σ as when assigned to τ. Then we derive from the above calculations applied on τ

$$\begin{aligned} P(\sigma_z \geq \sqrt{2zT} - |g_0| \; for \; all \; z \geq 0) &= P\left(\tau_z \geq \sqrt{2z\frac{T}{m}} - |g_0| \; for \; all \; z \geq 0\right) \\ &\geq \frac{1}{2} \exp\left(-\mu_u \frac{T(1+\varepsilon)}{2m}\right) \end{aligned}$$

b) The same argument yields here for large enough T

$$P(\sigma_z \geq \sqrt{2zT} - |g_0| \; for \; all \; z \geq 0) \geq \frac{1}{2} \left(\frac{T(1+\varepsilon)}{2}\right)^{-(\alpha-1)}$$

Replacing here T by T/m does not change the exponent, hence the result. □

Proposition 4 (Lower bounds) *Let σ be a subordinator having Lévy measure Π and finite mean $m = E(\sigma_1)$, and g_0 a negative random variable independent of σ satisfying $P(|g_0| > t) = \frac{\bar{I}(t)}{m}$*

a) If $\bar{\Pi}$ is at least exponentially decreasing with a lower exponent μ_ℓ, then $P(\sigma_z \geq \sqrt{2zT} - |g_0| \; for \; all \; z \geq 0)$ is at least exponentially decreasing as well. More precisely we denote by r_0 the unique positive location in $(2m - \gamma_1, m)$ such that $\beta(2m - r_0) = \mu_\ell r_0$, if *it exists,* $r_0 = 2m - \gamma_1$ *otherwise. Then we can find for all* $0 < \mu_0 < \beta(2m - r_0)/2m^2$ *a* $T_0 \geq 0$ *such that*

$$P(\sigma_z \geq \sqrt{2zT} - |g_0| \ for\ all\ z \geq 0) \leq e^{-\mu_0 T} \qquad for\ all\ T \geq T_0.$$

b) If $\bar{\Pi}$ *is at least polynomially decreasing with a lower exponent bound* $\alpha_\ell > 1$*, then for all* $\alpha < \alpha_\ell$ *there is a* $T_0 \geq 0$ *such that*

$$P(\sigma_z \geq \sqrt{2zT} - |g_0| \ for\ all\ z \geq 0) \leq T^{-(\alpha-1)} \qquad for\ all\ T \geq T_0.$$

Proof: a) In estimating the probability in question, we restrict our attention to a single location $z = \frac{T}{2m^2}$. This location suggests to be the best choice as here the concave square root has slope m which is the critical value, the mean slope of the subordinator. In order to hit the square root in front of or behind this location while being above it at this instant requires a behaviour that appears atypical.

Let $0 < \mu < \mu_\ell$. In the following we apply Lemma 3 and choose T_0 so large as to be able to use the according estimates for $T \geq T_0$. An application of Lemma 6 is possible when we exclude the deterministic subordinator and choose a $0 < r \in (2m - \gamma_1, m)$

$$\begin{aligned} & P(\sigma_z \geq \sqrt{2zT} - |g_0| \ for\ all\ z \geq 0) \\ \leq \ & P(\sigma_{\frac{T}{2m^2}} \geq \frac{T}{m} - |g_0|) \\ \leq \ & P\left(|g_0| \geq r\frac{T}{2m^2}\right) + P\left(\sigma_{\frac{T}{2m^2}} \geq \frac{T}{m} - |g_0| \geq (2m-r)\frac{T}{2m^2}\right) \\ \leq \ & \frac{1}{m}\exp\left(-\mu r\frac{T}{2m^2}\right) + \exp\left(-\beta(2m-r)\frac{T}{2m^2}\right). \end{aligned}$$

b) Let $1 < \alpha < \alpha_\ell$. We repeat the argument. Instead of applying Lemma 6 we now use Lemma 5. This yields for any $0 < r < m$ and all $T \geq T_0$

$$\begin{aligned} & P(\sigma_z \geq \sqrt{2zT} - |g_0| \ for\ all\ z \geq 0) \\ \leq \ & P\left(|g_0| \geq r\frac{T}{2m^2}\right) + P\left(\sigma_{\frac{T}{2m^2}} \geq (2m-r)\frac{T}{2m^2}\right) \\ \leq \ & \left(r\frac{T}{2m^2}\right)^{-(\alpha-1)} + \left(\frac{T}{2m^2}\right)^{-(\alpha-1)} \leq MT^{-(\alpha-1)} \end{aligned}$$

where M is a constant sufficiently large. □

5 The shock structure

5.1 Formulation and discussion of Theorem 3

Assume for the whole of this section an inviscid Burgers turbulence model initialized by a stationary regenerative impulse. In the sequel we exlude implicitly the trivial case $\Pi \equiv 0$ which corresponds to systems without shocks. Then our main result on the shock structure at positive times is the following

