B.4 Simulation

1. (a) Independent $A \sim \text{Gamma}(a,1)$ and $B \sim \text{Gamma}(b,1)$ have joint density

$$f_{A,B}(x,y) = \frac{x^{a-1}e^{-x}}{\Gamma(a)}\frac{y^{b-1}e^{-y}}{\Gamma(b)}$$

The transformation $(R,S) = T(A,B) = (A/(A+B), A+B)$ is bijective $T : (0,\infty)^2 \to (0,1)\times(0,\infty)$ with inverse transformation $(A,B) = T^{-1}(R,S) = (SR, S(1-R))$ that has the Jacobian

$$J(r,s) = \begin{pmatrix} s & r \\ -s & 1-r \end{pmatrix} \qquad \Rightarrow \qquad |\det(J(r,s))| = s$$

and so the transformation formula yields

$$\begin{aligned} f_{R,S}(r,s) &= |\det(J(r,s))| f_{A,B}(T^{-1}(r,s)) = s\frac{(sr)^{a-1}e^{-sr}}{\Gamma(a)}\frac{(s(1-r))^{b-1}e^{-s(1-r)}}{\Gamma(b)} \\ &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} r^{a-1}(1-r)^{b-1}\frac{s^{a+b-1}e^{-s}}{\Gamma(a+b)}, \end{aligned}$$

as required.

Vice versa, for $c = a+b$ and $p = a/(a+b)$, we recognise $T^{-1}(R,S) = (A,B)$, which has joint distribution

$$f_{A,B}(x,y) = \frac{x^{a-1}e^{-x}}{\Gamma(a)}\frac{y^{b-1}e^{-y}}{\Gamma(b)} = \frac{x^{cp-1}e^{-x}}{\Gamma(cp)}\frac{y^{c(1-p)-1}e^{-y}}{\Gamma(c(1-p))}.$$

and so any random variable $(\widetilde{R},\widetilde{S})$ with joint distribution as (R,S) will be such that $T^{-1}(\widetilde{R},\widetilde{S}) \sim T^{-1}(R,S) = (A,B)$.

(b) $\mathbb{P}(X \le x) = \mathbb{P}(U^{1/a} \le x) = \mathbb{P}(U \le x^a) = x^a$, $x \in (0,1)$ and so $f_X(x) = ax^{a-1}$, $x \in (0,1)$. We recognise $X \sim \text{Beta}(a,1)$.

(c) We calculate

$$\begin{aligned} \mathbb{P}\left(\frac{Y}{Y+Z} \le t, Y+Z \le 1\right) &= \mathbb{P}(\frac{Y(1-t)}{t} \le Z \le 1-Y) \\ &= \int_0^t \int_{y(1-t)/t}^{1-y} ay^{a-1}(1-a)z^{-a}dzdy \\ &= \int_0^t ay^{a-1}\left((1-y)^{1-a} - y^{1-a}(1-t)^{1-a}t^{a-1}\right)dy. \end{aligned}$$

We differentiate with respect to t to get

$$\begin{aligned} f_{W|Y+Z\le 1}(t) &= \frac{at((1-a)(1-t)^{-a}t^{a-1} - (a-1)(1-t)^{1-a}t^{a-2})}{\mathbb{P}(X+Y\le 1)} \\ &= \frac{a(1-a)t^{a-1}(1-t)^{-a}((1-t)+t)}{\mathbb{P}(X+Y\le 1)} \end{aligned}$$

and we recognise the density of $\text{Beta}(a,1-a)$, up to the normalisation constant, but we have calculated a conditional density which integrates to 1, so the normalisation constant must be the one of $\text{Beta}(a,1-a)$.

(d) Given $Y + Z \leq 1$, W is Beta$(a, 1-a)$-distributed. Since T is independent of (Y, Z, W), its conditional distribution given $Y+Z \leq 1$ is still Gamma$(1,1)$ and it is conditionally independent of W given $Y + Z \leq 1$. Therefore $\mathbb{P}(TW \leq h | Y + Z \leq 1) = \mathbb{P}(SR \leq h)$, and we can apply (a) for $c = 1$ and $p = a$ to deduce that, $SR \sim$ Gamma$(a, 1)$, i.e. the conditional distribution of WT given $Y + Z \leq 1$ is Gamma$(a, 1)$.

