BS4b/MScApplStats
Actuarial Science II

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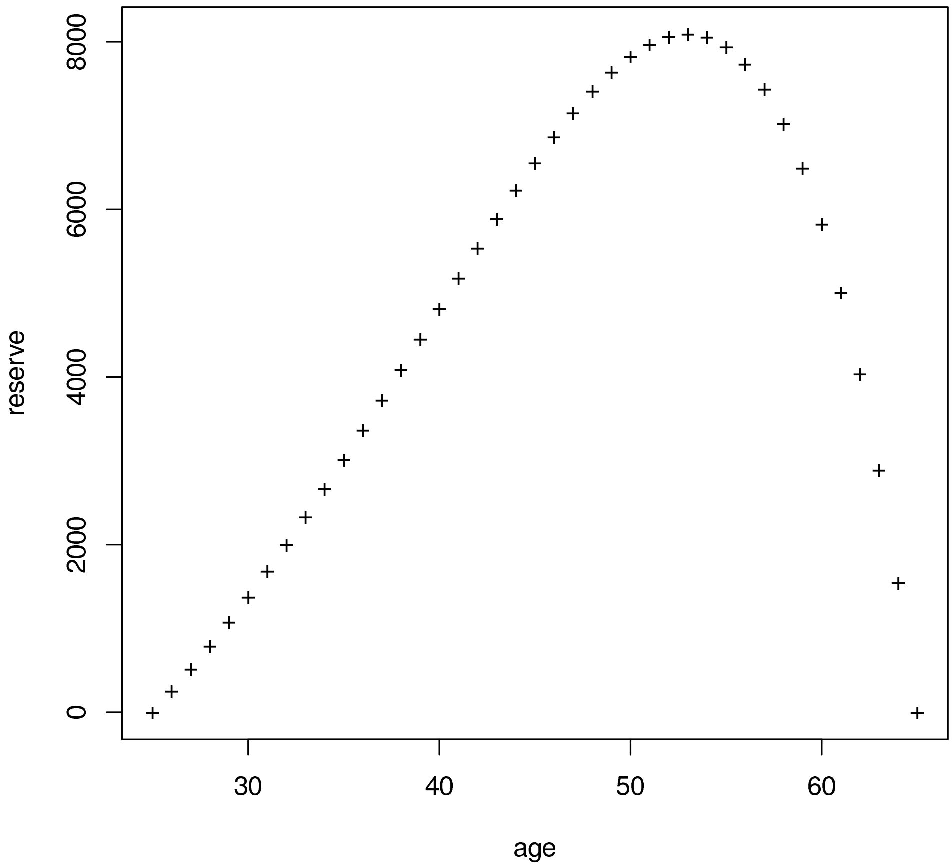
based in part on earlier hand-written notes by Daniel Clarke



HT 2011

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Matthias Winkel – 16 lectures HT 2011

Prerequisites

Part A Probability is useful, but not essential. If you have not done Part A Probability, make sure that you are familiar with Mods work on Probability.

Aims

This unit is supported by the Institute of Actuaries. It has been designed to give the undergraduate mathematician an introduction to the financial and insurance worlds in which the practising actuary works. Students will cover the basic concepts of risk management models for investment and mortality, and for discounted cash-flows. In the examination, a student obtaining at least an upper second class mark on this unit can expect to gain exemption from the Institute of Actuaries paper CT1, which is a compulsory paper in their cycle of professional actuarial examinations. An Independent Examiner approved by the Institute of Actuaries will inspect examination papers and scripts and may adjust the pass requirements for exemptions.

Synopsis

Uncertain payments, corporate bonds, fair prices and risk.

Simple stochastic interest rate models, mean-variance models, log-normal models. Mean, variance and distribution of accumulated values of simple sequences and payments.

Stability of investment portfolios, analysis of small changes in interest rates and Redington immunisation.

The no-arbitrage assumption, the law of one price, and arbitrage-free pricing. Price and value of forward contracts. Effect of fixed income or fixed dividend yield from the asset. Futures, options and other financial products.

Single decrement model. Present values and accumulated values of a stream of payments taking into account the probability of the payments being made according to a single decrement model. Annuity functions and assurance functions for a single decrement model. Risk and premium calculation.

Liabilities under a simple assurance contract or annuity contract. Premium reserves, Thieles differential equation. Expenses and office premiums.

Reading

All of the following are available from the Publications Unit, Institute of Actuaries, 4 Worcester Street, Oxford OX1 2AW

- Subject 102[CT1] Financial Mathematics Core Reading Faculty & Institute of Actuaries.
- Subject CT5[105] Contingencies Core Reading Faculty & Institute of Actuaries.
- J.J. McCutcheon and W.F. Scott: An Introduction to the Mathematics of Finance. Heinemann (1986)
- H.U. Gerber: Life Insurance Mathematics. 3rd edition, Springer (1997)
- N.L. Bowers et al: Actuarial mathematics. 2nd edition, Society of Actuaries (1997)

Contents

Lecture 1

Revision of MT material and introduction to HT material

Reading: Michaelmas Term notes

The material last term was largely deterministic in the sense that it was generally assumed that cash-flows as well as interest rates were known, not necessarily constant, but with variability given by a time-varying rate or force of interest. In practice, such situations arise when analysing the past or when doing a scenario analysis of the future. In the last few lectures you saw a few pointers at random models. One elementary example that was aimed at analysing the future was the discounted dividend model for pricing equity shares, where *expected* future dividend payments appeared. This can be expanded by the specification of probability distributions for the amounts of dividend payments. A more sophisticated example was the brief discussion of risk, particularly the notion of pooling of independent risks.

This term, we will look more generally at random models for cash-flows and interest rates, and have a particular focus on the important special case, where the randomness depends on a single random time, which can model an individual's time of death, but also a company's insolvency time.

In this lecture we recall the basic setup from last term and introduce the directions we take this term.

