Appendix B

Solutions

B.1 Infinite divisibility and limits of random walks

1. (a) Recall that for independent $A_1 \sim \text{Gamma}(\alpha_1, \beta)$ and $A_2 \sim \text{Gamma}(\alpha_2, \beta)$ we have $A_1 + A_2 \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$. A quick proof can be given using moment generating functions. The Gamma distribution has moment generating function

$$\mathbb{E}(\exp\{\gamma A\}) = \int_0^\infty e^{\gamma x} \frac{\beta^\alpha x^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta x} dx = \frac{\beta^\alpha}{(\beta - \gamma)^\alpha}, \qquad \gamma < \beta.$$

We see that

$$\mathbb{E}(\exp\{\gamma(A_1 + A_2)\}) = \mathbb{E}(\exp\{\gamma A_1\})\mathbb{E}(\exp\{\gamma A_2\}) = \frac{\beta^{\alpha_1+\alpha_2}}{(\beta - \gamma)^{\alpha_1+\alpha_2}}$$

and recognise the moment generating function of the $\text{Gamma}(\alpha_1 + \alpha_2, \beta)$ distribution. By the Uniqueness Theorem for moment generating functions, $A_1 + A_2 \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$.

If we now choose $Y_{n,1}, \ldots, Y_{n,n} \sim \text{Gamma}(\alpha/n, \beta)$ independent, we obtain, by induction in n, that $Y_{n,1} + \ldots + Y_{n,n} \sim \text{Gamma}(\alpha, \beta)$. Since this holds for all $n \geq 1$, a random variable $Y \sim \text{Gamma}(\alpha, \beta)$ has an infinitely divisible distribution.

(b) First calculate for $B_1, B_2 \sim \text{geom}(p)$ independent that

$$\begin{aligned}\mathbb{P}(B_1 + B_2 = n) &= \sum_{k=0}^{n} \mathbb{P}(B_1 = k, B_2 = n - k) = \sum_{k=0}^{n} p^k (1-p) p^{n-k} (1-p) \\ &= (n+1) p^n (1-p)^2,\end{aligned}$$

and, e.g. by induction, for $A_m = B_1 + \ldots + B_m = A_{m-1} + B_m$ a negative binomial distribution. Alternatively, consider independent Bernoulli trials until the mth success, then $\{A_m = n\}$ means there have been n failures and m successes, the $m-1$ first successes chosen from the first $n+m-1$ trials, and we get

$$\begin{aligned}\mathbb{P}(A_m = n) &= \binom{n+m-1}{m-1} p^n (1-p)^m = \frac{(n+m-1)!}{(m-1)!n!} p^n (1-p)^m \\ &= \frac{\Gamma(n+m)}{\Gamma(m)n!} p^n (1-p)^m.\end{aligned}$$

This formula makes sense for $m \in (0,\infty)$, and we refer to this probability mass function as $\mathrm{NB}(m,p)$. Then we calculate the probability generating function for $A \sim \mathrm{NB}(m,p)$

$$\mathbb{E}(s^A) = \sum_{n\ge 0} \frac{\Gamma(n+m)}{\Gamma(m)n!}(sp)^n(1-p)^m = \frac{(1-p)^m}{(1-sp)^m}, \qquad s \in [0,1],$$

and if $B \sim \mathrm{NB}(r,p)$ is independent, we obtain

$$\mathbb{E}(s^{A+B}) = \frac{(1-p)^{m+r}}{(1-sp)^{m+r}},$$

the probability generating function of the $\mathrm{NB}(m+r,p)$ distribution, so we conclude by the Uniqueness Theorem for probability generating functions that $A+B \sim \mathrm{NB}(m+r,p)$.

If we now choose $Y_{n,1},\ldots,Y_{n,n} \sim \mathrm{NB}(1/n,p)$ independent, we obtain, by induction in n, that $Y_{n,1}+\ldots+Y_{n,n} \sim \mathrm{NB}(1,p) = \mathrm{geom}(p)$. Since this holds for all $n \ge 1$, a random variable $Y \sim \mathrm{geom}(p)$ has an infinitely divisible distribution.