(e) This procedure generates a Gamma$(a, 1)$ random variable. Specifically, the conditioning on $Y + Z \leq 1$ is realised by repeated trials until $Y + Z \leq 1$, see Lemma 57. The procedure is easily implemented and gives a more efficient way of simulating Gamma random variables from uniform random variables than inverting the distribution function of the Gamma distribution numerically.

2. (a) From 1.(a) we take that we obain $A/(A+B) \sim$ Beta(a, b) for independent $A \sim$ Gamma$(a, 1)$ and $B \sim$ Gamma$(b, 1)$. Johnk's procedure works for $a \in (0, 1)$. To generate Gamma variables for higher parameters, we can write $a = [a]+\{a\}$ for integer part and fractional part and then represent

$$A = \sum_{k=1}^{[a]} E_k + A_0$$

where $(E_k)_{1 \leq k \leq [a]}$ is a sequence of independent Exp(1) random variables and $A_0 \sim$ Gamma$(\{a\}, 1)$. To summarise, the following procedure generates a Beta(a, b) random variable:

1.-5. Run Johnk's Gamma generator for parameter $\{a\}$. Set $A_0 = TY/(Y+Z)$.

6.-10. Independently of 1.-5., run Johnk's Gamma generator for parameter $\{b\}$. Set $B_0 = TY/(Y + Z)$.

11. Generate independent $U_1, \ldots, U_{[a]+[b]} \sim$ Unif$(0, 1)$ and set

$$A = A_0 - \ln\left(\prod_{k=1}^{[a]} U_k\right) \qquad \text{and} \qquad B = B_0 - \ln\left(\prod_{k=[a]+1}^{[a]+[b]} U_k\right).$$

12. Return the number $A/(A + B)$.

(b) The procedure generates a stochastic process successively at refining lattices of dyadic times. The key step (for $n = 0$ and then inductively for $n \geq 1$) is to take a 2^{-n}-increment $Y_{k,n} = X_{k2^{-n}} - X_{(k-1)2^{-n}} \sim$ Gamma$(2^{-n}, 1)$ and a $B_{k,n} \sim$ Beta(a_n, b_n) random variable to split $Y_{k,n}$ into two increments $B_{k,n}Y_{k,n}$ and $(1 - B_{k,n})Y_{k,n}$. Now 1.(a) applies with $a_n = 2^{-n-1}$ and $b_n = 2^{-n-1}$. Note the stationary independent increments of length 2^{-n-1}. Therefore, running this procedure up to stage n yields $X^{(1,\delta)}$ for $\delta = 2^{-n}$.

(c) Johnk's Gamma generator is more efficient than the inverse distribution function computation. The method is less liable to accumulating errors since time 1 is most accurate and errors only accumulate along the dyadic expansions, i.e. with local rather than global impact. Furthermore, we get an iterative procedure for which we do not have to fix the time lag δ in advance, but can continue to fill in extra points until a satisfactory result is obtained.

3. (a) Since $G_t \sim \text{Gamma}(\alpha_+ t, \beta_+)$ and $H_t \sim \text{Gamma}(\alpha_- t, \beta_-)$, we have

$$\mathbb{E}(at + G_t - H_t) = at + \frac{\alpha_+ t}{\beta_+} - \frac{\alpha_- t}{\beta_-} = 0 \iff a = \frac{\alpha_-}{\beta_-} - \frac{\alpha_+}{\beta_+}.$$