1.1 Cash-flow and interest rate models

The overarching framework was and is that of a cash-flow c in an interest rate model δ. More specifically, we often deal with

- cash-flows discrete $c = ((t_j, c_j), 1 \leq j \leq n)$ and continuous $c = (c(t), t \geq 0)$
- in a constant interest model of an interest rate $i > -100\%$, equivalently a constant force of interest $\delta = \log(1 + i) \in \mathbb{R}$, in which we express present (discounted) values

$$\mathrm{Val}_0(c) = \sum_{j=1}^{n} c_j v(t_j) \qquad \text{and, respectively,} \qquad \mathrm{Val}_0(c) = \int_0^\infty c(t)v(t)dt,$$

where $v(t) = v^t = (1 + i)^{-t} = e^{-\delta t}$ and $v = (1 + i)^{-1} = e^{-\delta}$ are discount factors,

- or in a time-varying interest model of interest rates i_j for the jth year, $j \geq 1$, so that $v(n) = (1+i_1)^{-1} \cdots (1+i_n)^{-1}$,
- or in a time-varying interest model of forces of interest $\delta(t)$ at time t, $t \geq 0$, so that

$$v(t) = \exp\left\{-\int_0^t \delta(s)ds\right\};$$

- regular cash-flows where $t_j = j/p$, $1 \leq j \leq n = mp$, and $c_j = 1/p$ in the constant-i-model have

$$a_{\overline{m}|}^{(p)} = \mathrm{Val}_0(c) = \frac{1-v^m}{i^{(p)}}, \qquad \text{where } \left(1+\frac{i^{(p)}}{p}\right)^p = 1+i$$

and

$$s_{\overline{m}|}^{(p)} = \mathrm{Val}_m(c) = \mathrm{Val}_0(c)(1+i)^m = \frac{(1+i)^m - 1}{i^{(p)}}.$$

For any cash-flow c with inflows $c_j > 0$ and outflows $c_j < 0$, which we can see as an investment deal or loan scheme, we define the yield to be the constant interest rate i for which $\mathrm{Val}_0(c) = 0$, provided that such $i > -1$ exists and is unique; otherwise the yield remains undefined.

1.2 Random cash-flows and stochastic interest rate models

In practice, at least some part of a cash-flow is unknown in advance, or at least subject to some uncertainty. The following definition provides a suitably general framework.

Definition 1 A random vector $((T_j, C_j), 1 \leq j \leq N)$ of *times* $T_j \in [0, \infty)$ and *amounts* $C_j \in \mathbb{R}$ possibly with random length N is called a (discrete) random cash-flow. A random locally Riemann integrable function $C : [0, \infty) \to \mathbb{R}$ of *payment rates* $C(t)$ at time $t \geq 0$ is called a (continuous) random cash-flow.

Similarly, future interest rates are often subject to change.

Definition 2 A sequence of random variables I_j, $j \geq 1$, with $\mathbb{P}(I_j > -1) = 1$, is called a stochastic interest rate model. A random locally Riemann integrable function $\Delta : [0, \infty) \to \mathbb{R}$ of forces of interest is called a stochastic force of interest model.

We can associate random discount factors $V(n) = (1+I_1)^{-1} \cdots (1+I_n)^{-1}$ or $V(t) = \exp\{-\int_0^t \Delta(s)ds\}$ and specify random values as before:

$$\mathrm{Val}_0(C) = \sum_{j=1}^{N} C_j V(T_j).$$

Difficulties arise both in specifying (joint!) distributions of random cash-flows and stochastic interest rates, and in analysing such models. We postpone the discussion of stochastic interest rates to week 2 and suppose for the remainder of this week that we are given a (constant or time-varying) rate of interest i or force of interest δ.

1.3 An example

Every company has a (usually small) risk of default that should be taken into account when assessing any corporate investments. This can be dealt probabilistically.

Example 3 Suppose, you are offered a zero-coupon bond of £100 nominal redeemable at par at time 1. Current market interest rates are 4%, but there is also a 10% risk of default, in which case no redemption payment takes place. What is the fair price?

The present value of the bond is 0 with probability 0.1 and $100(1.04)^{-1}$ with probability 0.9. The weighted average $A = 0.1 \times 0 + 0.9 \times 100(1.04)^{-1} \approx 86.54$ is a sensible candidate for the fair price.

1.4 Fair premiums and risk under uncertainty

Definition 4 In an interest model $\delta(\cdot)$, the *fair premium* for a random cash-flow (of fixed length)

$$C = ((T_1, C_1), \ldots, (T_n, C_n))$$

(typically of benefits $C_j \geq 0$) is the mean value

$$A = \mathbb{E}\left(\mathrm{DVal}_0(C)\right) = \sum_{j=1}^{n} \mathbb{E}\left(C_j v(T_j)\right)$$

where $v(t) = \exp\{-\int_0^t \delta(s)ds\}$ is the discount factor at time t.

Example 5 If the times of a random cash-flow are deterministic $T_j = t_j$ and only the amounts C_j are random, the fair premium is

$$A = \sum_{j=1}^{n} \mathbb{E}(C_j)v(t_j)$$

and depends only on the mean amounts, since the deterministic v^{t_j} can be taken out of the expectation. Such a situation arises for share dividends.

Example 6 If the amounts of a random cash-flow are fixed $C_j = c_j$ and only the times T_j are random, the fair premium is

$$A = \sum_{j=1}^{n} c_j \mathbb{E}(v(T_j))$$

and in the case of constant δ we have $\mathbb{E}(v(T_j)) = \mathbb{E}(\exp\{-\delta T_j\}) = \mathbb{E}(v^{T_j})$ which is the so-called Laplace transform or (moment) generating function. Such a situation arises for life insurance payments, where usually a single payment is made at the time of death.

The fair premium is just an average of possible values, i.e. the actual random value of the cash-flow C is higher or lower with positive probability each as soon as $\mathrm{DVal}_0(C)$ is truly random. In a typical insurance framework when $C_j \geq 0$ represents benefits that the insurer has to pay us under the policy, we will be charged a premium that is higher than the fair premium, since the insurer has (expenses that we neglect and) the risk to bear that we want to get rid of by buying the policy.

