B.5 Financial models

1. (a) We want to simulate from $f(x) = \lambda^{-1}Ce^{-Mx}x^{-Y-1}$, $x > 1$. Take $h(x) = \gamma e^{-\gamma(x-1)}$, $x > 1$. Then we can use the rejection method if there is $c > 0$ such that $h(x) \geq cf(x)$, so we calculate

$$c(\gamma) = \min\left\{\frac{h(x)}{f(x)} : x \in (1,\infty)\right\}.$$

For $\gamma > M$, the ratio decreases to zero as $x \to \infty$. For $0 < \gamma \leq M$, the function

$$x \mapsto \frac{h(x)}{f(x)} = \frac{\lambda\gamma e^{\gamma}}{C}x^{Y+1}e^{(M-\gamma)x}$$

is increasing on $(1,\infty)$, so the minimum is attained at $x = 1$. We obtain

$$c(\gamma) = \frac{\lambda\gamma e^{M}}{C}$$

and this is maximised for $\gamma = M$. The number of trials is geometric with parameter $c(M)$, so the expected number of trials is $1/c(M)$.

(b) Let $\lambda(a,b) = \int_a^b g(x)dx$. For an interval $(a,b]$, we can use $h(x) = 1/(b-a)$ to simulate from $f(x) = \lambda(a,b)Ce^{-Mx}x^{-Y-1}$, $a < x \leq b$. Since f is decreasing, we obtain $h(x) \geq cf(x)$ for

$$c = \frac{e^{Ma}a^{Y+1}}{(b-a)\lambda(a,b)}.$$

(c) To simulate a CGMY process, select $1 = a_1 > \ldots > a_k = \varepsilon$. Simulate a compound Poisson process $P^{(0)}$ according to (a) and compensated compound Poisson processes $P^{(j)}$ according to (b) applied to $(a,b) = (a_j, a_{j+1})$, $1 \leq j < k$. Similarly, simulate $N^{(0)}$ according to (a) $N^{(j)}$ according to (b) with M replaced by G. Then

$$X_t^{(2,\varepsilon)} = \sum_{j=0}^{k-1}(P_t^{(j)} - N_t^{(j)}) - b_\varepsilon t, \quad \text{where } b_\varepsilon = -a - \int_\varepsilon^1 xg(x)dx + \int_{-1}^{\varepsilon}|x|g(x)dx$$

is an approximation of X with jumps of sizes smaller than ε thrown away.

(d) This is bookwork.

2. (a) $W_0 = T_0 + U_0 + V_0$, $W_1(\omega_1) = T_0e^{\delta} + U_0B_1^{\text{up}} + V_0C_1^{\text{up}}$, $W_1(\omega_2) = T_0e^{\delta} + U_0B_1^{\text{up}} + V_0C_1^{\text{down}}$, $W_1(\omega_3) = T_0e^{\delta} + U_0B_1^{\text{down}} + V_0C_1^{\text{up}}$ and $W_1(\omega_4) = T_0e^{\delta} + U_0B_1^{\text{down}} + V_0C_1^{\text{down}}$.

(b) By general reasoning, there is arbitrage if one asset is uniformly better than another asset. In particular:

- If $B_1(\omega_1) \leq A_1$, then $(1,-1,0)$ is an arbitrage portfolio, since $W_0 = 0$ and $W_1 \geq 0$ with $W_1(\omega_3) = W_1(\omega_4) > 0$.
- If $A_1 \leq B_1(\omega_4)$, then $(-1,1,0)$ is an arbitrage portfolio, since $W_0 = 0$ and $W_1 \geq 0$ with $W_1(\omega_1) = W_1(\omega_2) > 0$.

- Similarly $(1, 0, -1)$ or $(-1, 0, 1)$ are arbitrage portfolios if $C_1(\omega_1) \leq A_1$ or $A_1 \leq C_1(\omega_4)$.

These can also be deduced from the standard two-asset binary model (A, B) or (A, C). Now let $B_1^{\text{up}} > A_1 > B_1^{\text{down}}$ and $C_1^{\text{up}} > A_1 > C_1^{\text{down}}$. Since the model (A, B) has no arbitrage, there is no arbitrage portfolio of the form $(T_0, U_0, 0)$. Assume that $(T_0, U_0, 1)$ is an arbitrage portfolio. Then $0 = W_0 = T_0 + U_0 + 1$, $W_1(\omega_1) > W_1(\omega_2) \geq 0$ and $W_1(\omega_3) > W_1(\omega_4) \geq 0$.

