o13 Hilary Term Lecture 1

Introduction

Last term dealt mainly with certainty and straightforward cases. For instance, annuities were payable for a set period of time. Rates of interest were constant over time. Payment of interest was guaranteed. By and large the mathematics delivered unique answers. World of life insurance has to deal with various uncertain issues. We briefly looked at life contingencies at the end of last term and we will return to that topic at the end of this term. The structure of lectures for the term is as follows:

Week 1 Investments

Week 2 Concept of matching assets and liabilities

Week 3 Term structure of interest rates

Week 4 No arbitrage assumption

Week 5 Stochastic Interest Rate Models

Week 6 Mortality

Week 7 Life Insurance

Week 8 Revision/Past exam questions

So we will start by looking at the uncertain world of investments.

Before the Second World War, most life offices invested wholly in fixed interest securities. Many actuarial papers looked very closely at the mathematics of fixed interest investments and its relationship to the liabilities of the life office. During the 1950s, offices began to invest more and more in equities – some went up to 90% of their investments in equities. The 3 years stock market performance (2000/1/2) has meant that there has been a strong move back to fixed interest.

At the end of 1998 life insurance companies invested some 35% of their assets in fixed interest securities. By the end of 2002 this had risen to 50%.

Fixed Interest securities

We looked at this topic last term so this is by way of reminder! Fixed interest securities are offered by the Government (where they are called "Gilts") and by companies (where it is called "Corporate debt"). Normally they are issued for a stated term (e.g. Exchequer 10.5% 2005) and at other times there is a range of dates (e.g. Treasury 11.75% 2003-7). Some are undated (e.g. War Loan 3.5%). The coupon represents the amount of interest income that will be paid each year related to the nominal amount of the stock.

Key issue is the certainty of payment of the coupon and the redemption proceeds. With UK Government stock the risk is considered zero. Risk varies with companies and that is where rating agencies play a role (e.g. Triple A ratings).

Don't forget tax – income tax and CGT.

Index linked bonds

Some years ago the UK Government started issuing index linked bonds. In these both the coupon and the redemption proceeds are linked to the RPI. These can be very useful when covering index-linked benefits (e.g. pensions).

Equities

Now we move to investments with uncertain payments. We start with ordinary shares (commonly called "equities"). These are issued by companies, normally "limited companies" where the shareholders have limited risk (as opposed to Lloyds members). Risk is limited to amount invested (so you can lose all that – but not your house as well!). A shareholder is a part owner of the company and has voting rights.

Company calculates profit each year (having paid interest on its corporate debt) and then declares a "dividend" for each ordinary share. Thus equities have highest risk. In bad times a company may decide not to declare a dividend (e.g. Britannic).

Return from investment is the amount of the dividend plus (or minus) any change in value of share. Large companies have shares quoted on the stock market and thus the value of the share can be tracked on a daily basis. Performance of the stock market as a whole is tracked by the FTSE Actuaries indices. For instance the FTSE 100 deals with the 100 largest companies quoted on the UK stock market. Actuaries started calculating indices over 70 years ago. The Actuaries Investment Index was calculated form 1928 – 1962. Thereafter it was jointly produced by the Financial Times and the actuarial profession.

The following table shows the FTSE index at the start of each year from 1983 to 2004:

02/01/1984	1000	02/01/1995	3065.5
01/01/1985	1232.2	01/01/1996	3689.3
01/01/1986	1412.6	01/01/1997	4118.5
01/01/1987	1679	01/01/1998	5135.5
01/01/1988	1712.7	01/01/1999	5882.6
02/01/1989	1793.1	03/01/2000	6930.2
01/01/1990	2422.7	01/01/2001	6222.46
01/01/1991	2143.5	01/01/2002	5217.35
01/01/1992	2493.1	01/01/2003	3940.36
01/01/1993	2846.5	02/01/2004	4510.2
03/01/1994	3418.4		

Preference Shares

These are a special sort of share. Assuming the company makes sufficient profits, they offer a fixed stream of income. No dividends can be paid on ordinary shares unless the preference share dividend has been paid first. In risk terms they rank between normal corporate debt and ordinary shares.

Convertibles

Some companies issue "Convertible Preference Shares". These start life as preference shares but contain the option to convert to ordinary shares at stated times in the future,

Property

There are 3 main types of investment used by insurance companies: Fixed Interest, Equities and Property. Property represents land plus buildings thereon.

Different types: Private/Commercial

Commercial: Office; Industrial; Retail

Return = Rental income + Capital growth

Rental terms specified in lease agreements (regular rent reviews). Likely to go up in line with inflation (but possibility of void periods with no tenants).

Drawbacks: Large sizes of investment (thus lack of flexibility/marketability); Valuation is difficult; Dealing costs high; Maintenance expenses.

Location, Location, Location. (e.g. the change in valuation of properties in Battersea)

Life insurance companies overall have invested just under 10% of their assets in Property.

Derivatives

Financial instruments dependent (in a variety of ways) on the value of another underlying asset (or assets). Start with the "vanilla" variety – many others devised by "rocket scientists".

Futures

Standardised, exchange tradable contract between 2 parties to trade specified asset on a set date in the future at a specified price. (Coffee futures, "Trading Places" with oranges, financial futures). 4 main categories of financial futures. In each case a margin is deposited with the clearing house and daily variation margins are paid if the underlying prices change.

Bond – Requires the physical delivery of the bond

Short interest rate – Contract based on the interest paid on a notional deposit for a specified period from the expiry of the future.
Stock index – Notional transfer of assets underlying the stock index.
Currency – Delivery of a set amount of currency on the given date.

Options

An option gives the right (but not the obligation) to buy or sell a specified asset on a specified future date. Call options give right to buy. Put options give right to sell.

American options can be exercised at any date before expiry.
European options can be exercised only on the expiry date.

Swaps

Interest rate swap exchanges a fixed series of payments for a variable series based on short term interest rates. There is no exchange of principal.

There are 2 aspects of counterparty risk:

Market risk – where market conditions change so that the present value of one part of the swap changes.
Credit risk – where one party defaults on its payments.

o13 Hilary Term Lecture 2

Asset/Liability Matching

Essence of an insurance company is that it is investing money (assets) to provide income to cover expected outgo (liabilities). Safest position is where the amount and timing of each item of income exactly matches each item of outgo. For example an annuity certain for 10 years in arrears could be exactly matched by 10 zero coupon bonds of terms 1,2…. 10. Annuity portfolios try to match as closely as possible but there are uncertainties about mortality and also the latest expected date of payment may well be after the longest dated loan stock. Currently the longest dated UK Government stock is Treasury 4.25% with a redemption date of 2036.

There are 2 undated stocks: War Loan 3.5% and Treasury 2.5% but these could be repaid at any date so are not suitable for exact matching.

Last week we considered other types of investment – equities, property and derivatives – all with some uncertain qualities. For this week we will return to fixed interest stock – and also assume that the rate of interest is constant across all terms. That is not normally the case as we shall see next week but it is useful for this introductory look at matching.

Effective Duration

Consider a series of cash flows $\{C_{t_k}\}$ for k = 1 to n.

Let A(i) be the present value of the series at rate i.

$$A(i) = \sum_{k=1}^{n} v^{t_k} C_{t_k}$$

Effective duration (or volatility) is defined as

$$\nu = -\frac{1}{A}\frac{dA}{di} = -\frac{A'}{A}$$

$$= \frac{1}{\sum_{1}^{n} v^{t_k} C_{t_k}} \left\{ \sum_{1}^{n} C_{t_k} t_k v^{t_k+1} \right\}$$

It is a measure of the rate of change in value for small changes in interest rate.

Macauley Duration

This is the mean term of the cash flows C, weighted by their present value.

$$\tau = \frac{\sum_{k=1}^{n} t_k C_{t_k} v^{t_k}}{\sum_{1}^{n} C_{t_k} v^{t_k}}$$

Comparing with the effective duration we see that

$$\tau = (1+i)\nu$$

Remembering that $\quad v^t = e^{-\delta t} \longrightarrow \dfrac{dA}{d\delta} = -\sum_1^n t_k C_{t_k} v^{t_k}$

$$\therefore \quad \tau = -\frac{1}{A}\frac{dA}{d\delta} = \frac{di}{d\delta}\nu$$

$$\text{But } i = e^{\delta} - 1 \qquad \therefore \frac{di}{d\delta} = e^{\delta} \qquad \therefore \tau = e^{\delta}\nu = (1+i)\nu$$

Example

Macauley duration for an n year bond with redemption price R and coupon D is

$$\tau = \frac{D \times (\bar{I}a)_{\overline{n|}} + Rnv^n}{Da_{\overline{n|}} + Rv^n}$$

Macauley duration for a zero coupon bond of term n is n.

Convexity

Going for the second differential the convexity of the cash flow C is defined as:

$$c = \frac{1}{A}\frac{d^2A}{di^2} = \frac{A''}{A} = \frac{1}{A}\left\{\sum_1^n C_{t_k} t_k (t_k+1) v^{t_k+2}\right\}$$

For small changes in interest rates we have:

$$\frac{A(i+\varepsilon) - A(i)}{A} = \frac{1}{A}\frac{\partial A}{\partial i}\varepsilon + \frac{1}{2A}\frac{\partial^2 A}{\partial i^2}\varepsilon^2 + \dots$$

$$\approx -\varepsilon\nu + \varepsilon^2 c/$$

Positive convexity means that if interest rates fall by a small amount, liabilities rise by a greater amount than they fall for an equivalent rise in interest rate.

Frank Redington

Frank Redington was the outstanding British actuary of his generation. He became Chief Actuary of the Prudential at the age of 46, President of the Institute of Actuaries and received the Institute's Gold Medal. In 1952 he presented a paper to the Institute entitled "A ramble through the actuarial countryside". In it he introduced the subject of immunisation.

Let's return to the fundamental question of matching assets and liabilities. Consider a fund with an asset cash flow $\{A_{t_k}\}$ and an equivalent liability cash flow $\{L_{t_k}\}$.

Let the present values of the 2 flows be $V_A(i)$ and $V_L(i)$ respectively and similarly the volatilities $\nu_A(i)$ and $\nu_L(i)$; and the convexities $c_A(i)$ and $c_L(i)$.

We assume that at interest rate i the fund is exactly balanced so that $V_A(i) = V_L(i)$.

