B.6 Time change

1. (a) First note that

$$\begin{aligned}\mathbb{E}\left(\sum_{j=1}^{[2^n y]}(Z_{j2^{-n}} - Z_{(j-1)2^{-n}})^2\right) &= \sum_{j=1}^{[2^n y]} \mathrm{Var}\left((B_{f(j2^{-n})} - B_{f((j-1)2^{-n})}\right) \\ &= \sum_{j=1}^{[2^n y]}(f(j2^{-n}) - f((j-1)2^{-n})) \\ &= f([2^n y]2^{-n}) - f(0) = f([2^n y]2^{-n}) \to f(y),\end{aligned}$$

as $n \to \infty$. For L^2-convergence we then calculate

$$\begin{aligned}&\mathbb{E}\left(\left(\sum_{j=1}^{[2^n y]}(Z_{j2^{-n}} - Z_{(j-1)2^{-n}})^2 - f(y)\right)^2\right) \\ &= \mathrm{Var}\left(\sum_{j=1}^{[2^n y]}(Z_{j2^{-n}} - Z_{(j-1)2^{-n}})^2\right) + (f([2^n y]2^{-n}) - f(y))^2 \\ &\le \left(\sum_{j=1}^{[2^n y]}(f(j2^{-n}) - f((j-1)2^{-n}))^2\right)\mathrm{Var}(B_1^2) + (f([2^n y]2^{-n}) - f(y))^2 \\ &\to [f]_y \mathrm{Var}(B_1^2) = 0,\end{aligned}$$

provided that f is continuous (and increasing). Convergence in L^2 implies convergence in probability.

(b) Note first that both Z and $\widetilde{Z}$ are continuous. For marginal distributions, note that $Z_y = B_{f(y)} \sim \mathrm{Normal}(0, f(y))$ and, for $y_j \le y < y_{j+1}$,

$$\widetilde{Z}_y = \sum_{i=1}^{j}\sigma_i(W_{y_i} - W_{y_{i-1}}) + \sigma_{j+1}(W_y - W_{y_j})$$

is the sum of independent $\sigma_i(W_{y_i} - W_{y_{i-1}}) \sim \mathrm{Normal}(0, \tau_i^2)$, where

$$\tau_i^2 = \sigma_i^2(y_i - y_{i-1}) = \int_{y_{i-1}}^{y_i} f'(s)ds = f(y_i) - f(y_{i-1}),$$

and these variances add up to $f(y)$, as well. As for joint distributions, Z and $\widetilde{Z}$ have independent increments: for $0 = u_0 < u_1 < \ldots < u_n$

$$Z_{u_k} - Z_{u_{k-1}} = B_{f(u_k)} - B_{f(u_{k-1})} \sim \mathrm{Normal}(0, f(u_k) - f(u_{k-1})),$$

are independent as increments of B; similarly, increments $\widetilde{Z}_{u_k} - \widetilde{Z}_{u_{k-1}}$, for $y_{l_k-1} < u_{k-1} \le y_{l_k}$ and $y_{r_k-1} < u_k < y_{r_k}$, are independent as linear combinations (for $l_k < r_k$, just a multiple for $l_k = r_k$) of increments of W

$$\begin{aligned}\widetilde{Z}_{u_k} - \widetilde{Z}_{u_{k-1}} &= \sigma_{l_k}(W_{y_{l_k}} - W_{u_{k-1}}) + \sum_{i=l_k+1}^{r_k-1}\sigma_i(W_{y_i} - W_{y_{i-1}}) + \sigma_{r_k}(W_{u_i} - W_{y_{r_i-1}}) \\ &\sim \mathrm{Normal}(0, f(u_k) - f(u_{k-1})).\end{aligned}$$

2. (a) Let $0 \leq y_0 \leq \ldots \leq y_n$. Since f is increasing with range $[0,\infty)$, this implies $0 \leq f(y_0) \leq \ldots \leq f(y_n)$. By the independence of increments of X, the following random variables are independent:

$$Z_{y_0} = X_{f(y_0)},\ Z_{y_1} - Z_{y_0} = X_{f(y_1)} - X_{f(y_0)}, \ldots, Z_{y_n} - Z_{y_{n-1}} = X_{f(y_n)} - X_{f(y_{n-1})}.$$

(b) Let $y_n \downarrow y_0$. Then by right-continuity and monotonicity of f, either $f(y_n) = f(y_0)$ for n large enough (if f is locally constant to the RHS of y_0) or $f(y_n) \downarrow f(y_0)$ (otherwise). In the first case trivially, in the second case by right-continuity of X, we obtain $Z_{y_n} = X_{f(y_n)} \to X_{f(y_0)} = Z_{y_0}$.