- If $U_0 \geq 0$, then we have $0 \leq W_1(\omega_4) = T_1 A_1 + U_0 B_1^{\text{down}} + C_1^{\text{down}} < (T_1 + U_0 + 1)A_1 = 0$, which is a contradiction.
- If $U_0 \leq 0$, then we have $0 \leq W_1(\omega_2) = T_1 A_1 + U_0 B_1^{\text{up}} + C_1^{\text{down}} < (T_1 + U_0 + 1)A_1 = 0$, which is a contradiction.

Similarly, now assume that $(T_0, U_0, -1)$ is an arbitrage portfolio, then $0 = W_0 = T_0 + U_0 - 1$, $W_1(\omega_2) > W_1(\omega_1) \geq 0$ and $W_1(\omega_4) > W_1(\omega_3) \geq 0$.

- If $U_0 \geq 0$, then we have $0 \leq W_1(\omega_3) = T_1 A_1 + U_0 B_1^{\text{down}} - C_1^{\text{up}} < (T_1 + U_0 - 1)A_1 = 0$, which is a contradiction.
- If $U_0 \leq 0$, then we have $0 \leq W_1(\omega_1) = T_1 A_1 + U_0 B_1^{\text{up}} - C_1^{\text{up}} < (T_1 + U_0 + 1)A_1 = 0$, which is a contradiction.

So there is no arbitrage portfolio.

(c) The contingent claim $W_1(\omega_1) = 1$, $W_1(\omega_2) = W_1(\omega_3) = W_1(\omega_4) = 0$ cannot be hedged, since we would require

$$\begin{aligned} 0 &= T_0 A_1 + U_0 B_1^{\text{up}} + V_0 C_1^{\text{down}} \\ &= T_0 A_1 + U_0 B_1^{\text{down}} + V_0 C_1^{\text{down}} = T_0 A_1 + U_0 B_1^{\text{down}} + V_0 C_1^{\text{up}}, \end{aligned}$$

for $\omega_2, \omega_3, \omega_4$, and these imply $T_0 = U_0 = V_0 = 0$, but then the fourth equation $1 = T_0 A_1 + U_0 B_1^{\text{up}} + V_0 C_1^{\text{up}}$ fails.

(d) Since the contingent claim does not change as C_1 varies, we should consider portfolios of the form $(T_0, U_0, 0)$. Since the model (A, B) with scenarios "up" and "down" is complete, the contingent claim $\widetilde{W}_1(\text{up}) = W_1(\omega_1)$, $\widetilde{W}_1(\text{up}) = W_1(\omega_3)$ can be hedged. Specifically,

$$\widetilde{W}_1(\text{up}) = T_0 A_1 + U_0 B_1^{\text{up}} \qquad \text{and} \qquad \widetilde{W}_1(\text{down}) = T_0 A_1 + U_0 B_1^{\text{down}}$$

has solution

$$T_0 = \frac{\widetilde{W}_1(\text{down}) B_1^{\text{up}} - \widetilde{W}_1(\text{up}) B_1^{\text{down}}}{A_1(B_1^{\text{up}} - B_1^{\text{down}})} \qquad \text{and} \qquad U_0 = \frac{\widetilde{W}_1(\text{up}) - \widetilde{W}_1(\text{down})}{B_1^{\text{up}} - B_1^{\text{down}}},$$

and so we read off from

$$\widetilde{W}_0 = T_0 + U_0 = \frac{A_1 - B_1^{\text{down}}}{A_1(B_1^{\text{up}} - B_1^{\text{down}})} \widetilde{W}_1(\text{up}) + \frac{B_1^{\text{up}} - A_1}{A_1(B_1^{\text{up}} - B_1^{\text{down}})} \widetilde{W}_1(\text{down}) \quad (1)$$

that

$$q_B = \mathbb{P}(B_1 = B_1^{\text{up}}) = \frac{A_1 - B_1^{\text{down}}}{B_1^{\text{up}} - B_1^{\text{down}}} \in (0, 1).$$

The martingale property is equation (1) for the contingent claim $\widetilde{W}_1(\text{down}) = B_1^{\text{down}}$ and $\widetilde{W}_1(\text{up}) = B_1^{\text{up}}$. The martingale probability q_B is unique and does not depend on $\widetilde{W}_1$.

