A.6 Time change

1. Consider Brownian motion $(B_t)_{t\geq 0}$ and a continuous increasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$. Set $Z_y = B_{f(y)}$, $y \geq 0$.

   (a) Show that Z has quadratic variation

   $$[Z]_y := \text{p-}\lim_{n\to\infty} \sum_{j=1}^{[2^n y]} (Z_{j2^{-n}} - Z_{(j-1)2^{-n}})^2 = f(y),$$

   where p-lim denotes a limit in probability of random variables.

   (b) Assume that f is piecewise differentiable on $[0,\infty)$ with piecewise constant derivative $\sigma^2(s) := f'(s)$, say taking values σ_j^2 on intervals $[y_{j-1}, y_j)$ for some $0 = y_0 < y_1 < \ldots < y_n < \ldots$. Let $(W_y)_{y\geq 0}$ be a Brownian motion. Show that the process

   $$\widetilde{Z}_y = \int_0^y \sigma(r)dW_r := \sum_{i=1}^{j} (W_{y_i} - W_{y_{i-1}})\sigma_i + (W_y - W_{y_j})\sigma_{j+1},$$

   $y_j \leq y < y_{j+1}$, has the same distribution as Z.
   This result holds in fact for a very wide class of stochastic processes σ. This is why both time-change models and models where the Brownian motion coefficient σ varies with time are called stochastic volatility processes.

2. Consider a Lévy process $(X_t)_{t\geq 0}$ with characteristics (a, σ^2, g) and let $f : [0,\infty) \to [0,\infty)$ be an increasing function. Set $Z_y = X_{f(y)}$, $y \geq 0$.

   (a) Show that the $(Z_y)_{y\geq 0}$ has independent increments.

   (b) Show that $y \mapsto Z_y$ is right-continuous with left limits if $y \mapsto f(y)$ is right-continuous with left limits.

   (c) Show that $(Z_y)_{y\geq 0}$ has stationary increments if and only if either f is linear or $X \equiv 0$.

   (d) Show that the distribution of Z_y is infinitely divisible. For each $y \geq 0$, specify the characteristics in the Lévy-Khintchine representation of the distribution of Z_y as a random variable.

3. Let $(X_t)_{t\geq 0}$ be a compound Poisson process and $f : [0,\infty) \to [0,\infty)$ right-continuous and increasing with $f(0) = 0$ and $f(\infty) = \infty$. Set $Z_y = X_{f(y)}$, $y \geq 0$.

   (a) Suppose first that f is differentiable. Show that

   $$N((a,b]\times(c,d]) = \#\{y \in (a,b] : \Delta Z_y \in (c,d]\}, \quad 0 \leq a < b, -\infty \leq c < d \leq \infty,$$

   is a Poisson counting measure and specify its intensity function $g : [0,\infty) \times [0,\infty) \to [0,\infty)$. Deduce that for all $y \geq 0$, we have $\mathbb{P}(\Delta Z_y = 0) = 1$. Explain how $(Z_y)_{y\geq 0}$ can be constructed from a Poisson point process with intensity function g.

(b) If $\Delta f(s) := f(s) - f(s-) > 0$, calculate the moment generating function of ΔZ_s. What is $\mathbb{P}(\Delta Z_s = 0)$?

(c) If the function

$$f_0(y) = f(y) - \sum_{0 \le s \le y} \Delta f(s), \qquad y \ge 0$$

is differentiable, show that $(Z_y)_{y \ge 0}$ has the same distribution as

$$Z_y^0 + \sum_{0 \le s \le y} J_s, \qquad y \ge 0,$$

where $(Z_y^0)_{y \ge 0}$ is constructed from a Poisson point process as in (a), and the J_s, $s \ge 0$, are independent with moment generating functions as in (b).

4. Let $(X_t)_{t \ge 0}$ be a Poisson process with jump times $(T_n)_{n \ge 1}$.

(a) Give examples of differentiable functions f_i as in A.6.1(b) for which $X_{f_i(y)}$ does and does *not* have the same distribution as

$$\int_0^y \sqrt{f_i'(s)} dX_s := \sum_{n=1}^{X_y} \sqrt{f_i'(T_n)}, \qquad y \ge 0.$$

(b) Find all such functions f_i in (a).

In fact, constant multiples of Brownian motion are the only Lévy processes for which the two distributions coincide for all such functions f.

5. Let $(X_t)_{t \ge 0}$ be a Lévy process with probability density function f_t and $(\tau_y)_{y \ge 0}$ a subordinator with characteristics $(0, g_\tau)$ (sum of jumps, no compensation!). Define

$$g(z) = \int_0^\infty f_t(z) g_\tau(t) dt, \qquad z \in \mathbb{R} \setminus \{0\}.$$

(a) In the case $\mathrm{Var}(X_1) < \infty$ and $\mathrm{Var}(\tau_1) < \infty$, show that g satisfies the requirements of a Lévy density of a Lévy process.

(b) In the case where either τ or X is compound Poisson, show that g also satisfies the requirements of a Lévy density of a Lévy process. More specifically, if X is a compound Poisson process with intensity λ, then we have $\mathbb{P}(X_t = 0) \ge e^{-\lambda t}$; assume that, in fact, $\mathbb{P}(X_t = 0) = e^{-\lambda t}$ and that $\mathbb{P}(X_t \in (a, b)) = \int_a^b f_t(x) dx$ for $(a, b) \not\ni 0$.