Now let $y_n \uparrow y_0$. Then by left limits and monotonicity of f, either $f(y_n) = f(y_0)$ for n large enough (if f is locally constant to the LHS of y_0) or $f(y_n) \uparrow f(y_0)$ (otherwise). In the first case, $Z_{y_n} = X_{f(y_n)} \to X_{f(y_0)} = Z_{y_0}$, in the second case $Z_{y_n} = X_{f(y_n)} \to X_{f(y_0)-} = Z_{y_0-}$.

(c) Note that $\mathbb{E}(e^{i\lambda Z_y}) = \mathbb{E}(e^{i\lambda X_{f(y)}}) = e^{-f(y)\psi(\lambda)}$. If $\psi(\lambda) = 0$ for all $\lambda \in \mathbb{R}$, then $X \equiv 0$. Otherwise, let $\lambda \in \mathbb{R}$ such that $\psi(\lambda) \neq 0$. Then stationarity of increments means for all $x \geq 0$ and $y \geq 0$ that

$$f(y+x) - f(y) = -\frac{1}{\psi(\lambda)} \log\left(\mathbb{E}(e^{i\lambda(Z_{y+x}-Z_y)})\right) = -\frac{1}{\psi(\lambda)} \log\left(\mathbb{E}(e^{i\lambda(Z_x)})\right) = f(x).$$

but this is linearity of f.

(d) Since $Z_y = X_{f(y)}$ and $X_{f(y)}$ is infinitely divisible, this is trivial. We have

$$\begin{aligned}\mathbb{E}(e^{i\lambda Z_y}) &= e^{-f(y)\psi(\lambda)} \\ &= \exp\left(-i\lambda f(y)a - \frac{1}{2}\lambda^2 f(y)\sigma^2 - \int_{-\infty}^{\infty}(1 - e^{i\lambda x} - i\lambda x 1_{\{|x|\leq 1\}})f(y)g(x)dx\right),\end{aligned}$$

so the characteristics are $(f(y)a, f(y)\sigma^2, f(y)g)$.

3. (a) Denote the jump intensity of X by λ and the jump density by h. Since f is differentiable, it is continuous and the jumps of Z are $\Delta Z_y = \Delta X_{f(y)}$. Then

$$\begin{aligned} N((a,b]\times(c,d]) &= \#\{y \in (a,b] : \Delta Z_y \in (c,d]\} \\ &= \#\{t \in (f(a),f(b)] : \Delta X_t \in (c,d]\} \\ &\sim \text{Poi}\left(\int_a^b f'(y)dy \int_c^d \lambda h(x)dx\right). \end{aligned}$$

We read off the intensity function as $g(y,x) = f'(y)\lambda h(x)$. Z can be constructed from a Poisson point process $(\Delta_y)_{y\geq 0}$ with intensity function g as $Z_y = \sum_{s\leq y}\Delta_s$, $y \geq 0$.

(b) If $\Delta f(s) > 0$, then $\Delta Z_s = X_{f(s)} - X_{f(s-)}$ is an increment of X of length $\Delta f(s)$ and so by stationarity of increments of X,

$$\mathbb{E}(e^{\gamma \Delta Z_s}) = \mathbb{E}(e^{\gamma X_{\Delta f(s)}}) = \exp\left\{\Delta f(s) \int_{\mathbb{R}} (e^{\gamma x} - 1)\lambda h(x)dx\right\}.$$

Since the jump sizes are continuously distributed, $\mathbb{P}(\Delta Z_s = 0)$ is the probability of no jump in the time interval $(f(s-), f(s))$, i.e. $e^{-\lambda \Delta f(s)}$. If the jump sizes are not continuously distributed, this probability may be bigger (if X can return to 0 after several jumps).

(c) Since Z and $\widetilde{Z}_y = Z^0_y + \sum_{0\le s\le y} J_s$ have independent increments, we just check

$$\begin{aligned}\mathbb{E}(e^{\gamma \widetilde{Z}_y}) &= \exp\left\{\int_0^y f_0'(s)ds \int_{\mathbb{R}} (e^{\gamma x}-1)\lambda h(x)dx\right\} \\ &\qquad \prod_{0\le s\le y} \exp\left\{\Delta f(s)\int_{\mathbb{R}}(e^{\gamma x}-1)\lambda h(x)dx\right\} \\ &= \exp\left\{\left(f_0(y)+\sum_{0\le s\le y}\Delta f(s)\right)\int_{\mathbb{R}}(e^{\gamma x}-1)\lambda h(x)dx\right\} = \mathbb{E}(e^{\gamma Z_y}).\end{aligned}$$