(e) By symmetry, contingent claims of the form $W_1(\omega_1) = W_1(\omega_3)$, $W_1(\omega_2) = W_1(\omega_4)$ can be hedged and priced as $e^{-\delta}\mathbb{E}(W_1)$, where

$$q_C = \mathbb{P}(C_1 = C_1^{\text{up}}) = \frac{A_1 - C_1^{\text{down}}}{C_1^{\text{up}} - C_1^{\text{down}}} \in (0,1).$$

The process $e^{-\delta t}C_t$, $t = 0, 1$, is a martingale under these probabilities.

(f) In order for both $e^{-\delta t}B_t$ and $e^{-\delta t}C_t$ to be martingales, we need

$$q_B = \mathbb{P}(B_1 = B_1^{\text{up}}, C_1 = C_1^{\text{up}}) + \mathbb{P}(B_1 = B_1^{\text{up}}, C_1 = C_1^{\text{down}}) = p_1 + p_2$$

and

$$q_C = \mathbb{P}(B_1 = B_1^{\text{up}}, C_1 = C_1^{\text{up}}) + \mathbb{P}(B_1 = B_1^{\text{down}}, B_1 = B_1^{\text{down}}) = p_1 + p_3.$$

Together with $p_1 + p_2 + p_3 + p_4 = 1$, we have three equations (of rank three) for four unknowns, so there is a one-dimensional solution space.

(g) The range of arbitrage-free prices $W_0 = e^{-\delta}p_1$ depends on q_B and q_C as follows.

- If $q_B + q_C \leq 1$, then p_1 can be arbitrarily close to zero, and then W_0 will be arbitrarily close to zero.
- If $q_B + q_C > 1$, then $q_B + q_C = 2p_1 + p_2 + p_3 < p_1 + 1$ and so $p_1 > q_B + q_C - 1$ and so $W_0 > e^{-\delta}(q_B + q_C - 1)$.
- Clearly $p_1 < \min\{q_B, q_C\}$ and so $W_0 < e^{-\delta}\min\{q_B, q_C\}$.

So we get $e^{\delta}W_0 \in (\max\{0, q_B + q_C - 1\}, \min\{q_B, q_C\})$. Note that this range is always non-empty.

3. (a) The direct proof is to calculate the moment generating function of $X_i^{(\varepsilon)}$

$$\mathbb{E}(e^{\gamma X_i^{(\varepsilon)}}) = e^{-\gamma\mu\varepsilon}e^{-\lambda\varepsilon} + e^{\gamma(1-\mu\varepsilon)}(1 - e^{-\lambda\varepsilon}) = e^{-\gamma\mu\varepsilon}(1 + (1 - e^{-\lambda\varepsilon})(e^{\gamma} - 1))$$

and to see

$$\mathbb{E}(e^{\gamma S_{[t/\varepsilon]}^{(\varepsilon)}}) = e^{-\gamma\mu\varepsilon[t/\varepsilon]}\left(1 + \frac{[t/\varepsilon](1 - e^{-\lambda\varepsilon})(e^{\gamma} - 1)}{[t/\varepsilon]}\right)^{[t/\varepsilon]} \to e^{-\gamma\mu t}e^{\lambda(e^{\gamma}-1)t}$$

which we recognise as being the moment generating function of $X_t = N_t - \mu t$.

(b) This is a special case of the n-step generalisation of the two-asset model (A, S) on two scenarios. Since $A_0 = S_0 = 1$, we have no arbitrage if and only if $S_1^{\text{down})} < A_1 < S_1^{\text{up}}$. Here, this is

$$e^{-\mu\varepsilon} < e^{\delta\varepsilon} < e^{1-\mu\varepsilon} \iff -\mu < \delta < 1/\varepsilon - \mu.$$

and the model is then also complete since the general binary n-step model is complete.

