A.1 Infinite divisibility and limits of random walks

Please hand in scripts by Monday 25 January 2010, 1.30pm, Department of Statistics. Starred problems are harder and optional. This problem sheet is based on week 1 only. MSc students who consider this option seriously, should attempt and hand in work for at least problems 1.(a), 2 and 3.(c). Undergraduates should hand in work for all problems.

1. (a) Show that $Y \sim \text{Gamma}(\alpha, \beta)$ with density

$$g(x) = \frac{\beta^\alpha x^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta x}, \quad x \geq 0,$$

is an infinitely divisible distribution and that the independent and identically distributed "divisors" $Y_{n,j}$ in $Y_{n,1} + \ldots + Y_{n,n} \sim Y$ are also Gamma distributed.

(b) Show that the $G \sim \text{geom}(p)$ distribution with probability mass function

$$\mathbb{P}(G = n) = p^n(1 - p), \quad n \geq 0,$$

is infinitely divisible and that "divisors" are not geometrically distributed. *Hint: Study sums of geometric variables and guess the "divisor" distribution.*

(c)* Show that the uniform distribution on $[0, 1]$ is *not* infinitely divisible.

2. (a) Let X be a Lévy process. Show that the distribution of $X_t - X_s$ is infinitely divisible for all $t \geq s \geq 0$.

(b) Let X and Y be independent Lévy processes and $a, b \in \mathbb{R}$. Show that $aX + bY$ is also a Lévy process.

(c) Let C and D be two independent Gamma Lévy processes with $C_1 \sim D_1 \sim \text{Gamma}(\alpha, \sqrt{2\mu})$. Determine the moment generating function of $C_s - D_s$, $s \geq 0$.

We will see later that the process $C - D$ has in fact the same distribution as $Z_s = B_{T_s}$, $s \geq 0$, for a Brownian motion B and a Gamma Lévy process T. It is called Variance Gamma process, because $\text{Var}(B_t) = t$ *implies* $\text{Var}(B_{T_s}|T_s) = T_s \sim \text{Gamma}(\alpha s, \mu)$. *It is a popular model for financial stock prices.*

3. A large number N of policy holders in a given time period make claims independently of one another with small probability p_N. Denote by S_N the total number of policy holders who make a claim in the time period. Assume that claim amounts $A_1, A_2, \ldots$ are independent and identically distributed.

(a) State the Poisson limit theorem and use probability generating functions to prove it.

(b) Explain why S_N is approximately Poisson distributed and give its parameter.

(c) Calculate the moment generating function of the total amount T_N of claims.

(d) Show that the distribution of T_N is well-approximated by a compound Poisson distribution, by showing that as $N \to \infty$ and $Np_N \to \lambda$,

$$T_N = \sum_{n=1}^{S_N} A_n \to T_\infty = \sum_{n=1}^{S_\infty} A_n \qquad \text{in distribution, as } N \to \infty,$$

where $S_\infty \sim \text{Poi}(\lambda)$ is independent of $A_1, A_2, \ldots$.

4. (a) Let $A_1, A_2, \ldots$ be independent and identically distributed random variables with $\mu = \mathbb{E}(A_1)$ and $\sigma^2 = \mathrm{Var}(A_1) \in (0, \infty)$. Define

$$Y_{n,k} = \frac{A_k - \mu}{\sigma\sqrt{n}} \qquad \text{and} \qquad V_n = \sum_{k=1}^{n} Y_{n,k}.$$

(i) Formulate the Central Limit Theorem for $A_1, A_2, \ldots$ in terms of V_n.

(ii)* Let $x > 0$. Apply Markov's inequality

$$\mathbb{P}(|X| > \varepsilon) \leq \varepsilon^{-2}\mathbb{E}(X^2)$$

to the random variable

$$X = |A_1 - \mu| 1_{\{|A_1-\mu| \geq \sigma x\sqrt{n}\}} = \begin{cases} |A_1 - \mu| & \text{if } |A_1 - \mu| \geq \sigma x\sqrt{n}, \\ 0 & \text{otherwise,} \end{cases}$$

to show that there is a sequence $\gamma_n(x) \to 0$ as $n \to \infty$ with

$$\mathbb{P}(|A_1 - \mu| > \sigma x\sqrt{n}) \leq \frac{\gamma_n(x)}{n}.$$

(iii) Define $M_n = \max\{|Y_{n,1}|, \ldots, |Y_{n,n}|\}$. Show that $\mathbb{P}(M_n \leq x) \to 1$ as $n \to \infty$, for all $x > 0$. Deduce that $M_n \to 0$ in probability.

(b) Consider an urn initially containing r red and s black balls, $r, s \geq 1$. One ball is drawn with replacement (stage 1). After this, a black ball is added to the urn and two balls are drawn, each with replacement (stage 2). After this, another black ball is added and three balls drawn with replacement (stage 3). Continue so that n balls are drawn at stage n followed by the addition of a single black ball. Let $Y_{n,k} = 1$ resp. 0 if the kth ball of stage n is red resp. black, $1 \leq k \leq n$ and $W_n = Y_{n,1} + \ldots + Y_{n,n}$.

(i) Show that $W_n \to \mathrm{Poi}(r)$.

(ii) Show that $\mathbb{P}(Y_{n,k} = 0) \to 1$ as $n \to \infty$.

(iii) Define $M_n = \max\{|Y_{n,1}|, \ldots, |Y_{n,n}|\}$. Show that $\mathbb{P}(M_n = 0) \to e^{-r}$ as $n \to \infty$. Deduce that $M_n \not\to 0$ in probability.

(c)* (i) Formulate Donsker's theorem and the process version of the Poisson limit theorem in the settings of (a) and (b). *Hint: Consider only* $t \in [0, 1]$ *and evaluate the discrete processes* V *and* W *at* $[nt]$, $t \in [0, 1]$.

(ii) Show that in both cases M_n converges in distribution to the size of the biggest jump of the limit process during the time interval $[0, 1]$.

Hint for 3.(c)-(d): If $(X_n)_{n\geq 0}$ *or* $(X_t)_{t\geq 0}$ *is a stochastic process and* N *or* T *an independent random time with probability mass function* $n \mapsto \mathbb{P}(N = n)$ *or probability density function* $t \mapsto f_T(t)$*, then for real-valued functions* g *for which the expectations exist, we have*

$$\mathbb{E}(g(X_N)) = \sum_{n=0}^{\infty} \mathbb{E}(g(X_n))\mathbb{P}(N = n) \qquad \textit{and} \qquad \mathbb{E}(g(X_T)) = \int_0^{\infty} \mathbb{E}(g(X_t)) f_T(t)dt.$$

Course website: http://www.stats.ox.ac.uk/~winkel/ms3b.html