A.3 Construction of Lévy processes

The first question is about martingales, which I have not "recalled" in lectures. The relevant section is Section 4.4 in the notes, which I'd like to encourage you to read. The other questions do not rely on material that I have not discussed in the lectures, but some parts do rely on parts (a) or (b) of the first question, specifically the fact that $\mathbb{E}(e^{\gamma X_t}) = e^{t\Psi(\gamma)}$ *or* $\mathbb{E}(e^{i\lambda X_t}) = e^{-t\psi(\lambda)}$ *for all* $t \geq 0$*, with notation as in the first question.*

1. Let $(X_t)_{t\geq 0}$ be a Lévy process with $\mathbb{E}(X_1^2) < \infty$. Denote $\mu = \mathbb{E}(X_1)$, $\sigma^2 = \mathrm{Var}(X_1)$ and $e^{-\psi(\lambda)} = \mathbb{E}(e^{i\lambda X_1})$. If $\mathbb{E}(e^{\gamma X_1}) < \infty$, denote $e^{\Psi(\gamma)} = \mathbb{E}(e^{\gamma X_1})$. Show that the following processes are martingales.

   (a) $\exp\{\gamma X_t - t\Psi(\gamma)\}$, if $\mathbb{E}(e^{\gamma X_1}) < \infty$. *Hint: First show that* $\mathbb{E}(\exp\{\gamma X_t\}) = e^{t\Psi(\gamma)}$ *for all* $t = 1/m$*, then for all* $t \in \mathbb{Q} \cap [0, \infty)$*, and finally, using right-continuity, for all* $t \in [0, \infty)$*.*

   (b) $\exp\{i\lambda X_t + t\psi(\lambda)\}$.

   (c) $X_t - t\mu$.

   (d) $(X_t - t\mu)^2 - t\sigma^2$.

2. Let $(\Delta_t)_{t\geq 0}$ be a Poisson point process with intensity function $g(x) = x^\kappa e^{-x}$, $x \in (0, \infty)$, for a parameter $\kappa \in \mathbb{R}$.

   (a) Let $\kappa \in (-1, \infty)$. Show that

   $$C_t = \sum_{s \leq t} \Delta_s$$

   is a compound Poisson process. Specify its jump rate λ and jump density h.

   (b) Let $\kappa \leq -1$. Show that $\Delta_t^{(n)} = \Delta_t 1_{\{\Delta_t > 1/n\}}$, $t \geq 0$, is a Poisson point process. Specify its intensity function. Show that

   $$C_t^{(n)} = \sum_{s \leq t} \Delta_s^{(n)}$$

   is a compound Poisson process.

   (c) For $C_t^{(n)}$ as defined in (b), show that $C_t^{(n)}$ converges to a limit $\to C_t < \infty$ as $n \to \infty$ if and only if $\kappa > -2$. Specify the moment generating function of C_t.

   (d) Show that for $\kappa > -2$, we have

   $$\sup_{s \leq t} |C_s^{(n)} - C_s| = |C_t^{(n)} - C_t|.$$

   Deduce that $C_t^{(n)} \to C_t$ a.s. (or in probability) locally uniformly.

   (e) Show that $C_t^{(n)} - \mathbb{E}(C_t^{(n)})$ converges for $\kappa > -3$. Show that the limit is a Lévy process.

3. Let $(X_t)_{t\geq 0}$ be a stable subordinator, that is an increasing Lévy process with $(c^{1/\alpha} X_{t/c})_{t\geq 0} \sim X$ for all $c > 0$ (scaling relation) for some $\alpha \in \mathbb{R}$.

(a) Show that for all $\mu \geq 0$ and $t \geq 0$, we have $\mathbb{E}(e^{-\mu X_t}) \in (0,1]$. Denote $\Phi_t(\mu) = -\ln(\mathbb{E}(e^{-\mu X_t}))$ and $\Phi = \Phi_1$.

(b) Show that $\Phi_t(\mu) = t\Phi(\mu)$ for all $t \geq 0$. Deduce from the scaling relation that $\Phi(\mu) = \Phi(1)\mu^\alpha$ for all $\mu \geq 0$.

(c) Show that $\frac{\partial}{\partial\mu}\mathbb{E}(e^{-\mu Y}) \leq 0$ and $\frac{\partial^2}{\partial\mu^2}\mathbb{E}(e^{-\mu Y}) \geq 0$ for any nonnegative random variable Y and for all $\mu > 0$ with equality if and only if $\mathbb{P}(Y = 0) = 1$. Deduce that $\alpha \in (0,1]$ or $\mathbb{P}(X \equiv 0) = 1$.

(d) By letting $\mu \downarrow 0$ in (c), show that $\mathbb{E}(X_t) = \infty$ for all $t > 0$ and $\alpha \in (0,1)$, unless $X \equiv 0$.

(e) For $\alpha \in (0,1)$, calculate $g : (0,\infty) \to (0,\infty)$ such that

$$\Phi(\mu) = \int_0^\infty (1 - e^{-\mu x})g(x)dx.$$

Hint: Apply a similar method as for Question A.2.2.(a).

(f) For every $\alpha \in (0,1]$ and $\Phi(1) = b > 0$, show that there exists a stable subordinator $(X_t)_{t\geq 0}$.

4.* (a) Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be independent stable subordinators with common index α and intensities $b_X = \Phi_X(1)$ and $b_Y = \Phi_Y(1)$. Show that $Z = X - Y$ is also a stable process with index α in the sense that $(c^{1/\alpha}X_{t/c})_{t\geq 0} \sim X$ for all $c > 0$.

(b) Let H be a real-valued random variable with symmetric distribution, i.e. $H \sim -H$. Show that $\mathbb{E}(e^{i\lambda H}) \in \mathbb{R}$ for all $\lambda \in \mathbb{R}$. *Hint:* $e^{ix} = \cos(x) + i\sin(x)$.

(c) In the setting of (a), for the special case $b_X = b_Y$, show that $\mathbb{E}(\exp\{i\lambda Z_t\}) = \exp\{-b_X|\lambda|^\alpha\}$, $\lambda \in \mathbb{R}$. *Hint: Show that Z_t is symmetric and that all symmetric stable processes have a characteristic function of this form for some $b > 0$. You may assume without proof that all characteristic functions are continuous in $\lambda \in \mathbb{R}$ and that those of infinitely divisible distributions have no zeros.*

(d) Fix $\widetilde{b} > 0$. Show that for $\alpha \in (0,2)$, the function

$$\widetilde{\psi}(\lambda) = \int_{-\infty}^\infty (\cos(\lambda x) - 1)\widetilde{b}|x|^{-\alpha-1}dx$$

has the property $\widetilde{\psi}(\lambda c^{1/\alpha}) = c\widetilde{\psi}(\lambda)$ for all $c > 0$ and $\lambda \in \mathbb{R}$. Deduce that

$$\widetilde{\psi}(\lambda) = b|\lambda|^\alpha$$

for some $b \in \mathbb{R}$.

Warning: Densities of stable processes are only known in closed form for some special cases $\alpha \in \{1/2, 1, 2\}$. It is known, however, that they all have smooth probability density functions.

You may use results from the lectures and previous assignment sheets without proof if you state them clearly, except that the Lévy density g of stable processes is to be derived here, and its form should not be assumed (although it may help to know what to aim for).

Question A.3.2 is the most relevant on this sheet for MSc MCF students – if pushed for time, please focus on this one. Question A.3.1 is also relevant.