B.3 Construction of Lévy processes

1. (a) First note that $e^{\Psi(\gamma)} = \mathbb{E}(e^{\gamma X_1})$ implies that $\mathbb{E}(e^{\gamma X_{1/m}}) = e^{\Psi(\gamma)/m}$ since stationarity and independence of increments implies $\mathbb{E}(e^{\gamma X_{1/m}})^m = e^{\Psi(\gamma)}$, then $\mathbb{E}(e^{\gamma X_q}) = e^{q\Psi(\gamma)}$, and then the right-continuity of sample paths implies that $X_q \to X_t$ almost surely and hence also in distribution, as $q \downarrow t$. Therefore, characteristic functions converge and $\mathbb{E}(e^{\gamma X_q}) = e^{q\Psi(\gamma)} \to e^{t\Psi(\gamma)}$.

   Now we use the independence and stationarity of increments to see

$$\begin{aligned}\mathbb{E}(\exp\{\gamma X_t\}|\mathcal{F}_s) &= \exp\{\gamma X_s\}\mathbb{E}(\exp\{\gamma(X_t - X_s)\}) \\ &= \exp\{\gamma X_s\}\exp\{(t-s)\Psi(\gamma)\}.\end{aligned}$$

   (b) The argument in (a) applies, with $\gamma = i\lambda$ and ψ instead of Ψ as appropriate. Recall that moment generating functions do not exist for all random variables, but characteristic functions always exist (because $x \mapsto e^{i\lambda x}$ is bounded).

   (c) The following argument can more easily be carried out for moment generating functions, but applies more generally if done for characteristic functions.

   Differentiate $\mathbb{E}(\exp\{i\lambda X_t\}) = e^{-t\psi(\lambda)}$ with respect to λ at $\lambda = 0$ to get $i\mathbb{E}(X_t) = -t\psi'(0)$ (see Grimmett-Stirzaker 5.7 for a statement and reference to the proof). The claim follows since $\mu = \mathbb{E}(X_1)$ must now be the slope of this linear function.

   Now, we use the independence and stationarity of increments to see

$$\mathbb{E}(X_t - t\mu|\mathcal{F}_s) = \mathbb{E}(X_s + (X_t - X_s) - t\mu|\mathcal{F}_s) = X_s + (t-s)\mu - t\mu = X_s - s\mu.$$

   (d) Differentiate $\mathbb{E}(\exp\{i\lambda X_t\}) = e^{-t\psi(\lambda)}$ twice with respect to λ at $\lambda = 0$ to get $-\mathbb{E}(X_t^2) = -t(\psi''(0) - t(\psi'(0))^2)$, so $\mathrm{Var}(X_t) = t\psi''(0)$, where now $\sigma^2 = \mathrm{Var}(X_1) = \psi''(0)$.

   Now we use the independence and stationarity of increments to see

$$\begin{aligned}\mathbb{E}((X_t - t\mu)^2|\mathcal{F}_s) &= \mathbb{E}((X_s - s\mu)^2 + 2(X_s - s\mu)(X_t - X_s - (t-s)\mu) \\ &\qquad + (X_t - X_s - (t-s)\mu)^2|\mathcal{F}_s) \\ &= (X_s - s\mu)^2 + 2(X_s - s\mu)\mathbb{E}(X_t - X_s - (t-s)\mu) \\ &\qquad + \mathrm{Var}(X_t - X_s) \\ &= (X_s - s\mu)^2 + (t-s)\sigma^2.\end{aligned}$$

2. (a) If $\kappa \in (-1, \infty)$, then

$$\int_0^\infty g(x)dx = \int_0^\infty x^\kappa e^{-x} dx = \Gamma(\kappa+1) < \infty.$$