If the interest rate moves to $i + \varepsilon$, consider the surplus $S = V_A - V_L$

$$S(i + \varepsilon) = S(i) + \varepsilon S'(i) + \frac{\varepsilon^2}{2} S''(i) + \ldots\ldots$$

For the fund to be "immunised" we require that $S(i+\varepsilon)$ is always positive whether is positive or negative.

Therefore, to achieve this we require that $S'(i) = 0$ i.e. $\nu_A(i) = \nu_L$

Effective duration is the same.

and also that $S''(i) > 0$ i.e. convexity of assets is greater than convexity of liabilities.

Practical Problems

Requires constant rebalancing (not to mention dealing expenses)

There may well be options and other uncertainties (e.g. mortality) in both the assets and the liabilities.

Relevant assets may not exist (e.g. not long enough terms)

o13 Hilary Term Lecture 3

Term structure of interest rates

Previous lectures have assumed that interest rates do not depend on the duration of the investment. However, consulting the financial pages shows that this is not the case. For instance, looking at the yields on UK Government stock at close of business on 2 dates shows the following yields to redemption:

Stock	*Redemption Date*	*Redemption yield* % 31.01.03	*Redemption yield* % 03.02.2004
Treasury 10%	2003	3.63	
Treasury 4%	2004		3.90
Treasury 5%	2008	4.15	4.70
Treasury 8%	2013	4.26	4.86
Treasury 6%	2028	4.36	4.78

<u>Discrete time spot rates</u>

Consider a "zero coupon bond" of term n (i.e. an agreement to pay 1 at the end of n years with no coupon payable during the term). This is also called a "pure discount bond". Denote the price at issue of this bond to be P_n.

The yield (y_n) on this zero coupon bond is called the "n-year spot rate of interest" which clearly satisfies the equation:

$$P_n = \frac{1}{(1+y_n)^n} \quad \Longrightarrow \quad (1+y_n) = P_n^{-1/n}$$

As demonstrated above, rates of interest normally depend on the term of the investment and so normally $y_s \neq y_t$ when $s \neq t$.

Each fixed interest investment can be considered as a string of zero coupon bonds.

Consider an n year bond, with coupon D and redemption price R. This is equivalent to n zero coupon bonds with maturity value D payable at durations 1,2,…,n-1 together with a zero coupon bond with maturity value R+D payable at duration n.

Price A is given by:

$$A = D.(P_1 + P_2 \ldots\ldots + P_n) + R.\ P_n$$

Define $v_{y_t} = (1+y_t)^{-1}$

Then $A = D.(v_{y_1} + v_{y_2}^2 + \ldots\ldots + v_{y_n}^n) + R.\ v_{y_n}^n$

<u>Discrete time forward rates</u>

Discrete time forward rate $f_{t,r}$ is the annual interest rate agreed at time 0 for an investment made at time t (>0) for a duration of r years.

In other words if an investor agrees at time 0 to invest 1 at time t for r years, the maturity amount (at time t+r) will be:

$$(1+ f_{t,r})^r$$

There are connections between forward rates, spot rates and zero coupon bonds.

Take an investment of 1 for t years and also agree at commencement that the accumulation at time t will be reinvested for a further r years.

Then final amount at time t+r will be:

$(1+ y_t)^t . (1+ f_{t,r})^r$

But 1 invested for t+r years accumulates to:

$(1+ y_{t+r})^{t+r}$

Also, using the zero coupon bond price, 1 invested for t+r years accumulates to:

$(P_{t+r})^{-1}$

Therefore $(1+ y_t)^t . (1+ f_{t,r})^r = (1+ y_{t+r})^{t+r} = (P_{t+r})^{-1}$

And $(1+ f_{t,r})^r = \dfrac{(1+ y_{t+r})^{t+r}}{(1+ y_t)^t} = \dfrac{P_t}{P_{t+r}}$

"One period forward rate" at time t (agreed at time 0) is denoted f_t and defined as $f_{t,1}$

So an amount of 1 invested at time 0 for t years at the spot rate y_t will be equivalent to the same 1 invested at time 0 for t one year forward rates. In other words:

$$(1+ y_t)^t = (1 + f_0) (1 + f_1) (1 + f_2)........ (1 + f_{t-1})$$

and, of course, $f_0 = y_1$

Continuous time spot rates

P_t is the price of a zero coupon bind of term t. The t year "spot force of interest" is Y_t which is defined as satisfying the equation:

$P_t = e^{-Y_t t} \implies Y_t = -\frac{1}{t} . \log P_t$

There exist the same linkages as between δ and i.

So after t years at the discrete spot rate, 1 accumulates to $(1+ y_t)^t$. At the spot force of interest (or continuous time spot rate) it accumulates to $e^{Y_t t}$
Thus $y_t = e^{Y_t} - 1$

Continuous time forward rates

Similarly the continuous time forward rate $F_{t,r}$ is the force of interest equivalent to the forward spot rate $f_{t,r}$

At time 0 there is an agreement to invest 1 at time t for r years. This will accumulate at the end of the term to

$$e^{F_{t,r}\, r}$$

As before: $f_{t,r} = e^{F_{t,r}} - 1$

As we did with the discrete rates we can consider the relationship between the spot and forward rates by considering the accumulation of 1 for t years at the spot rate followed by r years at the forward rate. Then we equate that to the accumulation of 1 for t+r years at the relevant spot rate:

$$e^{tY_t} \cdot e^{rF_{t,r}} = e^{(t+r)Y_{t+r}}$$

$$t\,Y_t + r\,F_{t,r} = (t+r)\,Y_{t+r}$$

$$F_{t,r} = \frac{(t+r)\,Y_{t+r} - t\,Y_t}{r}$$

Considering the zero coupon bond price P_n we can see that $Y_n = \frac{-1}{n} \log P_n$

Therefore $F_{t,r} = \frac{1}{r} \log \left\{ \frac{P_t}{P_{t+r}} \right\}$

Instantaneous forward rates

F_t is defined as $\lim_{r \to 0} F_{t,r}$

Thus $F_t = \lim_{r \to 0} \frac{1}{r} \log \left\{ \frac{P_t}{P_{t+r}} \right\}$

$$= \lim_{r \to 0} - \frac{\log P_{t+r} - \log P_t}{r}$$

$$= - \frac{d}{dt} \log P_t$$

Realising that $P_0 = 1$ + integrating $P_t = e^{-\int_0^t F_s \, ds}$

Redemption yield

This is also known as the "Yield to Maturity" is the effective rate at which the discounted value of future payments (coupon and redemption amount) equals the market price. It clearly depends on the coupon and so does not give a simple model of the relationship between term and yield.

For instance, again looking at the January 31 2003 yields:

Stock	*Redemption Date*	*Redemption yield* %
Treasury 5%	2012	4.28
Treasury 7.75%	2012-15	4.47
Treasury 9%	2012	4.31

Par Yield

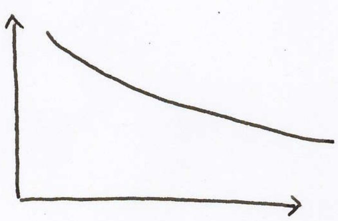
The *"Term t par yield"* is that coupon per 1 nominal on an n year bond redeemable at par which would give a current price of 1. In other words, if the term t par yield is yc_t:

$$1 = yc_t \left(v_{y_1} + v^2_{y_2} + v^3_{y_3} + \ldots\ldots\ldots + v^n_{y_n}\right) + v^n_{y_n}$$

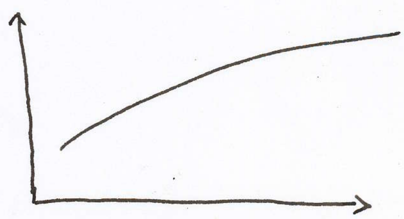
Yield Curves

Some examples of typical (spot rate) yield curves are as follows:

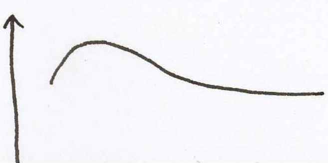
Decreasing yield curve



Increasing yield curve



Humped yield curve



There are 3 popular explanations for the variability of interest rates by duration:

- Expectations Theory
- Liquidity Preference
- Market Segmentation

Expectation. The relative attraction of short or long term investments varies according to the expectation of interest rate movements. If the expectation is that interest rates will fall, then there will be a preference for longer term investments. Investors will want to maintain the higher coupon for the longest possible time. An expectation of higher interest rates will mean a preference for shorter term investments.

Liquidity. Longer dated investments are more sensitive to interest rate movements than shorter dated investments. Risk averse investors will require compensation for the greater risk of loss on longer bonds. In general this means they will look for a higher yield.

Market. Different investors are interested in different durations. For instance pension schemes and annuity providers are generally looking for long duration investments; banks are normally looking for shorter duration investments (since bank depositors can often withdraw their funds with little notice).

o13 Hilary Term Lecture 4

"No arbitrage" assumption

Arbitrage in financial mathematics is generally defined as a risk-free trading profit. This can arise in 2 ways:

> An investor arranges a deal which yields an immediate profit with no risk of any future loss.
>
> An investor arranges a deal with no immediate cost, no risk of any future loss AND a positive probability of future profit.

2 examples of arbitrage opportunities:

2 securities A and B each with term 1 year. Prices at time 0 are $P_0(A)$ and $P_0(B)$. At time 1, the payout from each security depends on whether the stock market has risen or fallen. If it goes up, the payouts are $P_1(A,u)$ and $P_1(B,u)$; if it goes down, they are $P_1(A,d)$ and $P_1(B,d)$.

Investors can either buy or sell each security. If they buy a security, they pay the time 0 price and receive the time 1 payout; if they sell, they receive the time 0 price and pay the time 1 payout.

Consider the following price/payout schedule:

Security S	Time 0 price P_0	Market rises $P_1(S,u)$	Market falls $P_1(S,d)$
A	6	7	5
B	11	14	10

This produces an "immediate profit" arbitrage opportunity:

> Buy 1 unit of security B
> Sell 2 units of security A

This delivers the following income and outgo:

	Time 0	Time 1 Market rises	Time 1 Market falls
Income	12	14	10
Outgo	11	14	10
Total	1	0	0

A profit of 1 is made at time 0 with no prospect of loss (or profit) at time 1 – whatever happens to the stock market. Clearly security A is much less attractive than security B. Market pressures will act to ensure that equilibrium exists when:

$$P_0(A) = P_0(B)/2$$

Or take a different example:

Security S	Time 0 price P_0	Market rises $P_1(S,u)$	Market falls $P_1(S,d)$
A	6	7	5
B	6	7	4

which produces a "no loss" arbitrage possibility:

Buy 1 unit of security A
Sell 1 unit of security B

	Time 0	Time 1 Market rises	Time 1 Market falls
Income	6	7	5
Outgo	6	7	4
Total	0	0	1

Clearly investors will have a strong preference for security A and market pressures will work to increase the price of A. The arbitrage opportunity disappears as soon as:

$$P_0(A) > P_0(B)$$

Modern financial mathematics is based on the "no arbitrage assumption". In other words, in a developed financial market place, market pressures will ensure that arbitrage opportunities do not exist. A consequence of the "no arbitrage assumption" is the "Law of One Price" which states that any two securities or combinations of securities that have an identical payment schedule must have the same price.