(c) We need

$$1 = e^{\widetilde{S}_0^{(\varepsilon)}} = e^{-\delta\varepsilon}\mathbb{E}_q(e^{\widetilde{S}_1^{(\varepsilon)}}) = e^{-\varepsilon\delta}\left(e^{-\mu\varepsilon}(1-q_\varepsilon) + e^{1-\mu\varepsilon}q_\varepsilon\right)$$

and so

$$q_\varepsilon = \frac{e^{\delta\varepsilon} - e^{-\mu\varepsilon}}{e^{-\mu\varepsilon}(e-1)} = \frac{e^{\mu\varepsilon+\delta\varepsilon} - 1}{e-1}.$$

We can now check that $(e^{\widetilde{S}_n^{(\varepsilon)}})_{n\ge 0}$ is a martingale.

(d) This is in complete analogy to (a). We deduce this from the Poisson limit theorem considering $\widetilde{T}_n^{(\varepsilon)} = \widetilde{S}_n^{(\varepsilon)} + n\mu\varepsilon$, a Bernoulli random walk with success probability q_ε. Noting that

$$\frac{1}{\varepsilon}q_\varepsilon = \frac{1}{e-1}\frac{e^{\varepsilon(\delta+\mu)} - 1}{\varepsilon} \to \frac{\delta+\mu}{e-1} \quad \text{as } \varepsilon \downarrow 0,$$

we obtain $\widetilde{T}_{[t/\varepsilon]}^{(\varepsilon)} \to \widetilde{N}_t$ in distribution, as required. Now, clearly $[t/\varepsilon]\mu\varepsilon \to \mu t$, and taking differences in the two limit results completes the argument.

(e) Note from the moment generating function of the Poisson distribution that

$$\mathbb{E}(e^{\widetilde{N}_t}) = e^{t\frac{\delta+\mu}{e-1}(e-1)} = e^{\delta t+\mu t}$$

and so $M_t = e^{-\delta t}e^{\widetilde{N}_t-\mu t}$ is a martingale, because for $s < t$

$$\begin{aligned}\mathbb{E}(M_t|\mathcal{F}_s) &= \mathbb{E}(e^{-\delta s}e^{\widetilde{N}_s-\mu s}e^{-\delta(t-s)}e^{(\widetilde{N}_t-\widetilde{N}_s)-\mu(t-s)}|\mathcal{F}_s) \\ &= e^{-\delta s}e^{\widetilde{N}_s-\mu s}e^{-(\delta+\mu)(t-s)}\mathbb{E}(e^{\widetilde{N}_t-\widetilde{N}_s}) = M_s.\end{aligned}$$

Given $N_t = k$ or $\widetilde{N}_t = k$, the two processes $(e^{\widetilde{N}_s-\mu s})_{0\le s\le t}$ and $(e^{N_s-\mu s})_{0\le s\le t}$ have the same conditional distribution, since the k jump times of $\widetilde{N}$ and N occur at independent uniform times on $[0,t]$. Since also $\mathbb{P}(N_t = k) > 0$ if and only if $\mathbb{P}(\widetilde{N}_t = k) > 0$, the same paths are possible for the two processes. Since the discounted process $e^{-\delta t}e^{\widetilde{N}_t-\mu t}$ is a martingale, it provides martingale probabilities for the equivalent process $e^{N_t-\mu t}$.

(f) $(N_t)_{t\ge 0}$ only has jumps of size 1, all other jumps are impossible, and the only Lévy processes with this property are Poisson processes with drift. If $(Y_t)_{t\ge 0}$ is a Poisson process with drift $-\nu t$, then we have

$$\mathbb{P}((e^{Y_s})_{0\le s\le 1} \in D_\nu) = 1.$$

Since $D_\nu \cap D_\mu = \varnothing$ for $\mu \ne \nu$, we must have $\mu = \nu$ in order that e^{Y_t} has the same possible paths as $e^{N_t-\mu t}$. We can now check that of all intensities $\lambda > 0$ of Y, only $\lambda = (\delta+\mu)/(e-1)$ is such that $M_t = e^{-\delta t}e^{Y_t}$ is a martingale:

$$\begin{aligned}\mathbb{E}(M_t|\mathcal{F}_s) &= \mathbb{E}(e^{-\delta s}e^{Y_s}e^{-\delta(t-s)}e^{Y_t-Y_s}|\mathcal{F}_s) \\ &= e^{-\delta s}e^{Y_s}e^{-\delta(t-s)}\mathbb{E}(e^{Y_t-Y_s}) = M_s e^{-(\delta+\mu)(t-s)+\lambda(e-1)}.\end{aligned}$$