4. Take a Poisson process X of rate λ and a continuous function f with piecewise constant derivative. Then the process $Z = (X_{f(y)})_{y\ge 0}$ has jumps of size 1 only. However, if there is an interval $[y_{j-1}, y_j)$ with $f'(y) = \sigma_j \neq 1$, $y \in [y_{j-1}, y_j)$ and $\sigma_j \neq 1$ for some $j \ge 1$, then there is positive probability that $\widetilde{Z}_y = \int_0^y \sqrt{f'(s)}dX_s$ has jumps of size σ_j, specifically, there will be a $\mathrm{Poi}(\lambda(y_{j+1}-y_j))$ number of such jumps in the time interval $(y_j, y_{j+1}]$. Therefore, the processes have different distributions. So only for $f(y) = y$, the distributions of the processes will coincide.

   (a) To be specific, for $f_1(y) = y$ and $f_2(y) = 2y$, we obtain

$$X_{f_2(y)} \sim \mathrm{Poi}(2\lambda y) \qquad \text{and} \qquad \sum_{k=1}^{X_y}\sqrt{f_2'(T_k)} = \sqrt{2}X_y,$$

   only takes multiples of $\sqrt{2}$ as values.

   (b) The wording of the question suggests to compare distributions for fixed y. However, both processes have independent increments, so if $Z_y \sim \widetilde{Z}_y$, then for $0 \le x \le y$

$$\mathbb{E}(e^{\gamma Z_y}) = \mathbb{E}(e^{\gamma(Z_y - Z_x)})\mathbb{E}(e^{\gamma Z_x})$$

   and

$$\mathbb{E}(e^{\gamma(Z_y-Z_x)}) = \mathbb{E}(e^{\gamma Z_y})/\mathbb{E}(e^{\gamma Z_x}) = \mathbb{E}(e^{\gamma \widetilde{Z}_y})/\mathbb{E}(e^{\gamma \widetilde{Z}_x}) = \mathbb{E}(e^{\gamma(\widetilde{Z}_y - \widetilde{Z}_x)}).$$

   and similarly, finite-dimensional distributions coincide. Since both processes are right-continuous with left limits, they have the same distribution, so, by the reasoning at the beginning of the solution to this question, $f(y) = y$ is the only possible function.

Alternatively, one can study the distribution of the first jump time. The process $\widetilde{Z}$ has the same jump times as X (unless $f'(y) = 0$ for some y). For the time-changed process Z this is related to Question A.2.1, since by Question A.6.3 the jump counting measure is a Poisson counting measure.

5. (a) If $\mathrm{Var}(X_1) < \infty$ (and hence $\mathrm{Var}(X_t) = t\mathrm{Var}(X_1)$ and $\mathbb{E}(X_t) = t\mathbb{E}(X_1)$) and $\mathrm{Var}(\tau_1) = \int_0^\infty t^2 g_\tau(t)dt < \infty$, we check the stronger integrability condition

$$\int_{-\infty}^{\infty} z^2 g(z)dz = \int_{-\infty}^{\infty} z^2 \int_0^\infty f_t(z)g_\tau(t)dtdz$$

$$
\begin{aligned}
&= \int_0^\infty \int_{-\infty}^\infty z^2 f_t(z) dz g_\tau(t) dt \\
&= \int_0^\infty (\mathrm{Var}(X_t) + (\mathbb{E}(X_t))^2) g_\tau(t) dt \\
&= \int_0^\infty (t\mathrm{Var}(X_1) + t^2(\mathbb{E}(X_1))^2) g_\tau(t) dt < \infty.
\end{aligned}
$$

(b) If τ is a compound Poisson process, i.e. $\int_0^\infty g_\tau(t)dt < \infty$, then

$$\int_{-\infty}^\infty g(z)dz = \int_0^\infty \int_{-\infty}^\infty f_t(z) dz g_\tau(t) dt = \int_0^\infty g_\tau(t) dt < \infty.$$

If X is a compound Poisson process with intensity λ and such that $\mathbb{P}(X_t \in (a,b)) = \int_a^b f_t(x)dx$ for all $(a,b) \not\ni 0$ and $\mathbb{P}(X_t \neq 0) = 1 - e^{-\lambda t}$, then

$$\int_{\mathbb{R}\setminus\{0\}} g(z)dz = \int_0^\infty \int_{\mathbb{R}\setminus\{0\}} f_t(z) dz g_\tau(t) dt = \int_0^\infty (1 - e^{-\lambda t}) g_\tau(t) dt < \infty.$$

Note that (a) and (b) deal, respectively, with the integrability condition for small z and large z. The general case, when neither the conditions of (a) nor of (b) are satisfied, we know that we still obtain a Lévy density, from the calculation of characteristic functions in the lectures, but the integrability condition for Lévy densities is difficult to check directly.