Call A_+ the higher premium that is to be determined. Clearly, the insurer is concerned about his loss $(\mathrm{DVal}_0(C) - A_+)^+$, or expected loss $\mathbb{E}\,(\mathrm{DVal}_0(C) - A_+)^+$. In some special cases this can be evaluated, sometimes the quantity under the expectation is squared (so-called squared loss). A simpler quantity is the probability of loss $\mathbb{P}(\mathrm{DVal}_0(C) > A_+)$. One can use Tchebychev's inequality to demonstrate that the insurer's risk of loss gets smaller with the larger the number of policies.

1.5 Overview

This term's material will be organised as follows.

- In Lecture 2, we will discuss corporate bonds under risk of default and thereby recall some Probability that will also serve later in the course.
- In Lectures 3 and 4, we will discuss stochastic interest rate models.
- In Lectures 5-8, we discuss arbitrage-free pricing and immunisation against changes in interest rates in a framework of asset management to meet fixed liabilities.
- Lectures 9-14 or so will be on life insurance, where randomness enters via lifetimes. We will discuss life tables and their use to price life assurances, life annuities and related life products. We will also take the insurer's perspective, include expenses and office premiums, as well as associated risk. We will discuss premium reserves and liabilities.
- The last lecture or two will draw together some threads related to risk and premium calculation.

Lecture 2

Corporate bonds and insolvency risk

2.1 Uncertain payment

Before discussing pricing issues specific to corporate investments, we set out a general framework. The following is an important special case of Example 5.

Example 7 Let t_j be fixed and

$$C_j = \begin{cases} c_j & \text{with probability } p_j, \\ 0 & \text{with probability } 1 - p_j. \end{cases}$$

So we can say $C_j = B_j c_j$, where B_j is a Bernoulli random variable with parameter p_j, i.e.

$$B_j = \begin{cases} 1 & \text{with probability } p_j, \\ 0 & \text{with probability } 1 - p_j. \end{cases}$$

For the random cash-flow $C = ((t_1, B_1 c_1), \ldots, (t_n, B_n c_n))$, we have

$$A = \sum_{j=1}^{n} \mathbb{E}(B_j c_j v(t_j)) = \sum_{j=1}^{n} c_j v(t_j) \mathbb{E}(B_j) = \sum_{j=1}^{n} p_j c_j v(t_j).$$

Note that we have not required the B_j to be independent (or assumed anything about their dependence structure).

2.2 Pricing of corporate bonds

Let

$$c = ((1, DN), \ldots, (n-1, DN), (n, DN + N))$$

be a simple fixed-interest security issued by a company. If the company goes insolvent, all future payments are lost. So we may wish to look at the random cash-flow

$$((1, DNB_1), \ldots, (n-1, DNB_{n-1}), (n, (DN + N)B_n)),$$

where $B_j = I(T > j)$ and T is a random variable representing the insolvency time;

$$p_j = \mathbb{P}(B_j = 1) = \mathbb{P}(T > j) = \overline{F}_T(j).$$

Here and in the sequel, we will use the following notation for continuous probability distributions: let T be a continuous random variable taking values in $[0,\infty)$, modelling a lifetime or insolvency time. Its distribution function and survival function are given by

$$F_T(t) = \mathbb{P}(T \le t) \qquad \text{and} \qquad \overline{F}_T(t) = \mathbb{P}(T > t) = 1 - F_T(t),$$

while its density function satisfies

$$f_T(t) = \frac{d}{dt}F_T(t) = -\frac{d}{dt}\overline{F}_T(t), \quad F_T(t) = \int_0^t f_T(s)ds, \quad \overline{F}_T(t) = \int_t^\infty f_T(s)ds.$$

Definition 8 The function

$$\mu_T(t) = \frac{f_T(t)}{\overline{F}_T(t)}, \qquad t \ge 0,$$

specifies the *force of mortality* or *hazard rate* at (time) t.

Proposition 9 *We have* $\frac{1}{\varepsilon}\mathbb{P}(T \le t + \varepsilon | T > t) \to \mu_T(t)$ *as* $\varepsilon \to 0$.

Proof: We use the definitions of conditional probabilities and differentiation:

$$\begin{aligned}\frac{1}{\varepsilon}\mathbb{P}(T \le t+\varepsilon | T > t) &= \frac{1}{\varepsilon}\frac{\mathbb{P}(T > t, T \le t+\varepsilon)}{\mathbb{P}(T > t)} = \frac{1}{\varepsilon}\frac{\mathbb{P}(T \le t+\varepsilon) - \mathbb{P}(T \le t)}{\mathbb{P}(T > t)} \\ &= \frac{1}{\overline{F}_T(t)}\frac{F_T(t+\varepsilon) - F_T(t)}{\varepsilon} \longrightarrow \frac{F_T'(t)}{\overline{F}_T(t)} = \frac{f_T(t)}{\overline{F}_T(t)} = \mu_T(t).\end{aligned}$$

□

So, we can say informally that $\mathbb{P}(T \in (t, t+dt) | T > t) \approx \mu_T(t)dt$, i.e. $\mu_T(t)$ represents for each $t \ge 0$ the current "rate of dying" (or bankruptcy etc.) given survival up to t.

Example 10 The exponential distribution of rate μ is given by

$$F_T(t) = 1 - e^{-\mu t}, \quad \overline{F}_T(t) = e^{-\mu t}, \quad f_T(t) = \mu e^{-\mu t}, \quad \mu_T(t) = \frac{\mu e^{-\mu t}}{e^{-\mu t}} = \mu \quad \text{constant.}$$

Lemma 11 *We have* $\overline{F}_T(t) = \exp\left(-\int_0^t \mu_T(s)ds\right)$.

Proof: First note that $\overline{F}_T(0) = \mathbb{P}(T > 0) = 1$. Also

$$\frac{d}{dt}\log \overline{F}_T(t) = \frac{\overline{F}_T'(t)}{\overline{F}_T(t)} = \frac{-f_T(t)}{\overline{F}_T(t)} = -\mu_T(t).$$

So

$$\log \overline{F}_T(t) = \log \overline{F}_T(0) + \int_0^t \frac{d}{ds}\log \overline{F}_T(s)ds = 0 + \int_0^t -\mu_T(s)ds.$$

□

Let c be a deterministic cash-flow on $[0,\infty)$ and consider $c_{[0,T)}$, the cash-flow "killed" at time T: all events from time T on are cancelled. If T is random, $c_{[0,T)}$ is a random cash-flow.