(c)* Assume that a random variable $U \sim \mathrm{Unif}(0,1)$ can be written as $U = Y_1 + Y_2$ for some independent and identically distributed Y_1 and Y_2. Then for $x \in [0,1]$,

$$1-x = \mathbb{P}(U \ge x) \ge \mathbb{P}(Y_1 \ge x/2, Y_2 \ge x/2) \Rightarrow \mathbb{P}(Y_1 \ge x/2) \le \sqrt{1-x}$$

and

$$x = \mathbb{P}(U \le x) \ge \mathbb{P}(Y_1 \le x/2)^2 \Rightarrow \mathbb{P}(Y_1 \le x/2) \le \sqrt{x}.$$

For $x = 1$ and $x = 0$, respectively, we deduce $\mathbb{P}(Y_1 \ge 1/2) = 0 = \mathbb{P}(Y_1 \le 0)$. Now for $x \in (0,1/2)$

$$x = \mathbb{P}(U \le x) \le \mathbb{P}(Y_1 \le x, Y_2 \le x) \iff \mathbb{P}(Y_1 \le x) \ge \sqrt{x}$$

and the inequality on the left is an equality if and only if the inequality on the right is an equality. Similarly,

$$x = \mathbb{P}(U \ge 1-x) \le \mathbb{P}(Y_1 \ge 1/2 - x)^2 \iff \mathbb{P}(Y_1 \ge 1/2 - x) \ge \sqrt{x}$$

For $x = 1/4$, we get $\mathbb{P}(Y_1 \le 1/4) \ge 1/2$ and $\mathbb{P}(Y_1 \ge 1/4) \ge 1/2$. If both inequalities were equalities, we would deduce from the left-hand equalities that $\mathbb{P}(Y_1 \in (1/8,3/8)) = 0$ and this is incompatible with $\mathbb{P}(U \in (1/4,3/8)) > 0$, so the assumption that $U = Y_1 + Y_2$ must have been wrong.

2. (a) Stationarity of increments means $X_t - X_s \sim X_{t-s}$, so we check infinite divisibility of X_{t-s}. Note

$$X_{t-s} = \sum_{j=1}^{m} Y_j^{(m)}, \qquad \text{where } Y_j^{(m)} = X_{j(t-s)/m} - X_{(j-1)(t-s)/m},\ j = 1,\ldots,m.$$

By independence of increments, $Y_1^{(m)},\ldots,Y_m^{(m)}$ are independent. By stationarity of increments, $Y_j^{(m)} \sim X_{(t-s)/m}$ for all $j = 1,\ldots,m$. Since this holds for all $m \ge 1$, this proves infinite divisibility of the distribution of X_{t-s}.

(b) (i) Independence of increments. By the independence of increments of X and Y and by the independence of X and Y we have for all $0 \leq t_0 < t_1 < \ldots < t_n$ that the following random variables are all independent:

$$X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}} \qquad \text{and} \qquad Y_{t_0}, Y_{t_1} - Y_{t_0}, \ldots, Y_{t_n} - Y_{t_{n-1}}$$

Since functions of independent random variables are independent, we can add take linear combinations and deduce independence of

$$aX_{t_0} + bY_{t_0}, a(X_{t_1} - X_{t_0}) + b(Y_{t_1} - Y_{t_0}), \ldots, a(X_{t_n} - X_{t_{n-1}}) + b(Y_{t_n} - Y_{t_{n-1}}).$$

(ii) Stationarity of increments. We have that $X_{t+s} - X_t$ and $Y_{t+s} - Y_t$ are independent, and also that X_s and Y_s are independent. By the stationarity of increments we have that $X_{t+s} - X_t \sim X_s$ and $Y_{t+s} - Y_t \sim Y_s$ and so the joint distributions of $(X_{t+s} - X_t, Y_{t+s} - Y_t)$ is the same as the joint distribution of (X_s, Y_s). If we apply the same linear function to the random vectors, these will also have the same distribution, i.e.

$$a(X_{t+s} - X_t) + b(Y_{t+s} - Y_t) \sim aX_s + bY_s.$$

(iii) Right-continuity and left limits of paths. Linear combinations of such functions still have these properties.

(c) We calculated the moment generating function of the Gamma(α, β) distribution in Exercise 1 as

$$\mathbb{E}(\exp\{\gamma A\}) = \int_0^\infty e^{\gamma x} \frac{\beta^\alpha x^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta x} dx = \frac{\beta^\alpha}{(\beta - \gamma)^\alpha}, \qquad \gamma < \beta.$$

If $C_1 \sim D_1 \sim \text{Gamma}(\alpha, \sqrt{2\mu})$, then $C_s \sim D_s \sim \text{Gamma}(\alpha s, \sqrt{2\mu})$. Hence

$$\mathbb{E}(e^{\gamma(C_s - D_s)}) = \mathbb{E}(e^{\gamma C_s})\mathbb{E}(e^{-\gamma D_s}) = \frac{\sqrt{2\mu}^{\alpha s}}{(\sqrt{2\mu} - \gamma)^{\alpha s}} \frac{\sqrt{2\mu}^{\alpha s}}{(\sqrt{2\mu} + \gamma)^{\alpha s}} = \left(\frac{\mu}{\mu - \frac{1}{2}\gamma^2} \right)^{\alpha s}$$

for all $-\sqrt{2\mu} < \gamma < \sqrt{2\mu}$.

