A.5 Financial models

1. Consider the CGMY process $(X_t)_{t\geq 0}$ with characteristics $(a, 0, g)$, where

$$g(x) = C\exp\{-G|x|\}|x|^{-Y-1},\ x < 0, \quad g(x) = C\exp\{-M|x|\}|x|^{-Y-1},\ x > 0,$$

for some parameters $Y \in [0, 2)$, $C, G, M \geq 0$ and $a \in \mathbb{R}$.

   (a) Let $\lambda = \int_1^\infty g(x)dx$. Explain how to use the rejection method to simulate the compound Poisson process with jumps in $(1, \infty)$. Calculate the expected number of trials needed in the rejection algorithm. *Hint: We can simulate* $1 + E$ *for* $E \sim \text{Exp}(M)$. *Does any parameter* $\gamma < M$ *for the exponential distribution lead to an improvement?*

   (b) Explain how to use the rejection method to simulate the compensated compound Poisson process of jumps in $(a, b]$ for some $0 < a < b \leq 1$.

   (c) Use (a) and (b) to simulate $X_t^{(2,\varepsilon)}$, the approximation of X_t with small jumps of sizes below ε thrown away.

   (d) Show that for $Y > 0$, the Asmussen-Rosinski theorem allows to approximate by a multiple of Brownian motion the small jumps of sizes below ε thrown away.

2. Consider a one-period model with three assets, a risk-free asset that increases from $A_0 = 1$ to $A_1 = e^\delta$ and two risky assets B and C that can each move up to or down from $B_0 = C_0 = 1$, so that there are four scenarios $\omega_1 = (\text{up}, \text{up})$, $\omega_2 = (\text{up}, \text{down})$, $\omega_3 = (\text{down}, \text{up})$ and $\omega_4 = (\text{down}, \text{down})$. Suppose that $B_1^{\text{up}} = B_1(\omega_1) = B_1(\omega_2) > B_1(\omega_3) = B_1(\omega_4) = B_1^{\text{down}}$ and $C_1^{\text{up}} = C_1(\omega_1) = C_1(\omega_3) > C_1(\omega_2) = C_1(\omega_4) = C_1^{\text{down}}$.

   (a) For a portfolio (T_0, U_0, V_0) of T_0 units of A, U_0 units of B and V_0 units of C, specify the value W_0 and $W_1(\omega_i)$ of the portfolio at times 0 and 1, $i = 1, 2, 3, 4$.

   (b) Show that this model is arbitrage-free if and only if $B_1(\omega_1) > A_1 > B_1(\omega_4)$ and $C_1(\omega_1) > A_1 > C_1(\omega_4)$.

   Consider the arbitrage-free case now.

   (c) Give an example of a contingent claim $W_1(\omega_i)$ that cannot be hedged.

   (d) Show that contingent claims of the form $W_1(\omega_1) = W_1(\omega_2)$, $W_1(\omega_3) = W_1(\omega_4)$ can be hedged and priced as $e^{-\delta}\mathbb{E}(W_1)$, where you should specify

$$q_B = \mathbb{P}(B_1 = B_1^{\text{up}}) \qquad \text{and} \qquad 1 - q_B = \mathbb{P}(B_1 = B_1^{\text{down}}).$$

   In particular, $e^{-\delta t}B_t$, $t = 0, 1$, is then a martingale. Is q_B unique?

   (e) State the result analogous to (d) for contingent claims relating to C only rather than B only.

   (f) Show that there are infinitely many possibilities to choose

$$p_1 = \mathbb{P}(\omega_1) = \mathbb{P}(B_1 = B_1^{\text{up}}, C_1 = C_1^{\text{up}}),\ p_2 = \mathbb{P}(\omega_2),\ p_3 = \mathbb{P}(\omega_3) \text{ and } p_4 = \mathbb{P}(\omega_4)$$

   so that $e^{-\delta t}B_t$ and $e^{-\delta t}C_t$, $t = 0, 1$ are martingales.

(g) Consider the contingent claim $W_1(\omega_1) = 1$, $W_1(\omega_2) = W_1(\omega_3) = W_1(\omega_4) = 0$. Using the range of possibilities for $p = (p_1, p_2, p_3, p_4)$, give the range of proposed (arbitrage-free) prices $W_0 = e^{-\delta}\mathbb{E}_p(W_1)$.

3. Consider $X_t = N_t - \mu t$ for a Poisson process N of rate $\lambda \in (0, \infty)$ and a drift coefficient $\mu \in (0, \infty)$. Let

$$S_n^{(\varepsilon)} = \sum_{i=1}^{n} X_i^{(\varepsilon)},$$

where $(X_i^{(\varepsilon)})_{i\geq 1}$ is a sequence of independent random variables with common probability mass function given by

$$\mathbb{P}(X_i^{(\varepsilon)} = 1 - \mu\varepsilon) = 1 - e^{-\lambda\varepsilon} =: p_\varepsilon \quad \text{and} \quad \mathbb{P}(X_i^{(\varepsilon)} = -\mu\varepsilon) = 1 - p_\varepsilon.$$

(a) Show that $S_{[t/\varepsilon]}^{(\varepsilon)} \to X_t$ in distribution as $\varepsilon \downarrow 0$. *Hint: You can either prove this directly or consider* $T_n^{(\varepsilon)} = S_n^{(\varepsilon)} + n\mu\varepsilon$ *first.*

(b) Show that the market model $(e^{\delta\varepsilon n}, e^{S_n^{(\varepsilon)}})_{n\geq 0}$ is arbitrage-free if and only if $-\mu < \delta < 1/\varepsilon - \mu$ and then also complete.

Consider the arbitrage-free case now.

(c) Show that the martingale probabilities are

$$q_\varepsilon = \mathbb{P}(\widetilde{S}_1^{(\varepsilon)} = 1 - \mu\varepsilon) = \frac{e^{\varepsilon(\delta+\mu)} - 1}{e - 1}.$$

(d) Show that under the martingale probabilities

$$\widetilde{S}_{t/\varepsilon}^{(\varepsilon)} \to \widetilde{X}_t = \widetilde{N}_t - \mu t,$$

where $(\widetilde{N}_t)_{t\geq 0}$ is a Poisson process with rate $(\delta + \mu)/(e - 1)$.

(e) Show that in the notation of part (d), the discounted process $e^{-\delta t}R_t$ associated with $R_t = e^{\widetilde{N}_t - \mu t}$ is a martingale. By conditioning on N_t and $\widetilde{N}_t$, respectively, explain briefly why the distribution of $e^{\widetilde{N}_t - \mu t}$ can be seen as providing martingale probabilities (an equivalent martingale measure) for $e^{N_t - \mu t}$.

(f)* Using the following subsets of the set of right-continuous paths with left limits

$$D_\mu = \{f \in D([0, 1], (0, \infty)) : \Delta \log f(t) = 1 \text{ or } (\log f)'(t) = -\mu \text{ for all } t \leq 1\},$$

show that $(e^{\widetilde{N}_t - \mu t})_{0\leq t\leq 1}$ is the only exponential Lévy process that has the same set of possible paths as $(e^{N_t - \mu t})_{0\leq t\leq 1}$ and whose discounted process is a martingale. *Remark: In fact, it is the only process that has the same set of possible paths as* $(e^{N_t - \mu t})_{0\leq t\leq 1}$ *and whose discounted process is a martingale, so the market model is complete.*