   The Poisson point process is hence of the form of Example 18 and so $(C_t)_{t\geq 0}$ is a compound Poisson process with intensity $\Gamma(\kappa+1)$ and Gamma$(\kappa+1, 1)$ jump distribution with density

$$h(x) = \frac{1}{\Gamma(\kappa+1)} x^\kappa e^{-x}, \qquad x > 0.$$

(b) The counting measures associated to $(\Delta_t)_{t\ge 0}$ and $(\Delta_t^{(n)})_{t\ge 0}$ are

$$\begin{aligned} N((a,b]\times(c,d]) &= \#\{t\in(a,b]:\Delta_t\in(c,d] \\ &\sim \text{Poi}\left((b-a)\int_c^d g(x)dx\right), \qquad 0\le a<b, 0<c<d, \\ N_n((a,b]\times(c,d]) &= N((a,b]\times((c,d]\cap(1/n,\infty)) \\ &\sim \text{Poi}\left((b-a)\int_c^d g(x)1_{\{x>1/n\}}dx\right),\ 0\le a<b, 0<c<d. \end{aligned}$$

N_n inherits the properties of a Poisson counting measure from N. We read off the intensity function $g_n(x)=g(x)$, $x>1/n$, $g_n(x)=0$, $x\le 1/n$. The argument of (a) shows that $C_t^{(n)}$ is a compound Poisson process.

(c) $C_t^{(n)}$ increases as $n\to\infty$. We can study the limit of moment generating functions, whether or not the limit is finite. We get, as $n\to\infty$,

$$\mathbb{E}(e^{\gamma C_t^{(n)}})=\exp\left\{\int_{1/n}^\infty (e^{\gamma x}-1)g(x)dx\right\}\downarrow \exp\left\{\int_0^\infty (e^{\gamma x}-1)g(x)dx\right\}$$

and because for $\gamma<0$

$$\int_0^\infty (e^{\gamma x}-1)g(x)dx<\infty \iff \int_0^\infty (1\wedge x)g(x)dx<\infty,$$

and by Lemma 21, we need to investigate the right hand condition. We check that

$$\int_1^\infty g(x)dx<\infty, \quad \text{and} \quad \int_0^1 xg(x)dx<\infty \iff \kappa+1>-1,$$

as required.

(d) We can write

$$C_s-C_s^{(n)}=\sum_{r\le s}\Delta_r 1_{\{\Delta_r\le 1/n\}}\le \sum_{r\le t}\Delta_r 1_{\{\Delta_r\le 1/n\}}=C_t-C_t^{(n)},$$

and putting a supremum over $s\le t$ on the left hand side, we get the required estimate (as an equality because we can take $s=t$ on the left. Now we showed in (c) that $C_t^{(n)}\to C_t$ a.s., and so we deduce here that

$$\sup_{s\le t}|C_s^{(n)}-C_s|\to 0 \qquad \text{as } n\to\infty,$$

i.e. that the convergence is locally uniform.

(e) By Proposition 40(ii), we have for $m\le n$

$$\begin{aligned} \mathbb{E}(|C_t^{(n)}-\mathbb{E}(C_t^{(n)})-(C_t^{(m)}-\mathbb{E}(C_t^{(m)}))|^2) &= \text{Var}(C_t^{(n)}-C_t^{(m)}) \\ &= \int_{1/n}^{1/m} x^2 g(x)dx, \end{aligned}$$

and this decreases to zero as $n \geq m \to \infty$ if and only if $\int_0^1 x^2 g(x)dx < \infty$, i.e. $\kappa > -3$. In this case, $(C_t^{(n)} - \mathbb{E}(C_t^{(n)}))_{n\geq 1}$ is a Cauchy sequence that converges by completeness of $\mathbb{R}$ (and the associated L^2 space of $\mathbb{R}$-valued random variables).

The limiting process $(X_t, t \geq 0)$ is a Lévy process, since for $0 \leq t_0 < t_1 < \ldots < t_m$, we have that

$$\begin{aligned}
&\mathbb{E}\left(\exp\left\{\sum_{j=1}^m \gamma_j (X_{t_j} - X_{t_{j-1}})\right\}\right) \\
&= \lim_{n\to\infty} \mathbb{E}\left(\exp\left\{\sum_{j=1}^m \gamma_j \left(C_{t_j}^{(n)} - C_{t_{j-1}}^{(n)} - (t_j - t_{j-1})\mathbb{E}(C_1^{(n)})\right)\right\}\right) \\
&= \lim_{n\to\infty} \prod_{j=1}^n \mathbb{E}\left(\exp\left\{\gamma_j \left(C_{t_j}^{(n)} - C_{t_{j-1}}^{(n)} - (t_j - t_{j-1})\mathbb{E}(C_1^{(n)})\right)\right\}\right) \\
&= \prod_{j=1}^m \mathbb{E}\left(\exp\left\{\gamma_j (X_{t_j} - X_{t_{j-1}})\right\}\right)
\end{aligned}$$

and so X has (i) independent increments and (ii) the distributions in the third line only depend on $(t_j - t_{j-1})$ and this is preserved in the limit in the fourth line. (iii) Right-continuity and left limits are preserved under uniform convergence.