The "no arbitrage assumption" as developed into the "Law of One Price" enables us to calculate the price of complex financial instruments by "replicating" the payment schedules. In other words, if we can discover a portfolio of "simpler" assets that have exactly the same payment schedules as the complex instrument that we wish to value, then the value (price) of the instrument must equal the value (price) of the sum of the "simpler" assets. The aggregation of the simpler assets is called the "replicating portfolio".

Forward Contracts

A forward contract is an agreement at time 0 between 2 parties whereby one agrees to buy from the other a specified amount of an asset at a specified price at a specified FUTURE date. The buyer is said to hold a "long forward position" and the seller a "short forward position". In general, the future price of the asset is not known. For example, you might enter into a forward contract to buy a particular number of ordinary shares in BP in six months' time. The BP share price is known today but not in 6 months. Indeed the price will vary continuously for the next 6 months.

Let: S_r = the price of the asset (e.g. the BP share) at time r
K = the price agreed at time 0 to be paid at time T
δ = the force of interest available on a risk free investment from time 0 to T.

K is called the "forward price"; T is the maturity date of the forward contract and δ is known as the risk free force of interest.

At time 0 (when the contract is agreed), no money changes hands. K is so chosen that the present value of the forward contract equals zero. In general there will be a profit/loss at the maturity of the contract since it is extremely unlikely that K will equal S_T. The buyer of the contract will pay K and the seller will deliver S_T.

Replicating Portfolios

We will apply the "Law of One Price" to the valuation of forward contracts by finding "replicating portfolios".

1. Consider a simple forward contract for an asset where there are no intervening income payments (i.e. no dividends or coupons are payable before the maturity of the contract).

Look at 2 portfolios:

Portfolio A
Enter a forward contract to buy 1 unit of asset S at forward price K at time T
Invest $Ke^{-\delta T}$ in the risk free investment

Portfolio B
Buy 1 unit of asset S at the current price S_0

At time 0 the value of Portfolio A is $Ke^{-\delta T}$ (bear in mind that the price of a forward contract at inception is zero).

The value of Portfolio B is S_0

At time T the cash flow of Portfolio A is three fold:

Risk free investment yields $Ke^{-\delta T}.e^{\delta T} = K$
Payment for forward contract K
Receipt from forward contract S_T

Thus total amount is S_T (Risk free investment accumulates to contract payment)

At time T the cash flow of Portfolio B is a payout of S_T

In other words at time T the payout from both portfolios is identical. Applying the "Law of One Price" this means that the value (or price) at time 0 must be the same.

Therefore: $Ke^{-\delta T} = S_0 \Rightarrow K = S_0 .e^{\delta T}$

So the "no arbitrage assumption" means that we have derived a price for the forward contract without any model of how the asset price S will move during the period $0<t<T$.

2. Now assume that there is a fixed payment c due on the asset S at time t between 0 and T (for instance a coupon if the asset is Government stock). Again consider 2 portfolios:

Portfolio A
Enter a forward contract to buy 1 unit of asset S at forward price K at time T
Invest $Ke^{-\delta T} + ce^{-\delta t}$ in the risk free investment

Portfolio B
Buy 1 unit of asset S at the current price S_0
At time t invest the income of c in the risk free investment

At time T the cash flow of Portfolio A is three fold:

Risk free investment yields $K + ce^{\delta(T-t)}$
Payment for forward contract K
Receipt from forward contract S_T

Thus total amount is $S_T + c\, e^{\delta(T-t)}$

At time T the cash flow of Portfolio B is a payout of $S_T + ce^{\delta(T-t)}$

As before, using the "Law of One Price" :

$Ke^{-\delta T} + ce^{-\delta t} = S_0 \Rightarrow K = S_0\, e^{-\delta T} - ce^{\delta(T-t)}$

This can be extended to a series of coupon payments. Let the present value (at time 0) of the payments be I. Then $K = (S_0 - I)\, e^{\delta T}$

3. Finally assume there is a known dividend yield (D per annum) which is received continuously – and is immediately reinvested in the underlying security S. Starting at time 0 a unit investment accumulates to e^{DT} at time T.

Again consider 2 portfolios:

Portfolio A
Enter a forward contract to buy 1 unit of asset S at forward price K at time T
Invest $Ke^{-\delta T}$ in the risk free investment

Portfolio B
Buy e^{-DT} units of asset S at price S_0
Reinvest dividend income in asset S on receipt

At time T the cash flow of Portfolio A is three fold:

Risk free investment yields $Ke^{-\delta T}.e^{\delta T} = K$
Payment for forward contract K
Receipt from forward contract S_T

The payout from Portfolio B is $e^{-DT} e^{DT} S_T = S_T$ (equal to Portfolio A)

As before, using the "Law of One Price":

$Ke^{-\delta T} = S_0 e^{-DT} \Rightarrow K = S_0 e^{(\delta - D)T}$

The distinction between the last 2 examples is the basis on which the income is calculated. If the income is a fixed amount (e.g. a guaranteed coupon) regardless of the price of the underlying asset, then it is important to assume it is invested in a risk free asset. If it is expressed as a proportion of the price of the underlying asset, then it should be assumed to be reinvested in the underlying asset. In this way one can compute the accumulated amount at time T without knowing the intermediate performance of the price of the underlying asset.

Valuation of a Forward Contract

Consider a forward contract entered into at time 0 for 1 unit of security S with maturity at time T. Let the forward price be K_0.

What is the value of the long forward contract at time r where $0<r<T$?

Again consider 2 portfolios – both purchased at time r.

Portfolio A
Buy the existing forward contract at price V_l
Invest $K_0e^{-\delta(T-r)}$ in the risk free asset

Portfolio
Buy a new long forward contract with maturity at T, forward price
$K_r = S_r .e^{\delta(T-r)}$
Invest $K_re^{-\delta(T-r)}$ in the risk free asset

Price of portfolios at time r is:

A: $V_l + K_0e^{-\delta(T-r)}$

B: $K_re^{-\delta(T-r)}$

Payout at time T is:

A: S_T ; $+ K_0$; $-K_0$ Total S_T

B: S_T ; $+ K_r$; $-K_r$ Total S_T

Again using the "Law of One Price" we have:

$V_l + K_0 e^{-\delta(T-r)} = K_r e^{-\delta(T-r)}$ ➔ $V_l = (K_r - K_0)\, e^{-\delta(T-r)}$

Substituting for K gives $V_l = S_r - S_0\, e^{\delta r}$

By general reasoning the value of the short forward contract V_s is $- V_l$.

Hedging

This is the general term which describes the use of financial instruments (from straightforward bonds to highly sophisticated derivative products) which reduce or eliminate the future risk of loss.

For instance an investor may enter into a forward contract to sell an asset (S) at a future date (T) for price K. He need not be holding the asset at the start of the contract but he must have it at the maturity date. To hedge the risk he could borrow an amount $Ke^{-\delta T}$ at the risk free rate and buy asset S at price S_0. At time T the asset is available to sell to the counterparty. The payment of K (the forward price) by the counterparty in settlement exactly matches the amount required to repay the loan.

In this way the investor has avoided any possibility of loss on the contract (but also any possibility of profit!).

This is a "static hedge" because, once put in place, it remains untouched until the maturity of the contract. The more complicated the financial instruments, the more necessity there is for continual re-balancing – a "dynamic hedge".

Hilary Term Lecture 5

Stochastic interest rate models

So far, we usually assumed that we knew all interest rates, or we compared investments under different interest rate assumptions. Any uncertainty of investment proceeds was expressed modelling cash flows by random variables. In practice, interest rates themselves are uncertain, and we model here interest rates by random variables.

5.1 Basic model for one stochastic interest rate

We start off with an elementary example.

Example 1 Suppose, you invest £100 for 1 year at an interest rate I not known in advance. Say, interest rates are at 3% at the moment, you might expect one of three possibilities, a rise by 1%, no change or a fall by 1%, each equally likely, say:

$$P(I = 2\%) = P(I = 3\%) = P(I = 4\%) = 1/3$$

Then the investment proceeds $R = 100(1 + I)$ at the end of the year are random, as well

$$P(R = 102) = P(R = 103) = P(R = 104) = 1/3$$

You can calculate your expected proceeds $E(R) = 1/3(102 + 103 + 104) = 103$ and think, well, this can be calculated using the average interest rate $E(I) = 3\%$, and there is not much reason to study any further.

However, this only works for a term of 1 year. If the term is, say $t \in (0, \infty)$ years, $t \neq 1$, we get $R = 100(1 + I)^t$

$$P(R = 100(1.02)^t) = P(R = 100(1.03)^t) = P(R = 100(1.04)^t) = 1/3$$

and $E(R) = 100/3((1.02)^t + (1.03)^t + (1.04)^t) \neq 100(1.03)^t$.

Assuming that I can only take 3 values is of course an unnecessary restriction, and we can take any random variable I that ranges $(-1, \infty)$, discrete or continuous.

Definition 1 A *basic stochastic interest rate model* is described by a random interest rate I taking values in $(-1, \infty)$.

Proposition 1 *Given a basic stochastic interest rate model I. Let $(0, c)$ be an investment at time 0. Then its expected value at time t is given by*

$$E(I\text{-}Val_t((0,c))) = E(c(1+I)^t)$$

where, $P(I = E(I)) < 1$ and $t < 1$ ($t > 1$) imply that

$$E(I\text{-}Val_t((0,c))) < E(I)\text{-}Val_t((0,c)) \qquad (E(I\text{-}Val_t((0,c))) < E(I)\text{-}Val_t((0,c)))\,.$$

Proof: The proof of the inequalities is essentially an application of Jensen's inequality (see following lemma) for the function $f(x) = (1+x)^t$ which is convex if $t > 1$ and concave if $t < 1$, so that then $-f$ is convex. □

Lemma 1 (Jensen's inequality) *For any convex function $f : \mathbb{R} \to \mathbb{R}$ and any random variable X with $E(X) \in \mathbb{R}$, we have*

$$f(E(X)) \leq E(f(X)).$$

If f is strictly convex and $P(X = E(X)) < 1$, then the inequality is strict.