Proposition 12 *Let $\delta(\cdot)$ be an interest rate model, c a cash-flow and T a continuous insolvency time, then*

$$\mathbb{E}\left(\delta\text{-Val}_0\left(c_{[0,T)}\right)\right) = \widetilde{\delta}\text{-Val}_0(c)$$

where $\widetilde{\delta}(t) = \delta(t) + \mu_T(t)$ and $\mu(t) = f_T(t)/\overline{F}_T(t)$.

Proof: Let $c = ((t_j, c_j), j \geq 1)$. Then

$$\delta\text{-Val}_0(c) = \sum_{j\geq 1} c_j v(t_j), \qquad \text{where } v(t_j) = \exp\left(-\int_0^{t_j} \delta(s)ds\right),$$

and

$$\delta\text{-Val}_0(c_{[0,T)}) = \sum_{j\geq 1} c_j v(t_j) I(T > t_j),$$

so that

$$\mathbb{E}\left(\delta\text{-Val}_0(c_{[0,T)})\right) = \sum_{j\geq 1} c_j v(t_j)\mathbb{E}(I(T > t_j)), \quad \text{where } \mathbb{E}(I(T > t_j)) = \mathbb{P}(T > t_j) = \overline{F}_T(t_j).$$

By the previous lemma, the main expression equals

$$\sum_{j\geq 1} c_j \exp\left(-\int_0^{t_j} \delta(s)ds\right)\exp\left(-\int_0^{t_j} \mu_T(s)ds\right) = \sum_{j\geq 1} c_j \exp\left(-\int_0^{t_j} \widetilde{\delta}(s)ds\right) = \widetilde{\delta}\text{-Val}_0(c).$$

□

The important special case is when δ and μ are constant and c is a corporate bond whose payment stream stops at the insolvency time T.

Corollary 13 *Given a constant δ model, the fair price for a corporate bond C with insolvency time $T \sim Exp(\mu)$ is*

$$A = \mathbb{E}(\delta\text{-DVal}_0(C)) = \tilde{\delta}\text{-DVal}_0(c)$$

where $\tilde{\delta} = \delta + \mu$.

Often, problems arise in a discretised way. Remember that a geometric random variable with parameter p can be thought of as the first 0 in a series of Bernoulli 0-1 trials with success (1) probability p.

Proposition 14 *Let c be a discrete cash-flow with $t_k \in \mathbb{N}$ for all $k = 1, \ldots, n$, $T \sim$ geom(p), i.e. $\mathbb{P}(T = k) = p^{k-1}(1-p)$, $k = 1, 2, \ldots$. Then in the constant i model,*

$$\mathbb{E}\left(i\text{-Val}_0(c_{[0,T)})\right) = j\text{-Val}_0(c)$$

where $j = (1 + i - p)/p$.

Proof: $\mathbb{P}(T > k) = p^k$. Therefore

$$\mathbb{E}\left(i\text{-Val}_0(c_{[0,T)})\right) = \sum_{k=1}^{n} p^{t_k} c_k (1+i)^{-t_k}.$$

□

2.3 Expected yield

For deterministic cash-flows that can be interpreted as investment deals (or loan schemes), we defined the yield as an intrinsic rate of return. For a random cash-flow, this notion gives a random yield which is usually difficult to use in practice. Instead, we define:

Definition 15 Let C be a random cash-flow. The *expected yield* of C is the interest rate $i \in (-1, \infty)$, if it exists and is unique such that

$$\mathbb{E}(\text{NPV}(i)) = 0,$$

where $\text{NPV}(i) = i\text{-Val}_0(C)$ denotes the net present value of C at time 0 discounted at interest rate i.

This corresponds to the yield of the "average cash-flow". Note that this terminology may be misleading – this is *not* the expectation of the yield of C, even if that were to exist.

Example 16 An investment of £500,000 provides

- a continuous income stream of £50,000 per year, starting at an unknown time S and ending in 6 years' time;
- a payment of unknown size A in 6 years' time.

Suppose that

- S is uniformly distributed between [2years, 3years] (time from now);
- the mean of A is £700,000.

What is the expected yield?

We use units of £10,000 and 1 year. The time-0 value at rate y is

$$-50 + \int_{s=S}^{6} 5(1+y)^{-s} ds + (1+y)^{-6} A.$$

The expected time-0 value is

$$-50 + \int_{s=2}^{6} \mathbb{P}(S < s) 5(1+y)^{-s} ds + (1+y)^{-6}\mathbb{E}(A)$$
$$= -50 + \int_{s=2}^{3} (s-2) 5(1+y)^{-s} ds + \int_{s=3}^{6} 5(1+y)^{-s} ds + (1+y)^{-6} 70 =: f(y).$$

Set $f(y) = 0$ and find

$$f(10.45\%) = 0.1016... \qquad \text{and} \qquad f(10.55\%) = -0.1502...$$

So the expected yield is 10.5% to 1d.p. (note that we only need to use the *mean* of A).

As a consequence of Proposition 14, we note:

Corollary 17 *If $T \sim \text{geom}(p)$ and c a cash-flow at integer times with yield $y(c)$, then the expected yield of $c_{[0,T)}$ is $p(1 + y(c)) - 1$.*

Proof: Note that $y(c)\text{-Val}_0(c) = 0$, and by Proposition 14, we have $\mathbb{E}(i\text{-Val}_0(c_{[0,T)}) = 0$, if $y(c) = (1 + i - p)/p$, i.e. $1 + i = p(1 + y(c))$, as required. Also, this is the unique solution to the expected yield equation as otherwise the relationship $1 + i = p(1 + j)$ would give more solutions to the yield equation. □

Lecture 3

Stochastic interest-rate models I

Reading: McCutcheon-Scott Chapter 12, CT1 Unit 14

So far, we usually assumed that we knew all interest rates, or we compared investments under different interest rate assumptions. Any uncertainty of investment proceeds was expressed modelling cash-flows by random variables. In practice, interest rates themselves are uncertain, and we model here interest rates by random variables.

