A.7 Subordination and level passage events

1. (a) Let C and D be two independent Gamma Lévy processes with parameters α and $\sqrt{2\lambda}$ for $C_1 \sim D_1$. Let T be a Gamma process with parameters α and λ, and let B be an independent Brownian motion. Show that $X_s - Y_s \sim B_{T_s}$. *This result was mentioned in Question A.1.2. as an explanation for the name Variance Gamma process.*

   (b) Let B be Brownian motion, S an independent stable subordinator with index $\alpha \in (0,1)$. Show that $R_t = B_{S_t}$, $t \geq 0$, is a stable process with index 2α.

   (c) Write down procedures to simulate the processes in (a) and (b) using Method 3 (Subordination).

2. Let X be a Lévy process with $\mathbb{E}(X_1) = \mu$ and $\mathrm{Var}(X_1) = \sigma^2$. Let S be an independent subordinator with $\mathbb{E}(S_1) = m$ and $\mathrm{Var}(S_1) = q^2$. Denote $Z_t = X_{S_t}$, $t \geq 0$.

   (a) Show that $\mathbb{E}(Z_t) = m\mu t$, $t \geq 0$.

   (b) Show that $\mathrm{Var}(Z_t) = (\sigma^2 m + \mu^2 q^2)t$, $t \geq 0$. *Hint: Consider* $\mathbb{E}(Z_t^2)$ *first.*

   (c) Check this formula for the Variance Gamma process, using A.6.4.(a). For what values of α and λ is $\mathbb{E}(Z_1) = 0$ and $\mathrm{Var}(Z_1) = 1$? Show that $\mathbb{E}(Z_t^4) = 3\mathbb{E}(S_t^2)$ and deduce the range of $\mathbb{E}(Z_1^4)$ for these values of α and λ. *Standardized fourth moments (curtosis) give an indication of heavy tails. They reflect why Lévy processes such as the Variance Gamma process can better fit financial price processes.*

3. Let $(X_s)_{s\geq 0}$ be a Lévy process and $\underline{X}_t = \inf_{0\leq s\leq t} X_s$, $t \geq 0$.

   (a) For a fixed time $t > 0$, show that the process $(\widetilde{X}_s)_{0\leq s\leq t}$ given by

   $$\widetilde{X}_s^{(t)} = X_{t-} - X_{t-s-}, \qquad 0 \leq s \leq t,$$

   is a Lévy process with the same distribution as $(X_s)_{0\leq s\leq t}$.

   (b)* Show that this implies that for an *independent* random time τ with probability density function $f_\tau(x)$, $x \in (0,\infty)$, we have $(\widetilde{X}_s^{(\tau)})_{0\leq s\leq \tau} \sim (X_s)_{0\leq s\leq\tau}$ in the sense that for all $0 = s_0 < s_1 < \ldots < s_n < s_{n+1} = \infty$ and $0 \leq m \leq n$ we have

   $$\begin{aligned}&\mathbb{P}(X_{s_1} \in A_1, \ldots, X_{s_m} \in A_m, \tau \in [s_m, s_{m+1}) \cap B)\\ &\quad = \mathbb{P}(\widetilde{X}_{s_1}^{(\tau)} \in A_1, \ldots, \widetilde{X}_{s_m}^{(\tau)} \in A_m, \tau \in [s_m, s_{m+1}) \cap B)\end{aligned}$$

   for all intervals $A_1, \ldots, A_n \subset \mathbb{R}$ and $B \subset [0,\infty)$.

   (b) Using results and/or arguments from the lectures show that $\underline{X}_\tau$ is independent of $(X_\tau - \underline{X}_\tau)$ for an independent $\tau \sim \mathrm{Exp}(q)$.

   (c) Suppose now that X has no positive jumps. Calculate the distribution of $\underline{X}_\tau$.

4. Let $(X_t)_{t\geq 0}$ be an α-stable Lévy process with no positive jumps for some $\alpha \in (1,2]$, i.e. such that $\mathbb{E}(e^{\gamma X_t}) = e^{tc\gamma^\alpha}$. For $\alpha = 2$ this is Brownian motion, for $\alpha \in (1,2)$, we have $a_3 = 0$ and $g(y) = \widetilde{c}|y|^{-\alpha-1}$, $y < 0$. For $x \geq 0$ denote $T_x = \inf\{t \geq 0 : X_t > x\}$.

   (a) Using the strong Markov property of $(X_t)_{t\geq 0}$ at T_x, show that $(T_x)_{x\geq 0}$ is a stable subordinator with index $1/\alpha$.

   (b) Let Y have probability density function

   $$f_b(z) = \frac{b}{\sqrt{2\pi z^3}} e^{-b^2/(2z)}, \qquad z > 0.$$

   Calculate the distribution of aY and deduce that $(f_b)_{b\geq 0}$ is the family of densities of stable distributions on $(0,\infty)$ of index $1/2$.

   (c) Deduce that there is a constant $c > 0$ such that

   $$\int_0^\infty e^{\gamma x} f_b(x) dx = e^{-cb\sqrt{\gamma}}.$$

   In fact, $c = \sqrt{2}$.

5. (a) Let $A_1, A_2, \ldots$ be identically distributed and $S_n = A_1 + \ldots + A_n$ the associated random walk. Let $(N_m)_{m\geq 0}$ be an independent random walk. Denote the moment generating function of A_1 by $M(\gamma) = \mathbb{E}(\exp\{\gamma A_1\})$ and assume that it is finite for $\gamma \in (-\varepsilon, \varepsilon)$. Denote the probability generating function of N_1 by $G(s) = \mathbb{E}(s^{N_1})$. Show that $R_m = S_{N_m}$, $m \geq 0$, is also a random walk (with independent and identically distributed increments).

   (b) Let $(X_t)_{t\geq 0}$ be a Lévy process and $(T_s)_{s\geq 0}$ an independent increasing Lévy process. Show that $Y_s = X_{T_s}$, $s \geq 0$, is also a Lévy process.

   (c) (i) Let $(B_t)_{t\geq 0}$ be Brownian motion. For $s \geq 0$ define $T_s = \inf\{t \geq 0 : B_t + bt > s\}$, where $b \geq 0$ is fixed. Using the strong Markov property at T_s, show that $(T_s)_{s\geq 0}$ is an increasing Lévy process.

   (ii) Show that $\exp\{\gamma B_t - \frac{1}{2}\gamma^2 t\}$ is a martingale for all $\gamma \in \mathbb{R}$. Use the Optional Stopping Theorem to show that

   $$\mathbb{E}(\exp\{\rho T_s\}) = \exp\{s(b - \sqrt{b^2 - 2\rho})\}.$$

   This distribution is called the inverse Gaussian distribution (note that $B_{T_s} = s$ means that $s \mapsto T_s$ is the right inverse of $t \mapsto B_t$.) For an independent Brownian motion $(X_t)_{t\geq 0}$, the process $Z_s = X_{T_s}$, $s \geq 0$, obtained as in (b) has the so-called Normal Inverse Gaussian (NIG) distribution. This is another popular process to model financial price processes.

Feedback on the various topics and how you perceived them given your background (no BS3a, no B10a, MScMCF etc.) will be most gratefully received: winkel@stats.ox.ac.uk.