The result is still true if f is only defined on the interval $(\inf supp(X), \sup supp(X))$ and any boundary value a with $P(X = a) > 0$.

Proof: Recall that for (strictly) convex functions $f(x) \geq f(x_0) + (x - x_0)f_r'(x_0)$ for all $x, x_0 \in \mathbb{R}$ (strict inequality for $x \neq x_0$), where f_r' is the right derivative of f. Applying this, we obtain

$$E(f(X)) \geq E\left(f(E(X)) + (X - E(X))f_r'(E(X))\right) = f(E(X))$$

by linearity of E. □

Obviously, instead of modelling the interest rate I, we could model the force of interest $\Delta = \log(1 + I)$. This is particularly useful since valuation of cash flows then requires only knowledge of the so-called Laplace transforms $E(e^{-t\Delta})$ of Δ.

5.2 Independent annual interest rates

The model in the previous section is artificial, particularly for long terms. It is natural to allow the interest rate to change. The easiest such model is by independent annual interest rates.

Definition 2 An *iid interest rate model* is a collection of independent identically distributed (iid) random annual interest rates I_j, $j \geq 1$, taking values in $(-1, \infty)$, rate I_j being applied the jth year.

Proposition 2 *Given an iid interest rate model, any simple cash flow* (s, c), $s \in \mathbb{N}$ *has an expected value at time* $t \geq s$, $t \in \mathbb{N}$ *given by*

$$E(Val_t((s,c))) = c \prod_{j=s+1}^{t} (1 + E(I_j)) = c(1 + E(I_1))^{t-s}.$$

and at time $t \leq s$, $t \in \mathbb{N}$ *given by*

$$E(Val_t((s,c))) = c \prod_{j=t+1}^{s} E((1 + I_j)^{-1}) = c \left(E((1 + I_1)^{-1})\right)^{s-t}.$$

Proof: For the first statement we calculate

$$E(Val_t((s,c))) = E\left(c \prod_{j=s+1}^{t} (1 + I_j)\right) = c\left(\prod_{j=s+1}^{t} E(1 + I_j)\right)$$

by linearity of E and by the independence of the $(1+I_j)$ factors; remember that $E(XY) = E(X)E(Y)$ for independent random variables X and Y. Each of the $t - s$ factors is now equal to $(1 + E(I_1))$ since the I_j are identically distributed.

The second statement is analogous. Note that $E(1 + I_j) = 1 + E(I_j)$ above, but $E((1 + I_j)^{-1})$ cannot be simplified. □

As just seen, expected accumulated values can be computed fairly easily. Also similar formulas for variances exist, as one measure of risk. Useful formulas for loss probabilities as another measure of risk are only available in special cases. A very popular family of distributions for modelling interest rates is the log-normal distribution.

Definition 3 A random variable X is said to have a lognormal distribution if $Z = \log(X)$ is (well-defined) and Normal. The log-normal distribution $logN(\mu, \sigma^2)$ has two parameters $\mu = E(\log(X))$ and $\sigma^2 = Var(\log(X))$.

Proposition 3 *If* $1 + I$ *has a lognormal distributions with parameters* μ *and* σ^2*, then*

$$j = E(I) = \exp\left\{\mu + \frac{1}{2}\sigma^2\right\} - 1$$

and

$$s^2 = Var(I) = \exp\left\{2\mu + \sigma^2\right\}\left(\exp\{\sigma^2\} - 1\right).$$

Proof: The first statement follows from the formula for the moment generating function of the Normal distribution

$$E(I) = E\left(\exp\{\log(1 + I)\}\right) - 1 = E\left(\exp\{Z\}\right) - 1 = \exp\left\{\mu + \frac{1}{2}\sigma^2\right\} - 1.$$

The second statement follows from

$$\begin{aligned} Var(I) &= Var(1 + I) = E((1 + I)^2) - (E(1 + I))^2 = E\left(\exp\{2Z\}\right) - \left(E\left(\exp\{Z\}\right)\right)^2 \\ &= \exp\{2\mu + 2\sigma^2\} - \exp\{2\mu + \sigma^2\} \end{aligned}$$

and this factorizes as required. □

Note that if $1+I$ has a lognormal distribution, then $\Delta = \log(1+I)$ is Normal.

Example 2 Let $1+I_1, \dots, 1+I_n$ be independent lognormal random variables with common parameters μ and σ^2. We can calculate the distribution of the accumulated value at time n of a unit investment at time 0.

$$S_n = \prod_{j=1}^{n}(1+I_j) = \exp\left\{\sum_{j=1}^{n}\Delta_j\right\} \sim logN(n\mu, n\sigma^2)$$

since sums of independent Normal random variables are Normal with as parameters the sums of the individual parameters.

Assume that $\mu = 0.07$, $\sigma^2 = 0.006$ and $n = 10$. If we want to accumulate at least £600,000 with probability 99%, we have to invest A where

$$\begin{aligned} 0.99 &= P(AS_n > 600,000) = P(\log\{S_n\} > \log\{600,000/A\}) \\ &= P\left(Z > \frac{\log\{600,000/A\} - n\mu}{\sqrt{n\sigma^2}}\right) \\ \Rightarrow -2.33 &= \frac{\log\{600,000/A\} - n\mu}{\sqrt{n}\sigma} \Rightarrow A = 600,000\exp\{2.33\sqrt{n}\sigma - n\mu\} = 527242.59 \end{aligned}$$

Here we used that $P(Z > -2.33) = 0.99$ for a standard Normal random variable Z.

In practice, models for annual changes interest rates are too coarse, but by switching to the appropriate time unit, this problem can be easily overcome.

5.3 Dependent annual interest rates

In practice, interest rates do not fluctuate as strongly as in the iid model. In fact, when interest rates are high, the next year is quite likely to show another high interest rate, similarly with low rates. This can be modelled by centering the new interest rate around the current interest rate, or between the current and a general long term mean interest rate. In general, this leads into the theories of random walks, time series models, Markov chains etc. that lie beyond the scope of this course. We just give an examples that is quite accessible.

Example 3 (2-step dependent lognormal model) Let $1+I_1$ be a log-Normal variable with parameters μ and σ^2 as before, and $1+I_2$ a conditionally lognormal variable with parameters

$$\mu_2 = k\log(1+I_1) + (1-k)\mu$$

and σ^2 for some $k \in [0,1]$. Then given $\log(1+I_1)$, $\log(1+I_2)$ is Normal with these parameters, i.e.

$$\log(1+I_2) = k\log(1+I_1) + (1-k)\mu + N_2$$

where N_2 is independent $\mathcal{N}(0, \sigma^2)$. Therefore $\log(1 + I_2) \sim \mathcal{N}(\mu, (1 + k^2)\sigma^2)$, and

$$\log((1 + I_1)(1 + I_2)) = (1 + k)\log(1 + I_1) + (1 - k)\mu + N \sim \mathcal{N}(2\mu, (2 + 2k + k^2)\sigma^2)$$

This shows that the accumulated value at time 2 of a unit investment at time 0 has a lognormal distribution with parameters 2μ and $(2 + 2k + k^2)\sigma^2$.

Note that for $k = 0$, the model is the iid model studied previously with immediate falling back to mean μ, and for $k = 1$ the centering of the second rate is around the first rate whereas intermediate values of k represent different strengths of reversion to mean μ.

This model can be iterated, although formulas get more complicated. Analogous models for other distributions can be constructed. In their evaluation, one may have to simulate to deduce approximate distributions of accumulated values etc.

5.4 Modelling the force of interest

When reducing the time unit, one can also pass to continuous-time limits. Before we do this, we note that our previous models can be viewed as models with piecewise constant forces of interest.

In 5.1, the force of interest Δ was random, but constant for all time.

In 5.2 and 5.3, the force of interest was constant Δ_j during each time unit $(j - 1, j]$. Whereas in 5.2, forces of interst in different periods were independent, there was a Markovian dependency in 5.3, i.e. the distribution of the force of interest in the following period only depends on previous forces of interest via the current period. $(\Delta_j)_{j\geq 1}$ is a Markov chain.

In 5.3, we can get interesting limits as we let our time unit tend to zero.

E.g., for $k = 1$ we have $\Delta_0 \sim N(\mu, \sigma^2)$ and for $n \geq 0$

$$\Delta_{n+1} = \Delta_n + N_{n+1}, \qquad N_n \sim N(0, \sigma^2) \text{ independent.}$$

We can set up a model for $p$thly changing forces by $\Delta_0^{(p)} \sim N(\mu, \sigma^2)$ and for $n \geq 0$

$$\Delta^{(p)}_{\frac{n+1}{p}} = \Delta^{(p)}_{\frac{n}{p}} + N^{(p)}_{\frac{n+1}{p}}, \qquad N^{(p)}_{\frac{n}{p}} \sim N\left(0, \frac{\sigma^2}{p}\right) \text{ independent,}$$

where we mean that $\Delta^{(p)}_{n/p}$ applies during $((n-1)/p, n/p)$. Note that the two models are consistent in that $\Delta^{(p)}_n \sim \Delta_n$ for all n, p.

In the limit $p \to \infty$ we get a so-called Brownian motion $(\Delta_t)_{t\geq 0}$.

Similarly, for $k \in (0, 1)$, we can get diffusion limits that then retain the mean-reversion (reversion to μ) property of the approximations. These can be described by stochastic differential equations as

$$d\Delta_t = (\log(k))(\Delta_t - \mu)dt + \sigma dB_t.$$

One can also use instantaneous forward rates F_t as the forces of attraction

$$d\Delta_t = (\log(k))(\Delta_t - F_t)dt + \sigma dB_t.$$

5.5 What can one do with these models?

Pricing of derivative contracts (derived from interest rates) can be carried out (arbitrage-free in certain models, or as expectations under so-called martingale measures in more general models, etc.)

Assessmnent of interest rate risk in portfolios

o13 Hilary Term Lecture 6

Back to Life Contingencies

Recap from the end of last term:

Future lifetime of an individual aged x is a random variable T_x continuously distributed on the interval $[0,\omega-x]$ where ω is the "limiting age" – normally taken as 120. When x=0 this is written as T.