3. (a) Let $W_n \sim \text{Binomial}(n, p_n)$ with $np_n \to \lambda$, then $W_n \to \text{Poi}(\lambda)$ in distribution as $n \to \infty$. To prove this, check

$$\mathbb{E}(s^{W_n}) = \sum_{k=0}^n s^k \binom{n}{k} p_n^k (1 - p_n)^{n-k} = \left(1 - \frac{np_n(1-s)}{n} \right)^n \to e^{-\lambda(1-s)},$$

and this is the probability generating function of Poi(λ). By the Uniqueness Theorem and by the Continuity Theorem for probability generating functions, W_n converges in distribution to a Poi(λ) distribution.

(b) Since p_N is small, the Poisson limit theorem is appropriate, and since N is large, it will give a reasonably good approximation. As parameter of the Poisson distribution, Np_N is appropriate, since $Np_N \to \lambda$ in the limit theorem for a Poi(λ) limit.

(c) Denote by $B_1, \ldots, B_N$ the Bernoulli random variables so that $B_j = 1$ if policy holder j makes a claim. Then $S_N = B_1 + \ldots + B_N \sim \text{Binomial}(N, p_N)$. We calculate the moment generating function

$$\begin{aligned}
\mathbb{E}(\exp\{\gamma T_N\}) &= \mathbb{E}\left(\exp\left\{\gamma \sum_{j=1}^{S_N} A_j\right\}\right) \\
&= \sum_{k=0}^{N} \mathbb{E}\left(\exp\left\{\gamma \sum_{j=1}^{k} A_j\right\}\right) \binom{N}{k} p_N^k (1-p_N)^{N-k} \\
&= \sum_{k=0}^{N} \left(\mathbb{E}\left(e^{\gamma A_1}\right)\right)^k \binom{N}{k} p_N^k (1-p_N)^{N-k} \\
&= (1 - p_N + p_N \mathbb{E}(e^{\gamma A_1}))^N,
\end{aligned}$$

by the binomial theorem, for all $\gamma \in \mathbb{R}$ for which $\mathbb{E}(e^{\gamma A_1}) < \infty$.

(d) Consider the moment generating functions

$$\mathbb{E}(\exp\{\gamma T_N\}) = \left(1 - \frac{N p_N (1 - \mathbb{E}(e^{\gamma A_1}))}{N}\right)^N \to \exp\left\{-\lambda(1 - \mathbb{E}(e^{\gamma A_1}))\right\},$$

and this is the moment generating function of the compound Poisson distribution, which we calculate as follows

$$\begin{aligned}
\mathbb{E}\left(\exp\left\{\gamma \sum_{j=1}^{S_\infty} A_j\right\}\right) &= \sum_{k=0}^{\infty} \mathbb{E}\left(\exp\left\{\gamma \sum_{j=1}^{k} A_j\right\}\right) \frac{\lambda^k}{k!} e^{-\lambda} \\
&= \sum_{k=0}^{\infty} \left(\mathbb{E}\left(\exp\left\{\gamma A_1\right\}\right)\right)^k \frac{\lambda^k}{k!} e^{-\lambda} \\
&= e^{-\lambda} \exp\left\{\lambda \mathbb{E}(e^{\gamma A_1})\right\} = \exp\left\{-\lambda(1 - \mathbb{E}(e^{\gamma A_1}))\right\}.
\end{aligned}$$

4. (a) (i) Note that

$$\frac{\sum_{k=1}^{n} A_k - n\mathbb{E}(A_1)}{\sqrt{n\text{Var}(A_1)}} = \sum_{k=1}^{n} \frac{A_k - \mu}{\sigma\sqrt{n}} = \sum_{k=1}^{n} Y_{n,k} = V_n.$$

Thus, the Central Limit Theorem in terms of V_n states $V_n \to \text{Normal}(0, 1)$ in distribution as $n \to \infty$.