3. (a) Just note that for subordinators $0 \leq X_t < \infty$ a.s., and this implies that $1 \geq e^{-\mu X_t} > 0$ a.s. and then also $1 \geq \mathbb{E}(e^{-\mu X_t}) > 0$ as required. Therefore, Φ_t is well-defined.

(b) The first equality follows as in A.3.1(a), first for rational $t \geq 0$ and then, by right-continuity of paths and since a.s. convergence implies convergence in distribution, hence of moment generating functions. The scaling relation for fixed t translates to

$$\Phi_{t/c}(c^{1/\alpha}\mu) = -\ln(\mathbb{E}(\exp\{-\mu c^{1/\alpha} X_{t/c}\})) = -\ln(\mathbb{E}(e^{-\mu X_t})) = \Phi_t(\mu).$$

and therefore, for $t = 1$, $c = \mu^{-\alpha}$, we deduce the second equality from this and from the first equality

$$\mu^\alpha \Phi(1) = \frac{1}{c}\Phi(1) = \Phi_{1/c}(1) = \Phi(\mu).$$

(c) Clearly $\mu \mapsto e^{-\mu X_t}$ is a.s. decreasing and so is hence $\mu \mapsto \mathbb{E}(e^{-\mu X_t})$, strictly decreasing if $X_t > 0$ with positive probability. Now, $\Phi(\mu) = \Phi(1)\mu^\alpha$ is clearly differentiable for $\mu > 0$, and so

$$\frac{\partial}{\partial\mu}\mathbb{E}(e^{-\mu X_t}) = \frac{\partial}{\partial\mu} e^{-t\Phi(1)\mu^\alpha} = -t\Phi(1)\alpha\mu^{\alpha-1}e^{-t\Phi(1)\mu^\alpha}$$

and this is negative only for $\alpha > 0$ (or $\alpha = 0$ but then $\mu \mapsto \mathbb{E}(e^{-\mu X_t})$ is constant). To show that also $\alpha \leq 1$ note that $\mu \mapsto e^{-\mu X_t}$ is also a.s. convex, and hence so is $\mu \mapsto \mathbb{E}(e^{-\mu X_t})$. Now, $\Phi(\mu)$ is also twice differentiable so that

$$\frac{\partial^2}{\partial \mu^2}\mathbb{E}(e^{-\mu X_t}) = t\Phi(1)\alpha\mu^{\alpha-2}e^{-t\Phi(1)\mu^\alpha}(t\Phi(1)\alpha\mu^\alpha - (\alpha-1)),$$

and this is nonnegative for all $\mu > 0$ if and only if $\alpha \leq 1$.

(d) Note that, (by monotone convergence), as $\mu \downarrow 0$,

$$t\Phi(1)\alpha\mu^{\alpha-1}e^{-t\Phi(1)\mu^\alpha} = \mathbb{E}(X_t e^{-\mu X_t}) \uparrow \mathbb{E}(X_t),$$

where the left-hand side increases to ∞ for $\alpha \in (0,1)$.

(e) Note that $\Phi(0) = 0$ implies that the equation holds for $\mu = 0$ no matter what g is. Now differentiate both sides with respect to μ to get

$$\Phi(1)\alpha\mu^{\alpha-1} = \int_0^\infty e^{-\mu x} x g(x) dx.$$

Remember that the density of the Gamma$(1-\alpha,\mu)$ distribution is $f(x) = (\Gamma(1-\alpha))^{-1}\mu^{1-\alpha}x^{-\alpha}e^{-\mu x}$. Therefore, we can (and have to, by the Uniqueness Theorem for moment generating functions) take

$$g(x) = \frac{\Phi(1)\alpha}{\Gamma(1-\alpha)}x^{-\alpha-1}, \qquad x > 0.$$

(f) For $\alpha \in (0,1)$, the Construction Theorem for subordinators (Theorem 26) shows that we can construct the stable subordinator from a Poisson point process with intensity function g as specified in (e). Note that g satisfies the integrability condition

$$\int_0^\infty (1 \wedge x) g(x) dx < \infty$$

since $x^{-\alpha-1}$ is integrable at $x = \infty$ and $x^{-\alpha}$ is integrable at $x = 0$.