$F_x(t)$ is the distribution function of T_x

and $S_x(t) = P[T_x > t] = 1 - F_x(t)$ is the survival function of T_x

Actuarial notation is ${}_tq_x = F_x(t)$

and ${}_tp_x = 1 - {}_tq_x = S_x(t)$

Normally work in units of one year so that t=1 and the t is then dropped:

$q_x = {}_1q_x$ and $p_x = {}_1p_x$

Force of mortality is written as μ_x and defined as:

$$\mu_x = \lim_{h \to 0} \frac{1}{h} * P[T < x+h \,|\, T>x]$$

From definitions above $P[T< x+h \,|\, T>x] = F_x(h) = {}_hq_x$

For small h we can ignore the limit and ${}_hq_x \simeq h.\ \mu_x$

$$S_x(t) = P[T_x > t]$$

$$= P[T > x+t \quad T>x]$$

$$= \frac{P[T > x+t\,]}{P[T>x]}$$

$$= \frac{S(x+t)}{S(x)}$$

or

$${}_tp_x = \frac{{}_{x+t}p_0}{{}_xp_0}$$

$$= \frac{{}_{x+s}p_0}{{}_xp_0} \cdot \frac{{}_{x+s+t}p_0}{{}_{x+s}p_0}$$

$$= {}_sp_x * {}_tp_{x+s}$$

$$f_x(t) = \lim_{h\to 0} \frac{1}{h} \times \left(P[T_x \le t+h] - P[T_x \le t] \right)$$

$$= \lim_{h\to 0} \frac{1}{h} \times \left(P[T \le x+t+h \mid T > x] - P[T \le x+t \mid T > x] \right)$$

$$= \lim_{h\to 0} \frac{1}{S(x).h} \left(P[T \le x+t+h] - P[T \le x] - \left(P[T \le x+t] - P[T \le x] \right) \right)$$

$$= \lim_{h\to 0} \frac{1}{S(x).h} \left(P[T \le x+t+h] - P[T \le x+t] \right)$$

$$= \frac{S(x+t)}{S(x)} \lim_{h\to 0} \frac{1}{h} \frac{\left(P[T \le x+t+h] - P[T \le x+t] \right)}{S(x+t)}$$

$$= S_x(t) \lim_{h\to 0} \frac{1}{h} P\left(T \le x+t+h \mid T > x+t \right)$$

$$= S_x(t) \times \mu_{x+t}$$

$$f_x(t) = {}_tp_x \ \mu_{x+t}$$

$${}_tq_x = F_x(t) = \int_0^t f_x(s)\, ds = \int_0^t {}_sp_x \ \mu_{x+s}\, ds$$

$$\frac{\partial}{\partial s} {}_sp_x = -\frac{\partial}{\partial s} {}_sq_x = -f_x(s) = -{}_sp_x \mu_{x+s}$$

Now $\frac{\partial}{\partial s} \log {}_sp_x = \dfrac{\frac{\partial}{\partial s} {}_sp_x}{{}_sp_x}$

$$\therefore \quad \frac{\partial}{\partial s} \log {}_sp_x = -\mu_{x+s}$$

$$\Longrightarrow \int_0^t \frac{\partial}{\partial s} \log {}_sp_x\, ds = -\int_0^t \mu_{x+s}\, ds + c \quad \text{(integration constant)}$$

$$\text{LHS} = \left[\log {}_sp_x \right]_0^t = \log {}_tp_x \qquad (\text{since } {}_0p_x = 1)$$

Taking exponentials

$${}_tp_x = \exp\left\{ -\int_0^t \mu_{x+s}\, ds + c \right\}$$

Since ${}_0p_x = 1$ $\quad c$ must $= 0$

$$\therefore \quad {}_tp_x = \exp\left\{ -\int_0^t \mu_{x+s}\, ds \right\}$$

Simple Laws of Mortality

Gompertz' Law: $\mu_x = Bc^x$

Makeham's Law $\mu_x = A + Bc^x$

Life Table

Actuarial calculations use ${}_tq_x$ and ${}_tp_x$ extensively and to calculate these,life tables are produced. The basic life table is a tabulation of l_x where x goes from 0 to ω. If l_0 represents the number of lives at age 0, then l_x represents the expected number of those lives who will survive to age x.

Therefore, $l_x = l_0 * {}_xp_0$

d_x is defined as the number of lives expected to die between ages x and x+1.

Therefore $d_x = l_{x+1} - l_x$

$q_x = d_x / l_x$

The English Life Tables (ELT) are produced using population data obtained from the decennial census.

The Continuous Mortality Investigation Bureau (CMIB) produces life tables from data supplied by insurance companies and this relates only to so called "insured lives". In general this means that the rates of mortality will be lower for CMIB tables than for ELT.

The general pattern is:

High mortality just after birth (infant mortality)
Low mortality for the rest of childhood
Higher mortality around ages 18-25 (the "accident hump")
Increasing mortality from middle age onwards.

Many tables produced by CMIB include "select" figures. These figures relate to the first n years after an insurance contract has been underwritten. When someone has been underwritten for an insurance policy, their mortality rates are seen to be better for a period after the underwriting. In recent tables n is taken as 2 (in other words, 2 years after underwriting, mortality is assumed to depend only on age and not on duration since the start of the contract).

Different tables are also produced for the following:

Assured lives and annuitants
Men and women
Smokers and non-smokers

Insurance Contracts

We now bring together the compound interest discount factor v^n with the life contingency factors ${}_tq_x$ and ${}_tp_x$ in order to calculate premium rates for insurance contracts.

Whole Life Assurance

A whole life insurance contract is an agreement to pay a sum assured on the death of the life assured. We will start by considering the case where payment of 1 is made at the end of the year of death. At the start of the insurance let the insured be aged x.

Probability of death in year k is $d_{x+k} / l_x = {}_kp_x \, q_{x+k}$

Therefore the expected present value (EPV) of these payments at the end of year of death is:

$$\sum_{k=0}^{\infty} v^{k+1} \, {}_kp_x \, q_{x+k}$$

The actuarial terminology for this is A_x

If the sum assured is S then the EPV is $S.A_x$

Term Assurance

This is an agreement to pay a sum assured on the death of the life assured on condition that death occurs within a stated period. Let the insured be aged x; the term n years and payment of 1 be at the end of the year of death.

$$EPV = \sum_{k=0}^{n-1} v^{k+1} \, {}_kp_x \, q_{x+k}$$

The actuarial terminology is $A^{1}_{x\overline{n}|}$

Pure Endowment

A pure endowment provides a sum assured at the end of a fixed term on condition that the life assured survives to the end of the term. Let the insured be aged x , the term n and the endowment sum 1.

$$EPV = v^n \, {}_np_x$$

The actuarial terminology is $A_{x\overline{n}|}^{\;\;1}$

Endowment assurance

An endowment assurance is a combination of a pure endowment and a term assurance for the same term. In other words the sum insured is paid on death during the term of the contract or on survival to the end of the contract.

$EPV = A^{1}_{x\overline{n}|} + A_{x\overline{n}|}^{\;\;1}$ which in actuarial terminology is: $A_{x\overline{n}|}$

Life annuity

Consider a contract which pays a sum of 1 at the end of each year as long as a life now aged x survives.

EPV $= \sum_{k=1}^{\infty} v^k {}_kp_x$ which in actuarial terminology is a_x

If the sum is payable at the start of each year:

EPV $= \sum_{k=0}^{\infty} v^k {}_kp_x = a_x + 1$ which in actuarial terminology is $\ddot{a}_x$

Temporary annuity

This is similar to a life annuity but payments are limited to a specified term. Consider a contract which pays a sum of 1 at the end of each of the next n years as long as a life now aged x survives.

EPV $= \sum_{k=1}^{n} v^k {}_kp_x$ which in actuarial terminology is $a_{x\overline{n}|}$

If the sum is payable at the start of each year:

EPV $= \sum_{k=0}^{n-1} v^k {}_kp_x = a_{x\overline{n-1}|} + 1$ which in actuarial terminology is $\ddot{a}_{x\overline{n}|}$

Equivalence

$$A_x = \sum_{k=0}^{\infty} v^{k+1} {}_kp_x q_{x+k} = \sum_{k=0}^{\infty} v^{k+1} {}_kp_x (1 - p_{x+k}) = \sum_{k=0}^{\infty} v^{k+1} ({}_kp_x - {}_{k+1}p_x)$$

$$= v\ddot{a}_x - (\ddot{a}_x - 1)$$

$$= 1 - d\,\ddot{a}_x$$

Continuous functions

Finally we consider cases where payment is made at the date of death (in the case of whole life assurance) or payment is made continuously in the case of a life annuity.

The actuarial terminology for the EPV of a whole life assurance of unit sum assured to a life aged x payable at the date of death is:

$$\bar{A}_x = \int_0^{\infty} v^t {}_tp_x \mu_{x+t}\, dt$$

Similarly for an annuity of 1 per annum payable continuously to a life aged x for as long as he lives:

$$\bar{a}_x = \int_0^{\infty} v^t {}_tp_x\, dt$$

o13 Hilary Term Lecture 7

Premiums and Reserves

Last week we looked at life assurance functions A and a and the relationships between them.

We go on this week to look at the calculations of premiums for various insurance contracts and the reserves that an insurance company needs to hold in respect of its insurance contracts.

Premiums

There are 2 main ways of paying premiums:

Single premiums where one premium is paid at the start of the contract and no further premiums are payable.

Regular premiums where premiums are paid at regular intervals (yearly, half yearly, quarterly, monthly) for a specified term – often the length of term of the contract

Single premiums are straightforwardly the Expected Present Value of the future payment of the sum assured. Thus for a whole life assurance to an individual aged x, the single premium is merely A_x.

For the rest of the lecture we will concentrate on regular premium policies and furthermore, we will look only at annual premiums. We will also consider level premiums (i.e. the premium does not change year to year). Premiums are always payable in advance with the first premium due at the commencement of the contract.

Net Premiums

Net premiums (also known as pure premiums or risk premiums) make no allowance for any expenses. They are calculated using only a given mortality table and interest rate.

In the following cases we are looking at policies issued to a life aged x. Where relevant the term of the contract is n years:

The net premium notation is P.