3.1 Basic model for one stochastic interest rate

We start off with an elementary example.

Example 18 Suppose, you invest £100 for 1 year at an interest rate I not known in advance. Say, interest rates are at 3% at the moment, you might expect one of three possibilities, a rise by 1%, no change or a fall by 1%, each equally likely, say:

$$\mathbb{P}(I = 2\%) = \mathbb{P}(I = 3\%) = \mathbb{P}(I = 4\%) = 1/3$$

Then the investment proceeds $R = 100(1 + I)$ at the end of the year are random, as well

$$\mathbb{P}(R = 102) = \mathbb{P}(R = 103) = \mathbb{P}(R = 104) = 1/3$$

You can calculate your expected proceeds $\mathbb{E}(R) = 1/3(102+103+104) = 103$ and think, well, this can be calculated using the average interest rate $\mathbb{E}(I) = 3\%$, and there is not much reason to study any further.

However, this only works for a term of 1 year. If the term is, say $t \in (0, \infty)$ years, $t \neq 1$, we get $R = 100(1 + I)^t$

$$\mathbb{P}(R = 100(1.02)^t) = \mathbb{P}(R = 100(1.03)^t) = \mathbb{P}(R = 100(1.04)^t) = 1/3$$

and $\mathbb{E}(R) = 100/3((1.02)^t + (1.03)^t + (1.04)^t) \neq 100(1.03)^t$.

Assuming that I can only take 3 values is of course an unnecessary restriction, and we can take as stochastic interest rate any random variable I that ranges $(-1, \infty)$, discrete or continuous.

Proposition 19 *Given a stochastic interest rate I. Invest c at time 0. Then its expected accumulated value at time t is given by*

$$\mathbb{E}(R_t) = \mathbb{E}(I\text{-Val}_t((0,c))) = \mathbb{E}(c(1+I)^t).$$

Let $\lambda = \mathbb{E}(I)$ and $\mathbb{P}(I \neq \lambda) > 0$. Then

- *if $t < 1$, then $\mathbb{E}(R_t) < \lambda\text{-Val}_t((0,c))$,*
- *if $t > 1$, then $\mathbb{E}(R_t) > \lambda\text{-Val}_t((0,c))$.*

Proof: The proof of the inequalities is essentially an application of Jensen's inequality (see following lemma) for the function $f(x) = (1+x)^t$ which is convex if $t > 1$ and concave if $t < 1$, so that then $-f$ is convex. □

Lemma 20 (Jensen's inequality) *For any convex function $f : \mathbb{R} \to \mathbb{R}$ and any random variable X with $\mathbb{E}(X) \in \mathbb{R}$, we have*

$$f(\mathbb{E}(X)) \le \mathbb{E}(f(X)).$$

If f is strictly convex and $\mathbb{P}(X = \mathbb{E}(X)) < 1$, then the inequality is strict.

The result is still true if f is only defined on the interval $(\inf \mathit{supp}(X), \sup \mathit{supp}(X))$ *and any boundary value a with $\mathbb{P}(X = a) > 0$.*

Proof: Recall that for (strictly) convex functions $f(x) \ge f(x_0) + (x - x_0)f_r'(x_0)$ for all $x, x_0 \in \mathbb{R}$ (strict inequality for $x \neq x_0$), where f_r' is the right derivative of f. Applying this, we obtain

$$\mathbb{E}(f(X)) \ge \mathbb{E}\left(f(\mathbb{E}(X)) + (X - \mathbb{E}(X))f_r'(\mathbb{E}(X))\right) = f(\mathbb{E}(X))$$

by linearity of $\mathbb{E}$. □

Obviously, instead of modelling the interest rate I, we could model the force of interest $\Delta = \log(1+I)$. This is particularly useful since valuation of cash-flows then requires only knowledge of the so-called Laplace transforms $\mathbb{E}(e^{-t\Delta})$ of Δ.

3.2 Independent interest rates

The model in the previous section is artificial, particularly for long terms. It is natural to allow the interest rate to change. The easiest such model is by independent annual (or monthly) interest rates.

Definition 21 An *iid interest rate model* is a collection of independent identically distributed (iid) random interest rates I_j, $j \ge 1$, taking values in $(-1, \infty)$, rate I_j being applied the jth year (or month).

Proposition 22 *Given an iid interest rate model, any simple cash-flow* (s, c), $s \in \mathbb{N}$ *has an expected value at time* $t \geq s$, $t \in \mathbb{N}$ *given by*

$$\mathbb{E}(\mathrm{Val}_t((s,c))) = c \prod_{j=s+1}^{t} (1 + \mathbb{E}(I_j)) = c(1 + \mathbb{E}(I_1))^{t-s}.$$

and at time $t \leq s$, $t \in \mathbb{N}$, *given by*

$$\mathbb{E}(\mathrm{Val}_t((s,c))) = c \prod_{j=t+1}^{s} E((1 + I_j)^{-1}) = c\left(\mathbb{E}((1 + I_1)^{-1})\right)^{s-t}.$$

Proof: For the first statement we calculate

$$\mathbb{E}(\mathrm{Val}_t((s,c))) = \mathbb{E}\left(c \prod_{j=s+1}^{t} (1 + I_j)\right) = c\left(\prod_{j=s+1}^{t} \mathbb{E}(1 + I_j)\right)$$

by linearity of $\mathbb{E}$ and by the independence of the $(1+I_j)$ factors; remember that $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$ for independent random variables X and Y. Each of the $t - s$ factors is now equal to $(1 + \mathbb{E}(I_1))$ since the I_j are identically distributed.

The second statement is analogous. Note that $\mathbb{E}(1 + I_j) = 1 + \mathbb{E}(I_j)$ above, but $\mathbb{E}((1 + I_j)^{-1})$ cannot be simplified, in general. □

As just seen, expected accumulated values can be computed fairly easily. Also similar formulas for variances exist, as one measure of risk. Useful formulas for loss probabilities as another measure of risk are only available in special cases. A very popular family of distributions for modelling interest rates is the log-normal distribution.