Theorem 3 *When the regenerative impulse to the Burgers turbulence is light and* $\bar{\Pi}(t) \leq t^{-\rho}$ *near zero for some* $\rho < 1$*, then the shock structure at time* $t > 0$ *consists of a sequence of intervals of Eulerian regular points that have kept their initial position and between each successive two of which there is a finite number of Eulerian shock points.*

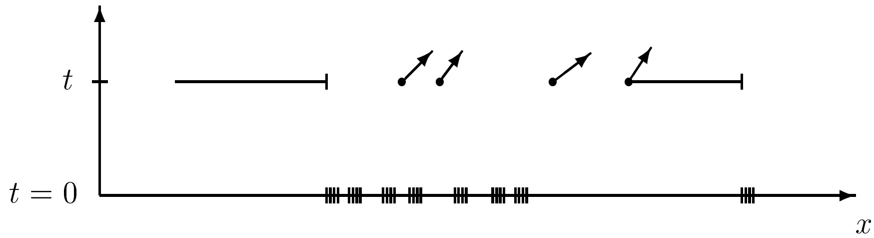

Figure 1 Shock structure

At time $t = 0$ particles in the regenerative set $\mathcal{R}$ ( ) receive the initial impulse. At time t, between intervals of non-moving particles, there are finitely many moving shock points , the rightmost of which is pushing against the non-moving particles.

Remark 4 **1.** For the first part of the statement no additional assumption is needed, i.e. a general stationary regenerative impulse leads to intervals of non-moving Eulerian regular points. This shall provide a powerful partition of the time axis in order to obtain ergodicity results on the shock structure.

2. In the heavy case, next to the intervals of non-moving Eulerian regular points, there may be moving regular points. As all initially moving particles have the same finite speed, this can happen for instance if the regenerative set contains intervals in which case it is no more than a discrete collection of intervals.

3. The regularity condition $\bar{\Pi}(t) \leq t^{-\rho}$ is weak because any Lévy tail fulfils the integrability condition $\int_0^1 \bar{\Pi}(t)dt < \infty$. More precisely, the possible cases $\limsup_{t\downarrow 0}(\ln(\bar{\Pi}(t))/\ln(1/t)) \in (-\infty, 1]$ are all admissible but the extreme value 1.

5.2 Proofs of the statements

The first lemma and the following proposition describe the set of non-moving Eulerian regular points. As stated in the remark, a restriction to light regenerative sets is not necessary here.

Lemma 7 *At any time* $t > 0$*, there are a.s. non-moving Eulerian regular points, i.e.* $\tau_0^+ := \sup\{a < 0 : x(a,t) = a\}$ *is a.s. finite.*

Proof: We shall use the parabola analysis described in subsection 2.2, transferred to the subordinator level via exchanging coordinate axes (cf. subsection 2.3). Then the event $\{\tau_0^+ = -\infty\}$ describes a behaviour of σ^- that requires for every $x \geq 0$ the parabola

$$z \mapsto \sigma_{x-}^- + \sqrt{2t(z-x)}, \qquad z > x,$$

to intersect with σ^-. Clearly the most critical of these $x \geq 0$ are the locations of large jumps ($\geq j_0$, say) as the parabolas in these locations start low in comparison to the subordinator and the asymptotic behaviour of the two (recall that $\sigma_z^- \sim \mu z$ by the law of large numbers) suggests an early hit if at all. Define now (possibly finite) sequences of locations T_n of large jumps and intersection points S_n of the subordinator and the parabola belonging to $x = T_n$ such that T_{n+1} is the first large jump after S_n. As $(T_n)_{n\geq 1}$ and $(S_n)_{n\geq 1}$ are sequences of stopping times (w.r.t. the canonical filtration of σ^-), constructed as successive occurrences of events, the subpaths $(\sigma_{S_{n-1}+u}^- - \sigma_{S_{n-1}}^-)_{S_{n-1}\leq u \leq S_n}$ are, conditionally under $S_{n-1} < \infty$, independent of $(\sigma_u^-)_{u \leq S_{n-1}}$ and identically distributed. The same is therefore true for $(\sigma_{T_n+u}^- - \sigma_{T_n-}^-)_{0\leq u\leq S_n - T_n}$ since the properties of S_{n-1} to be intersection time of σ^- with an increasing function and of T_n to be jump time entail $T_n > S_{n-1}$. As the probability that $(\sigma_u - \sigma_{T_1-})_{u \geq T_1}$ remains always above $\sqrt{2t(u-T_1)}$ is positive, there is a.s. an $n \geq 1$ such that $S_n = \infty$. Therefore we conclude