Thus for a whole life assurance:

$$P_x = A_x / \ddot{a}_x$$

For a term assurance:

$$P^1_{x:\overline{n}|} = A^1_{x:\overline{n}|} / \ddot{a}_{x:\overline{n}|}$$

For an endowment assurance:

$$P_{x\overline{n|}} = A_{x\overline{n|}} / \ddot{a}_{x\overline{n|}}$$

A slightly different type of policy is a whole life policy with premiums only payable for a limited term (say n years). In this case we have:

$$_nP_x = A_x / \ddot{a}_{x\overline{n|}}$$

The same notation is similarly used for continuous functions. Thus for a whole life assurance payable immediately on death, with premiums payable continuously:

$$\bar{P}_x = \bar{A}_x / \bar{a}_x$$

Reserves

For most assurance policies (but not all) the costs of paying benefits increases as the policy proceeds. For instance, in a whole life policy the chance of dying in a given year (q_x) increases with age and thus the expected payments increase year by year. At the beginning of the contract, the premium payable at the start of the year is more than sufficient to cover the expected benefit payments during the year. At later stages, the premium received each year is less than the expected benefit payments during the year. Therefore, the insurance company needs to build up reserves.

We start by looking at the "prospective policy value" for an in force life insurance contract. "In force" means that the policy has started and has not been completed either by death or the expiry of the stated term of the contract (or by surrender of the contract).

The prospective policy value is defined as:

> Expected present value of future outgo less expected present value of future income.

The notation for policy values is V.

For instance $_tV_x$ represents the policy value for a whole life issued to a life aged x at the end of t years. It will be seen that:

$$_tV_x = A_{x+t} - P_x\ddot{a}_{x+t}$$

$$\text{but } P_x = A_x / \ddot{a}_x$$

$$\text{and } A_{x+t} = 1 - d\,\ddot{a}_{x+t}$$

$$_tV_x = (1 - d\,\ddot{a}_{x+t}) - (1 - d\,\ddot{a}_x)\,\ddot{a}_{x+t} / \ddot{a}_x = 1 - \ddot{a}_{x+t} / \ddot{a}_x$$

Recursive Calculation

To prove: $({}_tV_x + P_x)(1+i) = q_{x+t} + p_{x+t} * {}_{t+1}V_x$

Proof: $\quad ({}_tV_x + P_x) = (A_{x+t} - P_x\ddot{a}_{x+t}) + P_x$

but $\quad A_{x+t} = v\,q_{x+t} + v\,p_{x+t}A_{x+t+1}$

and $\quad \ddot{a}_{x+t} = 1 + v\,p_{x+t}\ddot{a}_{x+t+1}$

so
$$({}_tV_x + P_x) = v\,q_{x+t} + v\,p_{x+t}A_{x+t+1} - P_x(1 + v\,p_{x+t}\ddot{a}_{x+t+1}) + P_x$$
$$= v(q_{x+t} + p_{x+t}(A_{x+t+1} - P_x\ddot{a}_{x+t+1}))$$
$$= v(q_{x+t} + p_{x+t} * {}_{t+1}V_x)$$

or $\quad ({}_tV_x + P_x)(1+i) = q_{x+t} + p_{x+t} * {}_{t+1}V_x$

This can also be seen from general reasoning.

The value held at the start of the year, plus the premium collected at the start of the year, accumulate with interest to provide at the end of the year sufficient to provide:

Death benefit (1 payable with probability q_{x+t})
Policy value at the start of the next year (with probability of survival p_{x+t})

Similar equations can be produced for other types of assurance contract.

Thiele's Equation

This is the equivalent for continuous functions.

We have $\quad {}_t\bar{V}_x = \bar{A}_{x+t} - \bar{P}_x\bar{a}_{x+t} = 1 - \bar{a}_{x+t}/\bar{a}_x$

Differentiating $\bar{a}_{x+t}$ we have

$$\frac{\partial}{\partial t}\bar{a}_{x+t} = \frac{\partial}{\partial t}\int_0^\infty e^{-\delta s}\,{}_sp_{x+t}\,ds = \int_0^\infty e^{-\delta s}\frac{\partial}{\partial t}\,{}_sp_{x+t}\,ds$$

$$= \int_0^\infty e^{-\delta s}\,{}_sp_{x+t}(\mu_{x+t} - \mu_{x+t+s})\,ds = \mu_{x+t}\bar{a}_{x+t} - \bar{A}_{x+t}$$

$$\therefore \frac{\partial}{\partial t}\,{}_t\bar{V}_x = \frac{-\mu_{x+t}\bar{a}_{x+t} + \bar{A}_{x+t}}{\bar{a}_x} = -\mu_{x+t}(1 - {}_t\bar{V}_x) + \frac{1 - \delta\bar{a}_{x+t}}{\bar{a}_x}$$

$$= -(1 - {}_t\bar{V}_x)\mu_{x+t} + \delta\left(1 - \frac{\bar{a}_{x+t}}{\bar{a}_x}\right) - \delta + \frac{1}{\bar{a}_x}$$

$$= -(1 - {}_t\bar{V}_x)\mu_{x+t} + \delta\,{}_t\bar{V}_x + \frac{1 - \delta\bar{a}_x}{\bar{a}_x}$$

$$\therefore \frac{\partial}{\partial t}\,{}_t\bar{V}_x = -(1 - {}_t\bar{V}_x)\mu_{x+t} + \delta\,{}_t\bar{V}_x + \bar{P}_x$$

Experience effects

So far we have assumed that events during a particular year are exactly as predicted (i.e. interest earned equals the i assumed and mortality follows the chosen table)

In practice this is not the case and surplus or deficit arises as a result. Consider the position with mortality.

The death strain at risk is the maximum cost to which the insurance company would be liable if an individual policyholder dies. Given that the insurance company holds the policy value at the end of the year ${}_{t+1}V_x$ and the sum assured is 1, the death strain at risk is $1 - {}_{t+1}V_x$.

We can recast the recursive equation as follows:

$$({}_tV_x + P_x)(1+i) = q_{x+t} + p_{x+t} * {}_{t+1}V_x = q_{x+t} + (1-q_{x+t}) * {}_{t+1}V_x = {}_{t+1}V_x + q_{x+t}(1 - {}_{t+1}V_x)$$

In other words, the value at the start of the year, accumulated with premium and interest is sufficient to provide the value at the end of the year plus the death strain at risk in respect of the probability of death during the year.

The "Expected Death Strain" is the amount the company expects to pay extra to the year end reserve (i.e. $q_{x+t}(1 - {}_{t+1}V_x)$)

Now consider a group of N identical policies each for unit sum assured.

The Expected Death Strain (EDS) is $N * (q_{x+t}(1 - {}_{t+1}V_x))$

The "Actual Death Strain" (ADS) is $\Sigma (1 - {}_{t+1}V_x)$ summed across the number of deaths during the year.

The Mortality Profit for the year is then EDS – ADS

Gross Premiums

In contrast to the "net premium" the office or gross premium allows for the expenses of conducting the insurance business.

Expenses can be expressed in a number of ways:

- Fixed £ amounts
- Amounts increasing with inflation
- Amounts as a percentage of the premium
- Amounts as a percentage of the sum assured

For example:

Set out the premium formula for a whole life assurance for a sum assured of £10,000 to a life aged 40 subject to the following expenses:

£100 to establish the policy
30% of the premium as commission at commencement
1.5% of the premium as renewal commission (after first year)
£10 per annum maintenance expenses (after first year)

Let premium be P

$P.\ \ddot{a}_{40} = 100 + 10000\ A_{40} + .3\ P + .015 P a_{40} + 10\ a_{40}$

$P = (100 + 10000\ A_{40} + 10\ a_{40})/(\ddot{a}_{40} - .3 - .015\ a_{40})$

If the maintenance expenses were assumed to increase with inflation at rate e, then the $10\ a_{40}$ item would be calculated at rate j where:

$1+j = (1+i)/(1+e)$ in the same way as the compound interest examples last term.

Hilary Term Lecture 8 Thursday March 11

The following pages are all from the Actuarial Profession's Website (www.actuaries.org.uk) where you will also be able to download other past papers and solutions.

But, on this page, for some light relief after the heavy demands of the term:

DEAR ECONOMIST

Resolving readers' dilemmas with the tools of Adam Smith

Dear Economist,
How do I choose the shortest queue at the supermarket?
Yours sincerely,
P.N., Aylesbury

Dear P.N.,
Mathematicians reckon the odds are against you. If you choose a queue at random, there will be a line on either side of you, and thus a two-thirds chance that one will be faster.

Economists take a more sophisticated view. David Friedman, for instance, argues that the relevant discipline is financial market theory. Choosing the right queue is like picking the right portfolio of shares: if it were obvious which shares were good value, they wouldn't be good value any more. If it were obvious which queue would be quickest, everyone would join it. Naïve attempts to "beat the market" will fail.

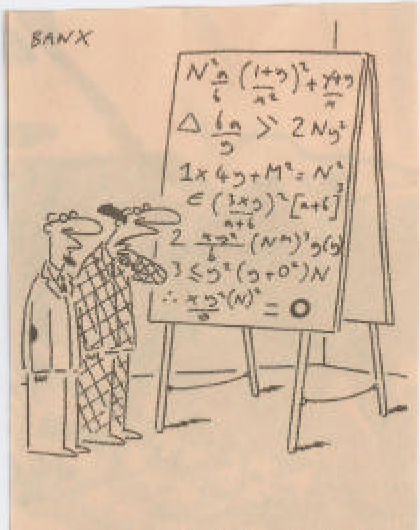
Then there is "efficient market" theory – you can't out-perform a random choice of shares because public information is immediately incorporated into share prices. In truth, most markets are not efficient and thus it is possible for an informed decision-maker to beat them. Even if supermarket queues were efficient, no queue would be a superior bet, because expert supermarket customers would quickly join any queue that was likely to be quicker.

More likely, queues are not efficient because few have much to gain from becoming expert queuers. Some have other considerations, such as minimising the distance walked, while others shop rarely, so the calculations are more trouble than they are worth.

And unlike the stock market, which a financial wizard can make more efficient by outweighing the foolish decisions of small traders, in the supermarket a single expert queuer has a limited effect on the distribution of queuing times.

I can advise you to steer clear of elderly ladies with vouchers, but more advice would be self-defeating. Too many of your rivals would read it.

This week's Economist is Tim Harford, the FT's 2003 Peter Martin fellow. Questions: economist@ft.com



'I've proved maths teachers don't exist'

EXAMINATIONS

3 April 2003 (pm)

Subject 102 — Financial Mathematics

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*
2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*
3. *Mark allocations are shown in brackets.*
4. *Attempt all 11 questions, beginning your answer to each question on a separate sheet.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available Actuarial Tables and your own electronic calculator.