Definition 23 A random variable X is said to have a lognormal distribution if $Z = \log(X)$ is (well-defined) and normally distributed. The log-normal distribution $\mathrm{logN}(\mu, \sigma^2)$ has two parameters $\mu = \mathbb{E}(\log(X))$ and $\sigma^2 = \mathrm{Var}(\log(X))$.

Proposition 24 *If* $1 + I \sim \mathrm{logN}(\mu, \sigma^2)$, *then*

$$\lambda := \mathbb{E}(I) = \exp(\mu + \sigma^2/2) - 1$$

and

$$s^2 = \mathrm{Var}(I) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1).$$

Proof: We need the moment generating function of $\Delta \sim \mathrm{N}(\mu, \sigma^2)$:

$$\begin{aligned}
\mathbb{E}(e^{t\Delta}) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{tx} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu+\sigma^2 t)^2}{2\sigma^2}\right) \exp\left(\frac{(\mu+\sigma^2 t)^2 - \mu^2}{2\sigma^2}\right) dx \\
&= \exp\left(\mu t + \frac{\sigma^2}{2} t^2\right).
\end{aligned}$$

So $\mathbb{E}(1 + I) = \mathbb{E}(e^{\Delta}) = \exp(\mu + \sigma^2/2)$ and

$$\begin{aligned}
\mathrm{Var}(I) = \mathrm{Var}(1 + I) = \mathrm{Var}(e^{\Delta}) &= \mathbb{E}(e^{2\Delta}) - \left(\mathbb{E}(e^{\Delta})\right)^2 \\
&= \exp(2\mu + 2\sigma^2) - \exp(2\mu + \sigma^2) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1).
\end{aligned}$$

□

Note that if $1 + I$ has a lognormal distribution, then $\Delta = \log(1 + I)$ is normally distributed.

Example 25 Let $1 + I_1, \ldots, 1 + I_n$ be independent lognormal random variables with common parameters μ and σ^2. We can calculate the distribution of the accumulated value at time n of a unit investment at time 0.

$$S_n = \prod_{j=1}^{n}(1 + I_j) = \exp\left\{\sum_{j=1}^{n}\Delta_j\right\} \sim \log\mathrm{N}(n\mu, n\sigma^2)$$

since sums of independent normal random variables are normal with as parameters the sums of the individual parameters.

Assume that $\mu = 0.04$, $\sigma = 0.02$ and $n = 5$. If we want to accumulate at least £600,000 with probability 99%, we have to invest A where

$$\begin{aligned} 0.99 &= \mathbb{P}(AS_n > 600,000) = \mathbb{P}(\log\{S_n\} > \log\{600,000/A\}) \\ &= \mathbb{P}\left(Z > \frac{\log\{600,000/A\} - n\mu}{\sqrt{n\sigma^2}}\right) \\ \Rightarrow -2.33 &= \frac{\log\{600,000/A\} - n\mu}{\sqrt{n}\sigma} \Rightarrow A = 600,000\exp\{2.33\sqrt{n}\sigma - n\mu\} = 545,187.90 \end{aligned}$$

Here we used that $\mathbb{P}(Z > -2.33) = 0.99$ for a standard normal random variable Z.

In practice, models for annual changes interest rates are too crude, but by switching to the appropriate time unit, this problem can be easily overcome.

If $1 + I_j$ are i.i.d. but not lognormal, we can approximate by the Central Limit Theorem. We have

$$S_n = (1 + I_1)(1 + I_2)\cdots(1 + I_n) = \exp(\Delta_1 + \cdots + \Delta_n).$$

So, let $\mu = \mathbb{E}(\Delta_1) = \mathbb{E}(\log(1+I_1))$ and $\sigma^2 = \mathrm{Var}(\Delta_1)$. Then $\Delta_1+\cdots+\Delta_n$ is approximately $\mathrm{N}(n\mu, n\sigma^2)$, so S_n is approximately $\log\mathrm{N}(n\mu, n\sigma^2)$, for large n.

Lecture 4

Stochastic interest-rate models II

Reading: McCutcheon-Scott Chapter 12

In this lecture we study some more realistic interest rate models.

4.1 Dependent annual interest rates

In practice, interest rates do not fluctuate as strongly as in the iid model. In fact, when interest rates are high, the next period is quite likely to show another high interest rate, similarly with low rates. In fact, the Monetary Policy Committee of the Bank of England meets every month to decide on changes to the Base Rate to which many commercial bank rates are coupled. Often, the rate remains unchanged: e.g.

- in 2000, 2002 and also more recently in 2010, there were no changes to the base rate, at all;
- in 2001, there were 6 reductions of 0.25%, 1 reduction of 0.5% and 5 meetings with unchanged rates;
- between October 2008 and March 2009 the monthly changes were -0.5%, -1.5%, -1.0%, -0.5%, -0.5%, -0.5% to reach a hard lower boundary of a base rate of 0.5%.

This can be modelled by centering the new interest rate around the current interest rate, or between the current and a general long term mean interest rate.

Example 26 (Random walk) Let $\Delta_0 = \mu$, $\Delta_j = \log(1 + I_j) = \Delta_{j-1} + \varepsilon_j$, where ε_j are i.i.d. $\mathrm{N}(0, \sigma^2)$, so

$$\Delta_j = \mu + \varepsilon_1 + \cdots + \varepsilon_j \sim \mathrm{N}(\mu, j\sigma^2)$$

and

$$\Delta_1 + \cdots + \Delta_n = n\mu + n\varepsilon_1 + (n-1)\varepsilon_2 + \cdots + 2\varepsilon_{n-1} + \varepsilon_n \sim \mathrm{N}(n\mu, \sigma^2(1+4+\cdots+(n-1)^2+n^2)).$$

The effect of random shocks ε_j on the interest rate is permanent. The variance of Δ_n grows like n, the variance of $\Delta_1 + \cdots + \Delta_n$ grows like n^3.