(b)
- Denote $F_\delta(x) = \mathbb{P}(V_\delta \leq x)$.
  1. Set $S_0 = 0$ and $n = 1$.
  2. Generate $U_n \sim \text{Unif}(0, 1)$.
  3. Set $S_n = S_{n-1} + F_\delta^{-1}(U_n)$. If enough steps have been performed, go to 4., otherwise increase n by 1 and go to 2.
  4. Return $(S_n)_{n\geq 0}$ as simulation of $(V_{\delta n})_{n\geq 0}$.
- Denote $F(x; \alpha, \beta) = \mathbb{P}(G \leq x)$ for $G \sim \text{Gamma}(\alpha, \beta)$.
  1. Set $S_0 = 0$ and $n = 1$.
  2. Generate two independent random numbers $U_{2n-1} \sim \text{Unif}(0, 1)$ and $U_{2n} \sim \text{Unif}(0, 1)$.
  3. Set $S_n = S_{n-1} + a\delta + F^{-1}(U_{2n-1}; \alpha_+\delta, \beta_+) - F^{-1}(U_{2n}; \alpha_-\delta, \beta_-)$. If enough steps have been performed, go to 4., otherwise increase n by 1 and go to 2.
  4. Return $(S_n)_{n\geq 0}$ as simulation of $(V_{\delta n})_{n\geq 0}$.
- Fix $t = 1$, iterate for further time units if needed. Denote $G(x; a, b) = \mathbb{P}(B \leq x)$ for $B \sim \text{Beta}(a, b)$.
  1. Set $V_0 = 0$ and $n = 0$.
  2. Generate 2 independent random numbers $U_1 \sim \text{Unif}(0, 1)$ and $U_2 \sim \text{Unif}(0, 1)$.
  3. Set $P_1 = F^{-1}(U_1; \alpha_+, \beta_+)$, $N_1 = F^{-1}(U_2; \alpha_-, \beta_-)$ and $V_1 = a + P_1 - N_1$.
  4. Generate 2^n independent random numbers $U_{2^{n+1}+k} \sim \text{Unif}(0, 1)$, $k = 1, \ldots, 2^n$.
  5. Set $B_{n,k} = G^{-1}(U_{2^{n+1}+k}; 2^{-n-1}\alpha_+, 2^{-n-1}\alpha_+)$, $k = 1, \ldots, 2^{n-1}$ and $C_{n,k} = G^{-1}(U_{2^{n+1}+2^{n-1}+k}; 2^{-n-1}\alpha_-, 2^{-n-1}\alpha_-)$, $k = 1, \ldots, 2^{n-1}$
  6. Set $P_{(2k-1)2^{-n}} = B_{n,k}P_{(2k-2)2^{-n}} + (1 - B_{n,k})P_{(2k)2^{-n}}$, $N_{(2k-1)2^{-n}} = C_{n,k}N_{(2k-2)2^{-n}} + (1 - C_{n,k})N_{(2k)2^{-n}}$ and $V_{(2k-1)2^{-n}} = (2k-1)2^{-n}a + P_{(2k-1)2^{-n}} - N_{(2k-1)2^{-n}}$ for $k = 1, \ldots, 2^{n-1}$. If the resolution is fine enough, go to 7., otherwise increase n by 1 and go to 4.
  7. Return $(V_{k2^{-n}})_{k=1,\ldots,2^n}$.

  Instead of F^{-1} and G^{-1}, one can use Johnk's Gamma generator of A.3.2. and the associated Beta generator of A.3.3.
- Denote $H(x; \beta) = \int_\varepsilon^x y^{-1}e^{-\beta y}dy / \int_\varepsilon^\infty y^{-1}e^{-\beta y}dy$. Also denote

$$\begin{aligned}
\lambda &= \alpha_+ \int_\varepsilon^\infty y^{-1}e^{-\beta_+ y}dy + \alpha_- \int_\varepsilon^\infty y^{-1}e^{-\beta_- y}dy \\
\text{and } p &= \lambda^{-1}\alpha_+ \int_\varepsilon^\infty y^{-1}e^{-\beta_+ y}dy.
\end{aligned}$$

  1. Set $V_0 = 0$, $T_0 = 0$ and $n = 1$.
  2. Generate three independent random numbers $U_{3n-2} \sim \text{Unif}(0, 1)$ and $U_{3n-1} \sim \text{Unif}(0, 1)$ and $U_{3n} \sim \text{Unif}(0, 1)$.

3. Set $Z_n = -\ln(U_{3n})/\lambda$.

4. If $U_{3n-1} > p$, let $J_n = -H^{-1}(U_{3n}; \beta_-)$, otherwise let $J_n = H^{-1}(U_{3n}; \beta_+)$.

5. Set $T_n = T_{n-1} + Z_n$ and $V_{T_n} = V_{T_{n-1}} + aZ_n + J_n$. If T_n is big enough, go to 6., otherwise increase n by 1 and go to 2.

6. Return $(V_{T_n})_{n \geq 0}$.

(c) Below are 9 simulations for $\alpha_+ \in \{1, 10, 100\}$ (rows) and $\alpha_- \in \{10, 100, 1000\}$ (columns). Note the big positive jumps for $\alpha_+ = 1$, the cases $\alpha_+ = \alpha_-$ with $a = 0$ and convergence to Brownian motion from top left to bottom right. The code is similar to the symmetric case and is available on the course website.