$$P(\tau_0^+ = -\infty) \leq P(S_n < \infty \ \textit{for all } n \geq 1) = 0.$$

$\square$

Proposition 5 *The random times*

$$\begin{aligned}
\tau_{n+1}^- &:= \inf\{a > \tau_n^+ : x(a,t) = a\} \\
\tau_{n+1}^+ &:= \inf\{a > \tau_{n+1}^- : x(a,t) > a\} \\
\tau_{-n}^- &:= \sup\{a < \tau_{-n}^+ : x(a,t) > a\} \\
\tau_{-n-1}^+ &:= \sup\{a < \tau_{-n}^- : x(a,t) = a\},
\end{aligned}$$

$n \geq 0$, are a.s. finite and satisfy $\tau_n^- \to \pm\infty$ as $n \to \pm\infty$. Then (τ_n^-, τ_n^+), $n \in \mathbb{Z}$, denote the successive intervals of non-moving particles.

Proof: By the previous lemma, all we need to show is that, positioned at the right end d (≥ 0, say) of an interval of Eulerian regular points, the next non-moving Eulerian regular point to the right is at a positive distance. Then the sequences of times are strictly increasing. Noting that τ_n^- and $\tau_n^+ = \inf\{a > \tau_n^- : a \in \mathcal{R}\}$ are stopping time w.r.t. the canonical filtration $\mathcal{F}^{\rightarrow}$ of $\mathcal{R}$, the regeneration property at τ_n^+ implies the independence and identical distribution of the increments $\tau_{n+1}^+ - \tau_n^+$, $n > 0$ (and similarly for $n < 0$).

Assume, there exists a sequence $\xi_j \downarrow d$ of Lagrangian regular points, i.e. for all $j \geq 1$

$$\psi(y,0) + \frac{(\xi_j - y)^2}{2t} \geq \psi(\xi_j, 0) \qquad \textit{for all } y \leq \xi_j.$$

For $y = d$ we obtain for all $j \geq 1$

$$\psi(\xi_j, 0) - \psi(d, 0) \leq \frac{(\xi_j - d)^2}{2t}.$$

The closure property of the regenerative set $\mathcal{R}$ of increase points of $\psi(\cdot, 0)$ implies that d is a regeneration point. Furthermore, d is a stopping time (w.r.t. the canonical forward filtration $\mathcal{F}^{\rightarrow}$ of $\mathcal{R}$). The right derivative is thus a.s. positive ($= k/\theta \geq k > 0$, cf. Proposition III.8 in Bertoin [3]), implying for all $j \geq j_0$

$$\psi(\xi_j, 0) - \psi(d, 0) \geq \frac{k}{2}(\xi_j - d)$$

which is incompatible with $\xi_j \downarrow d$. □

This provides a partition of the particle space into the parts between the non-moving regular points. As in the light case all initial velocities are zero or infinite, there cannot be any moving regular points because they keep their initial velocity. As an immediate consequence we note

Corollary 4 *$(x(\tau_n^+ + a, t) - \tau_n^+)_{0 \leq a < \tau_{n+1}^+ - \tau_n^+}$, $n \in \mathbb{Z}$, are independent. They are identically distributed for all $n \neq 0$.*

Proof: This is an immediate consequence of the regeneration property at τ_n^+, as is clear from the proof of the proposition. □

Let us now analyse the structure between two successive intervals of non-moving Eulerian regular points.