Example 27 (Autoregressive model) Let $\Delta_0 = \mu$, $\Delta_j = \theta\Delta_{j-1} + (1-\theta)\mu + \varepsilon_j$, where $\theta \in [0, 1)$ induces some "mean reversion". Conditional on Δ_{j-1}, we have

$$\Delta_j \sim \mathrm{N}\left(\theta\Delta_{j-1} + (1-\theta)\mu, \sigma^2\right).$$

This can also be expressed as $(\Delta_j - \mu) = \theta(\Delta_{j-1} - \mu) + \varepsilon_j$, say $D_j = \theta D_{j-1} + \varepsilon_j$. Now

$$\begin{array}{rcl} D_1 &=& \varepsilon_1 \sim \mathrm{N}(0, \sigma^2) \\ D_2 &=& \theta\varepsilon_1 + \varepsilon_2 \sim \mathrm{N}(0, (1+\theta^2)\sigma^2) \\ D_3 &=& \theta^2\varepsilon_1 + \theta\varepsilon_2 + \varepsilon_3 \sim \mathrm{N}(0, (1+\theta^2+\theta^4)\sigma^2) \\ \cdots & \vdots & \cdots \\ D_n &=& \theta^{n-1}\varepsilon_1 + \theta^{n-2}\varepsilon_2 + \cdots + \theta\varepsilon_{n-1} + \varepsilon_n \sim \mathrm{N}(0, (1+\theta^2+\cdots+\theta^{2(n-1)})\sigma^2) \\ D_1 + \cdots + D_n &=& (1+\theta+\cdots+\theta^{n-1})\varepsilon_1 + (1+\theta+\cdots+\theta^{n-2})\varepsilon_2 + \cdots + \varepsilon_n \sim \mathrm{N}(0, r_n^2) \end{array}$$

where $r_n^2 \le n\sigma^2/(1-\theta^2)$, since $1 + \theta + \cdots + \theta^{n-1} = (1-\theta^n)/(1-\theta) \le 1/(1-\theta)$. In particular, the variance of Δ_n is now of constant order and the variance of $\Delta_1 + \cdots + \Delta_n$ grows at rate n.

Both these models were Markov chains: the distribution of Δ_j depended on previous $(\Delta_1, \ldots, \Delta_{j-1})$ only through Δ_{j-1}.

4.2 Modelling the force of interest

When reducing the time unit, one can also pass to continuous-time limits. In fact, on credit markets, where credit is traded, market interest rates fluctuate much more than e.g. the Base Rate of the Bank of England. Such market rates react directly to supply and demand.

First note that our previous models can be viewed as models with piecewise constant forces of interest. We started with a single random force of interest Δ, constant for all time. We then considered forces of interest that were constant Δ_j during each time unit $(j-1, j]$. For the last two examples, we can get interesting limits as we let our time unit tend to zero.

E.g., in the random walk model, we had $\Delta_0 = \mu$ and $\Delta_n = \Delta_{n-1} + \varepsilon_n$ for i.i.d. $\varepsilon_n \sim \mathrm{N}(0, \sigma^2)$. We can model $p$thly changing forces by setting $\Delta_0^{(p)} = \mu$ and for $n \ge 1$

$$\Delta_{\frac{n}{p}}^{(p)} = \Delta_{\frac{n-1}{p}}^{(p)} + \varepsilon_{\frac{n}{p}}^{(p)}, \qquad \varepsilon_{\frac{n}{p}}^{(p)} \sim N\left(0, \frac{\sigma^2}{p}\right) \text{ independent,}$$

where we understand that $\Delta_{n/p}^{(p)}$ is to apply during $((n-1)/p, n/p)$. Note that the models are consistent in that $\Delta_n^{(p)} \sim \Delta_n$ for all n, p. In the limit $p \to \infty$ we get Brownian motion $(\Delta_t)_{t\ge0}$ (let time unity tend to zero). Brownian motion is a continuous Markov process such that $B(t) \sim \mathrm{N}(\mu, \sigma^2 t)$ for all $t \ge 0$. See Figure 4.1 for a simulation of Brownian motion.

Similarly, we can get continuous processes as limits of the mean-reverting walks. These are described by appropriate stochastic differential equations.

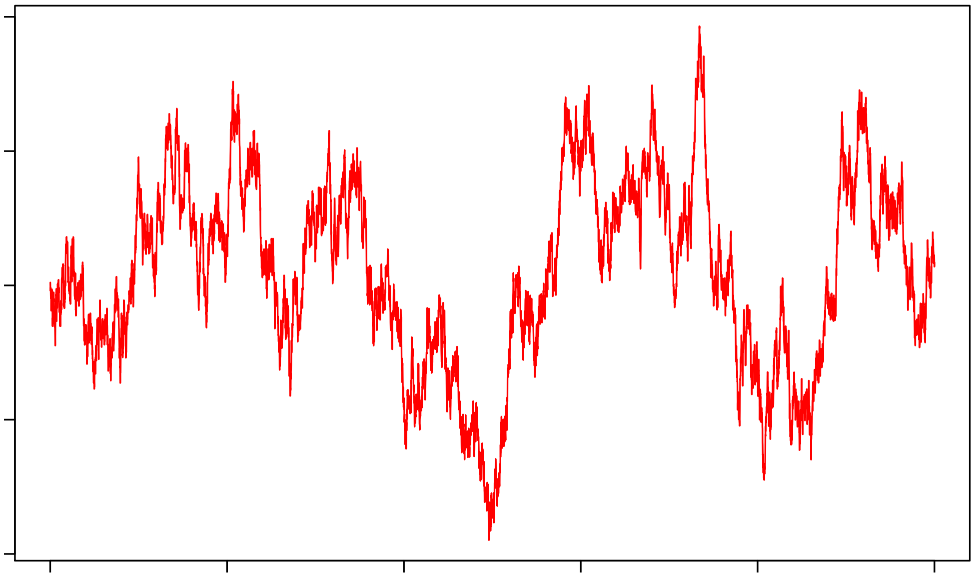

Figure 4.1: Brownian motion

4.3 What can one do with these models?

- Pricing of derivative contracts (derived from interest rates);
- assessment and quantification of interest rate risk in portfolios.