(ii)* Markov's inequality $\mathbb{P}(|X| > y) \leq \mathbb{E}(X^2)/y^2$ yields

$$\begin{aligned}
\mathbb{P}(|A_1 - \mu| > \sigma x \sqrt{n}) &= \mathbb{P}(|A_1 - \mu| 1_{\{|A_1 - \mu| \geq \sigma x \sqrt{n}\}} > \sigma x \sqrt{n}) \\
&\leq \frac{\mathbb{E}(|A_1 - \mu|^2 1_{\{|A_1 - \mu| \geq \sigma x \sqrt{n}\}})}{\sigma^2 x^2 n}.
\end{aligned}$$

Now note that, as $n \to \infty$,

$$\mathbb{E}\left(|A_1 - \mu|^2 1_{\{|A_1 - \mu| < \sigma x \sqrt{n}\}}\right) \to \mathbb{E}(|A_1 - \mu|^2) = \sigma^2,$$

(by monotone convergence) and so

$$\begin{aligned}\gamma_n(x) &:= \frac{1}{\sigma^2 x^2}\mathbb{E}(|A_1-\mu|^2 1_{\{|A_1-\mu|\geq\sigma x\sqrt{n}\}}) \\ &= \frac{1}{\sigma^2 x^2}\left(\sigma^2 - \mathbb{E}(|A_1-\mu|^2 1_{\{|A_1-\mu|<\sigma x\sqrt{n}\}})\right) \to 0.\end{aligned}$$

(iii) For all $x > 0$, calculate using (ii)

$$\begin{aligned}\mathbb{P}(M_n \leq x) &= \mathbb{P}(|Y_{n,1}| \leq x, \ldots, |Y_{n,n}| \leq x) = (\mathbb{P}(|Y_{n,1}| \leq x))^n \\ &\geq \left(1 - \frac{\gamma_n(x)}{n}\right)^n \to e^0 = 1.\end{aligned}$$

This implies that $\mathbb{P}(|M_n| > \varepsilon) = 1 - \mathbb{P}(|M_n| \leq \varepsilon) \to 0$ for all $\varepsilon > 0$, so $M_n \to 0$ in probability.

(b) (i) At stage n there are r red balls and $s+n-1$ black balls in the urn. So

$$Y_{n,k} \sim \text{Bernoulli}\left(\frac{r}{r+s+n-1}\right) \quad \Rightarrow \quad W_n \sim \text{Binomial}\,(n, p_n)\,,$$

where $p_n = r/(r+s+n-1)$. Note that $np_n \to r$, so that the Poisson limit theorem yields $W_n \to \text{Poi}(r)$.

(ii) Clearly $\mathbb{P}(Y_{n,k} = 0) = 1 - p_n = 1 - r/(r+s+n-1) \to 1$, as $n \to \infty$.

(iii) Now, as $n \to \infty$,

$$\mathbb{P}(M_n = 0) = \mathbb{P}(Y_{n,1} = 0, \ldots, Y_{n,n} = 0) = (1-p_n)^n = \left(1 - \frac{np_n}{n}\right)^n \to e^{-r},$$

If $M_n \to 0$, then $\mathbb{P}(|M_n| > \varepsilon) = 1 - \mathbb{P}(M_n = 0) \to 0$ for all $0 < \varepsilon < 1$, and this is incompatible with the limit above. So, $M_n \not\to 0$ in probability.

(c)* (i) Define $S_k^{(n)} = Y_{n,1} + \ldots + Y_{n,k}$, $k \geq 0$, $n \geq 1$.
Donsker's theorem says in the setting of (a), where $V_n = S_n^{(n)}$, that $S_{[nt]}^{(n)} \to B_t$ locally uniformly in distribution for a Brownian motion $(B_t)_{t\geq 0}$.
The process version of the Poisson limit theorem says in the setting of (b), where $W_n = S_n^{(n)}$, that $S_{[nt]}^{(n)} \to N_t$ in the Skorohod sense in distribution for a Poisson process $(N_t)_{0\leq t\leq 1}$ with rate r.

(ii) Clearly, the size of the biggest jump of Brownian motion is 0, and we have $M_n \to 0$ in probability, hence also in distribution.
The number of jumps of $(N_t)_{0\leq t\leq 1}$ is Poisson distributed with parameter r. The size J of the biggest jump of $(N_t)_{0\leq t\leq 1}$ is 1 if there is a jump, with probability $\mathbb{P}(J = 1) = 1 - e^{-r}$, and $\mathbb{P}(J = 0) = e^{-r}$ is the probability that there is no jump. This is the limit distribution that we wish to establish. We have shown that

$$\mathbb{P}(M_n = 0) \to e^{-r} = \mathbb{P}(J = 0)$$

and this implies $\mathbb{P}(M_n = 1) = 1 - \mathbb{P}(M_n = 0) \to 1 - e^{-r} = \mathbb{P}(J = 1)$, as required.