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1 State the main differences between a preference share and an ordinary share. [3]

2 An investor purchased a holding of ordinary shares two months before payment of the next dividend was due. Dividends are paid annually and it is expected that the next dividend will be a net amount of 12p per share. The investor anticipates that dividends will grow at a constant rate of 4% per annum in perpetuity.

Calculate the price per share that the investor should pay to obtain a net return of 7% per annum effective. [4]

3 A businessman is considering an investment which requires an initial outlay of £60,000 and a further outlay of £25,000 in eight months time.

Starting two years after the initial outlay, it is estimated that income will be received continuously for four years at a rate of £5,000 per annum, increasing to £9,000 per annum for the next four years, then increasing to £13,000 per annum for the following four years and so on, increasing by £4,000 per annum every four years until the payment stream stops after income has been received for 20 years (i.e. 22 years after the initial outlay). At the point when the income ceases, the investment can be sold for £50,000.

Calculate the net present value of the project at a rate of interest of 9% per annum effective. [7]

4 (i) Explain what is meant by a "forward contract". Your answer should include reference to the terms "short forward position" and "long forward position". [3]

(ii) An investor entered into a long forward contract for £100 nominal of a security seven years ago and the contract is due to mature in three years time. The price per £100 nominal of the security was £96 seven years ago and is now £148. The risk-free rate of interest can be assumed to be 4% per annum effective during the contract.

Calculate the value of the contract now if the security will pay a single coupon of £7 in two years time and this was known from the outset. You should assume no arbitrage. [5]

[Total 8]

5 A new management team has just taken over the running of a finance company. They discover that the company has liabilities of £15 million due in 13 years time and £10 million due in 25 years time. The assets consist of two zero-coupon bonds, one paying £12.425 million in 12 years time and the other paying £12.946 million in 24 years time. The current interest rate is 8% per annum effective.

Determine whether the necessary conditions are satisfied for the finance company to be immunised against small changes in the rate of interest. [8]

6 An individual is investing in a market in which a variety of spot rates and forward contracts are available.

If at time $t = 0$ he invests £1,000 for two years, he will receive £1,118 at time $t = 2$. Alternatively, if at time $t = 0$ he agrees to invest £1,000 at time $t = 1$ for two years, he will receive £1,140 at time $t = 3$. However, if at time $t = 0$ he agrees to invest £1,000 at time $t = 1$ for one year, he will receive £1,058 at time $t = 2$.

(i) Calculate the following rates per annum effective, implied by this data:

(a) The one-year spot rate at time $t = 0$.
(b) The two-year spot rate at time $t = 0$.
(c) The three-year spot rate at time $t = 0$. [5]

(ii) Calculate the three-year par yield at time $t = 0$ in this market. [3]
[Total 8]

7 The force of interest $\delta(t)$ is a function of time and at any time t, measured in years, is given by the formula:

$$\delta(t) = \begin{cases} 0.05 & 0 \le t \le 3 \\ 0.09 - 0.01t & 3 < t \le 8 \\ 0.01t - 0.03 & 8 < t \end{cases}$$

(i) If £500 is invested at $t = 2$ and a further £800 is invested at $t = 9$, calculate the accumulated amount at $t = 10$. [7]

(ii) Determine the constant effective rate of interest per annum, to the nearest 1%, which would lead to the same result as in (i) being obtained. [3]
[Total 10]

8 A pension fund had assets totalling £40 million on 1 January 2000. It received net income of £4 million on 1 January 2001 and £2 million on 1 July 2001. The value of the fund totalled:

£43 million on 31 December 2000
£49 million on 30 June 2001
£53 million on 31 December 2001

(i) Calculate for the period 1 January 2000 to 31 December 2001, to 3 decimal places:

(a) the time weighted rate of return per annum [3]

(b) the linked internal rate of return, using sub-intervals of a calendar year [5]

(ii) State both in general, and in this particular case, when the linked internal rate of return will be identical to the time weighted rate of return. [2]
[Total 10]

9 £1,000 is invested for 10 years. In any year the yield on the investment will be 4% with probability 0.4, 6% with probability 0.2 and 8% with probability 0.4 and is independent of the yield in any other year.

(i) Calculate the mean accumulation at the end of 10 years. [2]

(ii) Calculate the standard deviation of the accumulation at the end of 10 years. [5]

(iii) Without carrying out any further calculations, explain how your answers to (i) and (ii) would change (if at all) if:

(a) the yields had been 5%, 6% and 7% instead of 4%, 6% and 8% per annum, respectively; or

(b) the investment had been made for 12 years instead of 10 years [4]
[Total 11]

10 A fixed interest security pays coupons of 8% per annum half yearly on 1 January and 1 July. The security will be redeemed at par on any 1 January between 1 January 2006 and 1 January 2011 inclusive, at the option of the borrower.

An investor purchased a holding of the security on 1 January 2001, immediately after the payment of the coupon then due, at a price which gave him a net yield of at least 5% per annum effective. The investor pays tax at 40% on interest income and 30% on capital gains. On 1 January 2003 the investor sold the holding, immediately after the payment of the coupon then due, to a fund which pays no tax at a price to give the fund a gross yield of at least 7% per annum effective.

(i) Calculate the price per £100 nominal at which the investor bought the security. [5]

(ii) Calculate the price per £100 nominal at which the investor sold the security. [3]

(iii) Calculate the net yield per annum convertible half yearly which the investor actually received over the two years the investor held the security. [6]
[Total 14]

11 (i) Prove

$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}. \qquad [3]$$

A loan is repayable by an increasing annuity payable annually in arrears for 15 years. The repayment at the end of the first year is £3,000 and subsequent payments increase by £200 each year. The repayments were calculated using a rate of interest of 8% per annum effective.

(ii) Calculate the original amount of the loan. [3]

(iii) Construct the capital/interest schedule for years nine (after the eighth payment) and ten, showing the outstanding capital at the beginning of the year, the interest element and the capital repayment. [6]

(iv) Immediately after the tenth payment of interest and capital, the interest rate on the outstanding loan is reduced to 6% per annum effective.

Calculate the amount of the eleventh payment if subsequent payments continue to increase by £200 each year, and the loan is to be repaid by the original date, i.e. 15 years from commencement. [5]
[Total 17]

REPORT OF THE BOARD OF EXAMINERS

April 2003

Subject 102 — Financial Mathematics

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

J Curtis
Chairman of the Board of Examiners

3 June 2003

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EXAMINERS' COMMENT

Please note that differing answers may be obtained depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this.

However, candidates maybe penalised where excessive rounding has been used or where insufficient working is shown.

1 A preference share pays a dividend which is generally fixed. An ordinary share pays a dividend out of residual profits which is at the discretion of the company.

A preference share dividend is a prior charge so that, in general, no ordinary share dividend can be paid if a preference share dividend is outstanding.

A preference share holder may have no voting rights and is likely to get prior ranking in a winding up.

Credit was also given for any other relevant points.

2 Let Price = P

$$P = v^{\frac{1}{6}} \times 12(1 + 1.04v + 1.04^2 v^2 + \ldots) \text{ at } 7\%$$

$$= \frac{12}{(1.07)^{\frac{1}{6}}} \times \frac{1}{\left(1 - \frac{1.04}{1.07}\right)}$$

$$= 423.20$$

3 PV of outlay $= 60{,}000 + 25{,}000\, v^{\frac{8}{12}}$ at 9%

$= 83{,}604.19$

PV of income (in '000s)

$$v^2(5\bar{a}_{\overline{4}|} + 9v^4\bar{a}_{\overline{4}|} + 13v^8\bar{a}_{\overline{4}|} + 17v^{12}\bar{a}_{\overline{4}|} + 21v^{16}\bar{a}_{\overline{4}|})$$

$$= v^2\bar{a}_{\overline{4}|}(5 + 9v^4 + 13v^8 + 17v^{12} + 21v^{16})$$

and $\bar{a}_{\overline{4}|} = \frac{i}{\delta} \times a_{\overline{4}|} = 1.044354 \times 3.2397$

$= 3.38339$

$\Rightarrow PV$ of income

$= 0.84168 \times 3.38339\ (5 + 9 \times 0.70843 + 13 \times 0.50187 + 17 \times 0.35553 + 21 \times 0.25187)$

$= 0.84168 \times 3.38339 \times 29.23346$

$= 83.24905$ (i.e. £83,249.05)

PV of final cash flow $= 50{,}000v^{22}$ at 9%

$= 7{,}509.09$

PV of project $= 83{,}249.05 + 7{,}509.09 - 83{,}604.19$

$= £7{,}153.95$

4 (i) A forward contract is an agreement made between two parties under which one agrees to buy from the other a specified amount of an asset at a specified price on a specified future date.

The investor agreeing to sell the asset is said to hold a "short forward position" in the asset, and the buyer is said to hold a "long forward position".

(ii) The forward price at the outset of the contract was:

$$(96 - 7\underset{4\%}{v^9}) \times (1.04)^{10} = 134.82$$

The forward price that should be offered now is:

$$(148 - 7\underset{4\%}{v^2}) \times (1.04)^3 = 159.20$$

Hence, the value of the contract now is:

$$(159.20 - 134.82)\ \underset{4\%}{v^3} = 21.67$$

This result can also be obtained directly from:

$$148 - 96 \times (1.04)^7 = 21.67$$

since the coupon of £7 is irrelevant in this calculation.

5 We will consider the three conditions necessary for immunisation.

(1) $V_A = V_L$

$$V_A = 12.425v^{12} + 12.946v^{24} \quad \text{at 8\% p.a.}$$
$$= 6.9757$$

$$V_L = 15v^{13} + 10v^{25}$$
$$= 6.9756$$

Condition (1) satisfied.

(2) $V'_A = V'_L$ where $V'_A = \dfrac{dV_A}{d\delta}$ and $V'_L = \dfrac{dV_L}{d\delta}$

$$V'_A = 12 \times 12.425v^{12} + 24 \times 12.946v^{24} \quad \text{at 8\% p.a.}$$
$$= 108.207$$

$$V'_L = 13 \times 15v^{13} + 25 \times 10v^{25}$$
$$= 108.206$$

Condition (2) satisfied.