Lemma 8 *Any right endpoint of an interval of non-moving Eulerian regular points is not regular itself.*

Proof: Assume, the right endpoint a of an interval of Eulerian regular points is regular itself. Then $a(\xi, 1) > a$ for all $\xi > a$, i.e. for all $\xi > a$ there is a touch location $\eta \in (a, \xi]$, i.e. in particular, for a sequence $\xi_j \downarrow a$ we find another sequence $\eta_j \downarrow a$ such that

$$\psi(a, 0) + \frac{(\xi_j - a)^2}{2t} \geq \psi(\eta_j, 0) + \frac{(\xi_j - \eta_j)^2}{2t}$$

Now like in the proof of Proposition 5 we use the regenerative property in a to obtain

$$\frac{k}{2}(\eta_j - a) \leq \psi(\eta_j, 0) - \psi(a, 0) \leq \frac{(\eta_j - a)(2\xi_j - \eta_j - a)}{2t}$$

contradicting $\xi_j \downarrow a$. □

In order to show the discreteness of the shock structure between the regular points, we shall need a preliminary lemma on the path behaviour of subordinators.

Lemma 9 *Let σ be a subordinator with zero drift component and whose Lévy tail fulfils $\bar{\Pi}(t) \leq t^{-\rho}$ for some $\rho < 1$ and all $t \leq t_0$. Then we have*

$$P\left(\sup_{t>0} \liminf_{h \downarrow 0} \frac{\sigma_{t-} - \sigma_{t-h}}{h} = 0\right) = 1$$

Proof: If σ is a stable subordinator, this result is an immediate consequence of Theorem 1 of Fristedt [10]. The idea to reduce the general to the stable case was employed by Marsalle [16] in a similar setting. If σ is not stable, we first note that large jumps do not influence the result, so we may assume w.l.o.g. that the Lévy measure Π of σ has compact support. Now the condition posed on its tail allows to find $c > 0$ such that $\bar{\Pi}(t) \leq ct^{-\rho}$ for all $t > 0$ where the upper bound is the Lévy tail of a stable subordinator τ with index ρ.

This enables us to couple the Poisson point processes $\Delta\sigma$ and $\Delta\tau$, which describe the jumps of the two processes, in such a way that $\Delta\sigma(t) \leq \Delta\tau(t)$ for all $t \geq 0$. This implies $\sigma_t - \sigma_s \leq \tau_t - \tau_s$ for all $0 \leq s \leq t$. This reduces the assertion for σ to the stable case already known. □

Denoting now the points of $\mathcal{R}$ isolated to the left by $\mathcal{G}$ and those isolated to the right by $\mathcal{D}$, we deduce from Lemma 9 by elementary calculus considerations

Corollary 5 *In the case of a light regenerative set $\mathcal{R}$ and under the condition $\bar{\Pi}(t) \leq t^{-\rho}$ near zero, we have*

$$P\left(\inf_{x\in\mathcal{R}-\mathcal{G}} \limsup_{h\downarrow 0} \frac{\psi(x,0)-\psi(x-h,0)}{h} = \infty\right) = 1.$$

Proof: The corresponding statement when the infimum is taken over $x \in \mathcal{R} - \mathcal{G} - \mathcal{D}$ is a consequence of Lemma 9. Since $\mathcal{D}$ only contains a countable number of stopping times (w.r.t. the natural filtration $\mathcal{F}^{\leftarrow}$ of $\mathcal{R}$ in reversed time) where the behaviour of $\psi(\cdot, 0)$ is well-known to be likewise (cf. Proposition III.8 in Bertoin [3]), the infimum can be extended to $\mathcal{R} - \mathcal{G}$. □

Proposition 6 *When the regenerative set is light, there is only a finite number of Eulerian shock points between two successive intervals of Eulerian regular points.*

Proof: By Lemma 8, a right endpoint $b_0 := a$ of an interval of Eulerian regular points, contributes to a Eulerian shock point q_1, say, corresponding to a Lagrangian shock interval $[b_0, b_1)$. If b_1 is regular (i.e. $b_1 \in \mathcal{R}^c$ and $b_1 = q_1$), this part of the proof is finished. If not, we continue along the same lines, defining q_2 to be the Eulerian position corresponding to the shock interval $[b_1, b_2)$, etc. constructing a sequence $q_n \uparrow r < \infty$ of Eulerian shock points (the finiteness of r stems from the fact that all this happens before the next interval of Eulerian regular points). Now $b_n \uparrow c \leq r$.

