A.4 Simulation

Question A.4.3 is the most relevant on this sheet for MSc MCF students. Since it builds on the others, their statements, particularly A.4.1(e), A.4.2(b)-(c) are also relevant.

1. (a) Let $(X_t)_{t\geq 0}$ be a Gamma process with $X_t \sim \text{Gamma}(t,1)$ for all $t > 0$. Consider $A = X_a$ and $B = X_{a+b} - X_a$ for some $a > 0$ and $b > 0$. Show that $R = A/(A+B)$ and $S = A + B$ are independent and that $R \sim \text{Beta}(a,b)$, where Gamma and Beta densities are recalled as follows:

$$f_S(s) = \frac{s^{a+b-1}e^{-s}}{\Gamma(a+b)},\ s \in (0,\infty), \quad f_R(r) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} r^{a-1}(1-r)^{b-1},\ r \in (0,1).$$

Deduce, vice versa, the distribution of $(SR, S(1-R))$ for independent $S \sim \text{Gamma}(c,1)$ and $R \sim \text{Beta}(cp, c(1-p))$, for some $c > 0$ and $p \in (0,1)$.

(b) Let $U \sim \text{Unif}(0,1)$ and $a > 0$. Show that $X = U^{1/a} \sim \text{Beta}(a,1)$.

(c) Let $U \sim \text{Unif}(0,1)$ and $V \sim \text{Unif}(0,1)$ be independent and $a \in (0,1)$. Calculate for $Y = U^{1/a}$ and $Z = V^{1/(1-a)}$

$$\mathbb{P}\left(\frac{Y}{Y+Z} \leq t, Y + Z \leq 1\right)$$

and deduce that the conditional distribution of $W = Y/(Y+Z)$ given $Y+Z \leq 1$ is $\text{Beta}(a, 1-a)$. *Hint: Write both inequalities as constraints on Z to find the bounds when writing the probability as a double integral.*

(d) In the setting of (c), show that the conditional distribution of TW given $Y + Z \leq 1$, for an independent $T \sim \text{Exp}(1) = \text{Gamma}(1,1)$ random variable, is $\text{Gamma}(a,1)$.

(e) Consider the following procedure due to Johnk. Let $a \in (0,1)$.

   1. Generate two independent random numbers $U, V \sim \text{Unif}(0,1)$.
   2. Set $Y = U^{1/a}$ and $Z = V^{1/(1-a)}$.
   3. If $Y + Z \leq 1$ go to 4., else go to 1.
   4. Generate an independent $C \sim \text{Unif}(0,1)$ and set $T = -\ln(C)$.
   5. Return the number $TY/(Y+Z)$.

What is this procedure doing? Explain its relevance for simulations.

2. (a) Based on Question A.4.1, explain how to generate a $\text{Beta}(a,b)$ random variable from a sequence of $\text{Unif}(0,1)$ random variables, for any $a > 0$ and $b > 0$. *Hint: Consider $a \in (0,1)$ first and use the additivity of Gamma variables to generate $\text{Gamma}(a,1)$ variables, from which the Beta variable can be constructed.*

(b) Set $X_0 = 0$ and generate $X_1 \sim \text{Gamma}(1,1)$. For $n \geq 0$, having generated $X_{k2^{-n}}$, $k = 0,\ldots,2^n$, generate $B_{k,n} \sim \text{Beta}(2^{-n-1}, 2^{-n-1})$ and set $X_{(2k-1)2^{-n-1}} = X_{(k-1)2^{-n}} + B_{k,n}(X_{k2^{-n}} - X_{(k-1)2^{-n}})$, $1 \leq k \leq 2^n$. Show that this approximates a Gamma process on the time interval $[0,1]$.

(c) What are the advantages of this method when compared with the plain version of the time discretisation method (Method 1)?

3. Consider a variant of the Variance Gamma process of the form $V_t = at + G_t - H_t$ where $a \in \mathbb{R}$, $G_1 \sim \text{Gamma}(\alpha_+, \beta_+)$ and $H_1 \sim \text{Gamma}(\alpha_-, \beta_-)$

   (a) For what values of $a, \alpha_\pm, \beta_\pm$ is V a martingale?

   (b) Write out the steps needed to simulate V_t

   - by Method 1 (using a random walk with increment distribution $\sim V_\delta$)
   - by Method 1 (applied to G and H separately)
   - by the refinement of Method 1 given in A.4.3
   - by Method 2 (simulating the Poisson point process of jumps truncated at ε)

   (c) Carry out 9 simulations for a range of parameters $\alpha_+ \in \{1, 10, 100\}$ and $\alpha_- \in \{10, 100, 1000\}$, $\beta_\pm = \alpha_\pm^2/2$ and a such that V is a martingale. *This part of this question is optional.*

Warning: The incomplete Gamma function $\Gamma_t(a) = \int_0^t x^{a-1}e^{-x}dx$ cannot be simplified into closed form (nor expressed in terms of the Gamma function), except for some special values of a such as $a \in \mathbb{N}$. There are, however, numerical procedures to evaluate $\Gamma_t(a)$, which we will not address in this course.

If you have not used R, but would like to, you will find the "First steps with R" at

http://www.stats.ox.ac.uk/~myers/stats_materials/R_intro/WA5_R.pdf

useful. Following are brief explanations of the commands used in the sample file

http://www.stats.ox.ac.uk/~winkel/gammavgamma.R

This is a script file, which has to be run in the command window "R Console" to make the new commands available, e.g. select "Run all" in the drop-down menu "Edit".

- `runif(n,a,b)` *generates an n-vector of uniform variables on $[a, b]$.*
- `qgamma(u,a,b)` *evaluates $F^{-1}(u)$ for the Gamma(a, b) inverse distribution function F^{-1} at u. If* `u` *is a vector,* `qgamma` *is applied to each component.*
- `1:n` *generates the vector $(1, 2, \ldots, n)$. Multiplication of vectors* `v` *by scalars* `a` *can be written as* `a*v`*, similarly for addition and subtraction of vectors.*
- `plot(x,y,pch=".",sub=paste("text"))` *produces a scatter plot of pairs (x_i, y_i) for vectors* `x` *and* `y`*, with* `.` *marking the points, and* `text` *in the caption.*
- `psum <- function(vector){...}` *defines a new command* `psum` *that takes a vector* `vector` *as an argument. When this line is executed, the command is just made available. To execute the command, type* `psum(v)` *for a vector* `v` *to get the partial sums of* `v` *displayed, or* `s=psum(v)` *to create a new vector* `s` *containing the partial sums of* `v`*.*