(3) $V''_A > V''_L$

$$V''_A = 12^2 \times 12.425v^{12} + 24^2 \times 12.946v^{24} \quad \text{at 8\% p.a.}$$
$$= 1{,}886.46$$

$$V''_L = 13^2 \times 15v^{13} + 25^2 \times 10v^{25}$$
$$= 1{,}844.73$$

Condition (3) satisfied.

Thus the company is immunised against small changes in the rate of interest.

Candidates could, instead, have differentiated the expressions for V_A and V_L with respect to the interest rate (rather than the force of interest) to earn full marks.

6 (i) Let i_r = the r-year spot rate

$f_{t,r}$ = the r-year forward rate at time t years

$1000(1 + i_2)^2 = 1118$

$1000(1 + f_{1,2})^2 = 1140$

$1000(1 + f_{1,1}) = 1058$

$\Rightarrow i_2 = 5.73552\%$ p.a.

and $(1 + i_1)(1 + f_{1,1}) = (1 + i_2)^2$

$$\Rightarrow 1 + i_1 = \frac{1.118}{1.058} = 1.0567108$$

$\Rightarrow i_1 = 5.67108\%$ p.a.

and $(1 + i_1)(1 + f_{1,2})^2 = (1 + i_3)^3$

$\Rightarrow (1 + i_3)^3 = 1.0567108 \times 1.140$

$= 1.20465$

$\Rightarrow i_3 = 6.40295\%$ p.a.

(ii) Let $C\%$ be the 3-year par yield at time $t = 0$.

$$\text{Then } 100 = \frac{C}{1+i_1} + \frac{C}{(1+i_2)^2} + \frac{C}{(1+i_3)^3} + \frac{100}{(1+i_3)^3}$$

$$\Rightarrow 100 = C\left(\frac{1}{1.0567108} + \frac{1}{(1.0573552)^2} + \frac{1}{(1.0640295)^3}\right) + \frac{100}{(1.0640295)^3}$$

$\Rightarrow 100 = C \times 2.6709035 + 83.0116$

$\Rightarrow C = 6.3605\%$

7 (i) Accumulated amount at $t = 10$ is:

$$500\exp\left(\int_2^{10}\delta(s)ds\right)+800\exp\left(\int_9^{10}\delta(s)ds\right)$$

$$\int_2^{10}\delta(s)ds = \int_2^3 0.05ds+\int_3^8(0.09-0.01s)ds+\int_8^{10}(0.01s-0.03)ds$$

and $\int_2^3 0.05ds = [0.05s]_2^3 = 0.05$

$$\int_3^8(0.09-0.01s)ds = \left[0.09s-\frac{0.01s^2}{2}\right]_3^8$$

$$= 0.40 - 0.225 = 0.175$$

$$\int_8^{10}(0.01s-0.03)ds = \left[\frac{0.01s^2}{2}-0.03s\right]_8^{10}$$

$$= 0.20 - 0.08 = 0.12$$

Hence £500 accumulates to:

$$500e^{0.05+0.175+0.12} = 500e^{0.345}$$

$$= 705.99$$

and for 2nd term (i.e. the accumulation of £800):

$$\int_9^{10}\delta(s)ds = \int_9^{10}(0.01s-0.03)ds = \left[\frac{0.01s^2}{2}-0.03s\right]_9^{10}$$

$$= 0.20 - 0.135$$

$$= 0.065$$

So, £800 accumulates to $800e^{0.065} = 853.73$.

Therefore overall accumulation $= 705.99 + 853.73$

$$= 1{,}559.72$$

(ii) Let i = annual effective rate.

$$500(1 + i)^8 + 800(1 + i) = 1{,}559.72$$

approx: $1300(1 + i)^4 = 1{,}559.72$

$$\Rightarrow i \simeq 4.7\%$$

Try 5%, LHS = 1,578.73

Try 4%, LHS = 1,516.28

$\Rightarrow$ answer = 5% to nearest %

8 (i) (a) $(1 + i)^2 = \dfrac{43}{40} \times \dfrac{49}{43+4} \times \dfrac{53}{49+2}$

$$= 1.164695$$

$\Rightarrow i = 7.921\%$ p.a.

(b) 1st sub-interval = 1/1/2000–31/12/2000

IRR for the interval, i_1 given by

$$1 + i_1 = \frac{43}{40} \Rightarrow i_1 = 7.5\%$$

2nd sub-interval = 1/1/2001–31/12/2001

IRR for this interval, i_2, given by

$$(43 + 4)(1 + i_2) + 2(1 + i_2)^{½} = 53$$

Let $1 + i_2 = x^2$

$$x = \frac{-2 \pm \sqrt{4 + 4 \times 53 \times 47}}{2 \times 47}$$

$$= \frac{-2 \pm 99.8399}{94} = 1.04085 \text{ (taking +ve root)}$$

$\Rightarrow 1 + i_2 = x^2 = 1.08337$

i.e. $i_2 = 8.337\%$ p.a.

Hence linked IRR is i where

$$(1 + i)^2 = 1.075 \times 1.08337$$

$\Rightarrow i = 7.918\%$ p.a.

(ii) LIRR = TWRR when the sub-intervals chosen for the LIRR coincide with the intervals between net cash flows being received.

In this instance, the sub-intervals would need to be:

1/1/2000–31/12/2000, 1/1/2001–30/6/2001

and 1/7/2001–31/12/2001

9 (i) $j = 0.04 \times 0.4 + 0.06 \times 0.2 + 0.08 \times 0.4$

$= 0.06$

$\Rightarrow$ mean accumulation $= 1{,}000 \times (1 + j)^{10}$

$= 1{,}000 \times (1.06)^{10}$

$= £1{,}790.85$

(ii) $s^2 = 0.04^2 \times 0.4 + 0.06^2 \times 0.2 + 0.08^2 \times 0.4 - 0.06^2$

$= 0.00392 - 0.00360$

$= 0.00032$

Var(accumulation) $= 1{,}000^2\{(1 + 2j + j^2 + s^2)^{10} - (1 + j)^{20}\}$

$= 1{,}000^2\{1.12392^{10} - (1.06)^{20}\}$

$= 9{,}145.60$

SD(accumulation) $= \sqrt{9145.60} = £95.63$

(iii) (a) By symmetry $j = 0.06$ (as in (i))

Hence, mean(accumulation) will be same as in (i) (i.e. £1,790.85).

The spread of the yields around the mean is lower than in (i). Hence, the standard deviation of the accumulation will be lower than £95.63.

(b) Mean(accumulation) > £1,790.85 since the investment is being accumulated over a longer period.

SD(accumulation) > £95.63 since investing over a longer term than in (i) will lead to a greater spread of possible accumulated amounts.

10 (i) $g(1 - t_1) = 0.08 \times 0.6 = 0.048 < i^{(2)}_{5\%} = 0.04939$

$\Rightarrow$ Capital gain $\Rightarrow$ so assume redeemed as late as possible

$$P = 0.6 \times 8a^{(2)}_{\overline{10|}} + 100v^{10} - 0.3(100 - P)v^{10} \quad \text{at } 5\%$$

$$P = 4.8 \times 1.012348 \times 7.7217 + 100 \times 0.61391 - 30 \times 0.61391 + 0.3P \times 0.61391$$

$$\Rightarrow P = \frac{98.9128 - 18.4173}{1 - 0.184173} = 98.67$$

(ii) $g = 0.08 > i^{(2)}_{7\%} = 0.068816$

$\Rightarrow$ assume redeemed as early as possible

Let P' = Price at which investor sold the security.

Then $P' = 8a^{(2)}_{\overline{3|}} + 100v^3 \quad$ at 7%

$= 8 \times 1.017204 \times 2.6243 + 100 \times 0.81630$

$= 102.99$

(iii) Work in half years — CGT payable

CGT = 0.3(102.99 − 98.67) = 1.30

So $98.67 = 0.6 \times 4a_{\overline{4}|} + (102.99 - 1.30)v^4$

$= 2.4a_{\overline{4}|} + 101.69v^4]$

Try 3%, RHS $= 2.4 \times 3.7171 + 101.69 \times 0.88849$

$= 99.272$

Try 4%, RHS $= 2.4 \times 3.6299 + 101.69 \times 0.85480$

$= 95.636$

$$i = 0.03 + \frac{99.272 - 98.670}{99.272 - 95.636} \times 0.01$$

$= 3.166\%$

$\Rightarrow$ Answer $= 2 \times 3.166 = 6.33\%$ p.a. convertible half yearly

11 (i) $(Ia)_{\overline{n}|} = v + 2v^2 + 3v^3 + \ldots + nv^n$

$(1+i)(Ia)_{\overline{n}|} = 1 + 2v + 3v^2 + \ldots + nv^{n-1}$

$\Rightarrow i(Ia)_{\overline{n}|} = (1 + v + v^2 + \ldots v^{n-1}) - nv^n$

$$\Rightarrow (Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$$

(ii) Loan $= 200(Ia)_{\overline{15}|} + 2800a_{\overline{15}|}$ @ 8%

$$(Ia)_{\overline{15}|} = \frac{\ddot{a}_{\overline{15}|} - 15v^{15}}{0.08} = \frac{1.08 \times 8.5595 - 15 \times 0.31524}{0.08}$$

$= 56.4458$

$\Rightarrow$ Loan $= 200 \times 56.4458 + 2,800 \times 8.5595$

$= 35,255.76$

(iii) Capital o/s after 8th payment

$= 200(Ia)_{\overline{7}|} + 4,400a_{\overline{7}|}$ @ 8%

$$(Ia)_{\overline{7}|} = \frac{\ddot{a}_{\overline{7}|} - 7v^7}{0.08} = \frac{1.08 \times 5.2064 - 7 \times 0.58349}{0.08}$$

$= 19.2310$

$\Rightarrow$ Cap o/s = 200 × 19.2310 + 4,400 × 5.2064

= 26,754.36

Year	*Loan o/s at start*	*Repayment*	*Interest element*	*Capital element*
9	26,754.36	4,600	2,140.35	2,459.65
10	24,294.71	4,800	1,943.58	2,856.42

(iv) Loan o/s after 10th payment

= 24,294.71 − 2,856.42 = 21,438.29

Let 11th payment be X then

$$200(Ia)_{\overline{5}|} + (X - 200)a_{\overline{5}|} = 21,438.29 \quad @\ 6\%$$

$$(Ia)_{\overline{5}|} = \frac{1.06 \times 4.2124 - 5v^5}{0.06} = 12.1476$$

Hence 200 × (12.1476 − 4.2124) + X × 4.2124 = 21,438.29

$\Rightarrow X = 4,712.57$