By the Hopf-Cole formulas we have for all $j \geq 1$

$$\psi(b_j, 0) - \psi(b_{j-1}, 0) = \frac{(b_j - b_{j-1})(2q_j - b_j - b_{j-1})}{2t}$$

which yields for all $\varepsilon > 0$ when summing j from large enough $n+1$ to infinity

$$\psi(c, 0) - \psi(b_n, 0) \geq \frac{(r-\varepsilon)(c-b_n)}{t} - \frac{c^2 - b_n^2}{2t}$$

and hence

$$\frac{\psi(c,0)-\psi(b_n,0)}{c-b_n} \geq \frac{2r-2\varepsilon-c-b_n}{2t} \to \frac{r-\varepsilon-c}{t}$$

Since $a(r-,t)=c$, the parabola $x \mapsto k-(r-x)^2/(2t)$ for $k=\psi(c,0)-(r-c)/(2t)$ rests always below $\psi(x,0)$ and touches in $x=c$ where its slope is $(r-c)/t$. We conclude

$$\limsup_{h\downarrow 0} \frac{\psi(c,0)-\psi(c-h,0)}{h} = \frac{r-c}{t}$$

Clearly, there are regeneration points within every shock interval, hence arbitrarily close to left of c. Therefore, c is not a left endpoint of $\mathcal{R}$, and by Corollary 5 we obtain a contradiction. □

A A class of subordinators of Vershik and Yor

An interesting class of processes, which satisfy the conditions of Theorems 1 and 3 and the exponential condition of Theorem 2, correspond to subordinators having a zero drift coefficient and a Lévy measure of the form

$$\Pi(dt) = m\frac{\mu^{1-\alpha}}{\Gamma(1-\alpha)}\frac{e^{-\mu t}}{t^{\alpha+1}}dt$$

where $m>0$, $\mu>0$ and $\alpha<1$ - the latter is to ensure the integrability condition w.r.t. Π at $0+$ that every Lévy measure of a subordinator must fulfil. This class of processes has been studied previously by Vershik and Yor [22]. The case $\alpha=0$ corresponds to a *Gamma subordinator*, $\alpha=\frac{1}{2}$ to an *inverse Gaussian subordinator*. σ is a compound Poisson process if $\alpha<0$. The processes having $\alpha\in(0,1)$ converge in law when the coefficient of the Lévy measure is held constant, to a stable subordinator of index α as μ tends to zero.

It is easy to see that

$$\mu = \lim_{t\to\infty} -\frac{1}{t}\ln(\bar{\Pi}(t)).$$

The Laplace exponent of the Gamma subordinator is well known to be

$$\Phi(\lambda) = m\mu\log\left(1+\frac{\lambda}{\mu}\right).$$

For $\alpha\neq 0$ we can calculate the Laplace exponent of σ by partial integration and the Gamma integral

$$\begin{aligned}\Phi(\lambda) &= \int_0^\infty (1-e^{-\lambda t})m\frac{\mu^{1-\alpha}}{\Gamma(1-\alpha)}\frac{e^{-\mu t}}{t^{\alpha+1}}dt \\ &= m\frac{\mu^{1-\alpha}}{\Gamma(1-\alpha)}\int_0^\infty (e^{-\mu t}-e^{-(\lambda+\mu)t})t^{-\alpha-1}dt\end{aligned}$$

$$
\begin{aligned}
&= m\frac{\mu^{1-\alpha}}{\Gamma(1-\alpha)}\left[(e^{-\mu t}-e^{-(\lambda+\mu)t})\frac{t^{-\alpha}}{-\alpha}\right]_0^\infty \\
&\qquad +m\frac{\mu^{1-\alpha}}{\Gamma(1-\alpha)}\int_0^\infty(\mu e^{-\mu t}-(\lambda+\mu)e^{-(\lambda+\mu)t})\frac{t^{-\alpha}}{-\alpha}dt \\
&= 0-m\frac{\mu^{1-\alpha}}{\alpha\Gamma(1-\alpha)}\left(\frac{\Gamma(1-\alpha)}{\mu^{-\alpha}}-\frac{\Gamma(1-\alpha)}{(\lambda+\mu)^{-\alpha}}\right) \\
&= m\frac{\mu}{\alpha}\left(\left(\frac{\lambda+\mu}{\mu}\right)^\alpha-1\right).
\end{aligned}
$$

This entails in particular that the parametrisation of the Lévy measure above was done in such a way as to yield $E(\sigma_1)=\Phi'(0)=m$.

It is now easy to show that

$$
\lim_{\lambda\to\infty}\frac{\ln(\Phi(\lambda))}{\ln(\lambda)}=\alpha\vee 0.
$$

Furthermore, we can read off the domain of Φ as $(-\mu,\infty)$ if $\alpha\leq 0$ and $[-\mu,\infty)$ if $0<\alpha<1$.

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