For $\alpha = 1$ note that $\Phi_t(\mu) = \Phi(1)t\mu$. The associated subordinator is the deterministic drift $X_t = \Phi(1)t$.

4. (a) Just note that

$$c^{1/\alpha}Z_{t/c} = c^{1/\alpha}X_{t/c} - c^{1/\alpha}Y_{t/c} \sim X_t - Y_t = Z_t$$

for fixed t, and that, as processes in $t \geq 0$, both the left-hand side and the right-hand side are Lévy processes. Therefore, the distributions as processes coincide.

(b) $H \sim -H$ implies

$$\mathbb{E}(\cos(\lambda H)) + i\mathbb{E}(\sin(\lambda H)) = \mathbb{E}(e^{i\lambda H}) = \mathbb{E}(e^{-i\lambda H}) = \mathbb{E}(\cos(\lambda H)) - i\mathbb{E}(\sin(\lambda H))$$

and so the imaginary part $\mathbb{E}(\sin(\lambda H))$ must vanish for all $\lambda \in \mathbb{R}$.

(c) Clearly $Z_t = X_t - Y_t \sim Y_t - X_t = -Z_t$, so Z_t has a symmetric distribution. By (b), its characteristic function $\varphi_t(\lambda) = \mathbb{E}(e^{i\lambda Z_t})$ is real-valued. By the hint, we may assume that φ_t is continuous, and since Z_t is infinitely divisible, that $\varphi_t(\lambda) \neq 0$, so it must stay positive everywhere (note that $\varphi(0) = 1$). Define

$$\psi_t(\lambda) = -\ln(\varphi_t(\lambda)), \qquad \psi(\lambda) = \psi_1(\lambda), \qquad \lambda \in \mathbb{R}.$$

By A.3.1.(b), we have $\psi_t(\lambda) = t\psi(\lambda)$. The scaling relation implies

$$\psi_{t/c}(c^{1/\alpha}\lambda) = -\ln(\mathbb{E}(\exp\{i\lambda c^{1/\alpha} Z_{t/c}\})) = -\ln(\mathbb{E}(e^{i\lambda Z_t})) = \psi_t(\lambda),$$

and as in A.3.3.(b), this implies $\psi(\lambda) = \psi(1)\lambda^\alpha$ for all $\lambda \geq 0$. For $\lambda < 0$ note that

$$\psi(\lambda) = -\ln(\mathbb{E}(e^{i\lambda Z_t})) = -\ln(\mathbb{E}(e^{-i\lambda Z_t})) = \psi(-\lambda),$$

so we have $\psi(\lambda) = \psi(1)|\lambda|^\alpha$.

(d) Before we start, note that the integral defining $\widetilde{\psi}(\lambda)$ converges for $\alpha \in (0,2)$ since the integrand behaves like $x^{1-\alpha}$ at $x = 0$ and like $x^{-\alpha-1}$ at $|x| = \infty$. We then check, by change of variables $y = c^{1/\alpha}x$ (hence $x^{-1}dx = y^{-1}dy$), that

$$\begin{aligned} \widetilde{\psi}(\lambda c^{1/\alpha}) &= \int_{-\infty}^{\infty} (\cos(\lambda c^{1/\alpha}x) - 1)\widetilde{b}|x|^{-\alpha-1}dx \\ &= \int_{-\infty}^{\infty} (\cos(\lambda y) - 1)\widetilde{b}c|y|^{-\alpha-1}dy = c\widetilde{\psi}(\lambda). \end{aligned}$$

The argument of (c) shows that this implies $\widetilde{\psi}(\lambda) = b|\lambda|^\alpha$ for some $b \geq 0$ – the argument did not depend on $\alpha \in (0,1)$.