Lecture 5

The no-arbitrage assumption and forward prices

Reading: CT1 Unit 12

Arbitrage is a risk-free trading profit. We will work under a very general assumption of no arbitrage. For the next three lectures we will explore this framework to price derivative securities, i.e. securities that in some way depend on another, "underlying", security whose price evolution we model. In practice, arbitrage opportunities do exist, but they are usually very quickly eliminated, since markets are driven by supply and demand and particularly financial markets are highly efficient: whenever there is an arbitrage opportunity, arbitrageurs buy a product at a cheap price in one market, this extra demand meets the cheapest supply thereby increasing the remaining supply price; they sell the product at a higher price typically in another market, this extra supply meets the highest demand thereby decreasing the remaining demand price; arbitrageurs exploit such opportunities until all supply prices exceed all demand prices. With arbitrageurs constantly removing arbitrage opportunities, all other market participants act in virtually arbitrage-free markets. In our models, we will assume equal supply and demand prices.

5.1 Arbitrage and the law of one price

The usual definition of arbitrage is as follows:

Definition 28 We say that an *arbitrage opportunity* exists if either

- an investor can make a deal giving an immediate profit, with no risk of future loss, or
- an investor can make a deal that has zero initial cost, no risk of future loss, and a non-zero probability of a future profit.

We postpone a discussion of the second bullet point to when we discuss stochastic models that give a precise mathematical meaning to the notion of "non-zero probability". An important consequence (that only relies on the first bullet point) is that no arbitrage implies the Law of One Price. Its proof can be seen as an illustration of the exploitation of arbitrage opportunities.

Definition 29 The *Law of One Price* (LOOP) stipulates that any two assets with identical cash-flows in all possible scenarios must trade at the same price on all markets.

Proposition 30 *An assumption of no arbitrage implies the Law of One Price.*

Proof: Assume that there are two assets with identical cash-flows, but different prices $A < B$, say. An arbitrageur will buy for A and sell for B. An immediate profit is made, the net future cash-flow is zero leaving no risk of future loss – this is arbitrage. □

The Law of One Price was implicitly or explicitly applied all the time last term. We have now shown that failure of the Law of One Price implies arbitrage. In a mathematical model, arbitrage essentially means $0 = 1$, and there is no point going any further from there. But we will present some powerful developments when arbitrage is not possible. We do not claim in this generality that the Law of One Price implies no-arbitrage – we would have to be more precise about the class of mathematical models under consideration.

No-arbitrage arguments are fundamentally deterministic. However, "scenarios" refer to different possibilities that we ought to collect in a set Ω of all scenarios. We will see later several ways of connecting the ideas of no arbitrage with stochastic models. The first way is to set up a stochastic model for security prices that, in particular, identifies the possible scenarios. The second (and more subtle) way is the calculation of arbitrage-free prices as expectations under a very specific (so-called risk-neutral) stochastic model.

Example 31 Last term, the concept of a forward rate of interest was introduced. To recall this, suppose here that zero-coupon bonds of just two terms t and $t+r$ are currently priced at $P_0^{(t)}$ and $P_0^{(t+r)}$, respectively. We want to invest 1 at time t and agree now on an appropriate amount I of interest payable at time $t + r$, i.e.

A: Pay 1 at time t and receive $1 + I$ at time $t + r$.

Compare this with the following transactions:

B: Sell a term-t zero-coupon bond at time 0 to receive $P_0^{(t)}$, and pay $(1 + I)P_0^{(t+r)}$ to buy $1 + I$ units of the term-$(t + r)$ zero-coupon bond at time 0.

The future cash-flows under A and B are equal. If LOOP holds, they must have the same price at time 0, so that $0 = P_0^{(t)} - (1 + I)P_0^{(t+r)}$, i.e. $1 + I = P_0^{(t)}/P_0^{(t+r)}$.

By definition, the forward rate $f_{t,r}$ is the annual effective interest rate earned on an investment between times t and $t + r$, as implied by current zero-coupon bond prices $P_0^{(t)}$ and $P_0^{(t+r)}$, i.e. $f_{t,r}$ such that $(1 + f_{t,r})^r = 1 + I = P_0^{(t)}/P_0^{(t+r)}$. We will capture the important detail of "agree now" or "implied by current prices" in the concept of a forward *contract*. Note that the argument is robust to changes to the term structure of interest rates.

5.2 Standard form of no arbitrage pricing argument

Pricing a derivative security under an assumption of no arbitrage usually makes use of replicating portfolio arguments, as in Example 31. We consider two portfolios involving

A: the derivative security (plus other assets on the market, e.g. riskfree),

B: a portfolio of underlying securities (plus other assets on the market, e.g. riskfree),

where B is constructed to provide exactly the same cash-flow as A in every scenario.

Under the no-arbitrage assumption, which implies the Law of One Price, A and B must have the same price at any time. But we know the price of portfolio B as it is a weighted sum of assets we know the prices of. This gives the price of the derivative security. Sometimes, the unknown is not the price of the derivative security but some other quantity associated with the derivative security/cash-flow in portfolio A, which will then also appear in the replicating portfolio B, and LOOP gives us an equation for the unknown. An example for this is the a priori unknown I in Example 31.

In practice, there are transaction costs to buy derivative securities, fulfil the derivative security contract, buy or sell the underlying security. We ignore transaction costs to focus on the key part of the no-arbitrage argument. We also assume that we can hold any positive or negative, integer or fractional number of units of any asset on the market.

5.3 No-arbitrage computation of forward prices

Example 32 (Forward contract to buy a security with no income) Let

- S_t be the market price of the underlying security at time $0 \leq t \leq T$; consider as known the present market price S_0, but not future market prices S_t, $0 < t \leq T$,
- δ a known constant force of interest on risk-free investments over the term of the contract; we may think of a bank account that pays/charges interest at force δ,
- K the forward price to be determined, i.e. the price agreed at time $t = 0$ to be paid at time $t = T$ to purchase the underlying security at time $t = T$,

Note that under the forward contract, no money changes hands until time $t = T$, i.e. the forward contract has no initial cost, it's just setting a price K at time $t = 0$ for a purchase at time $t = T$. To compute the (unique arbitrage-free) forward price K, consider

A: enter into the forward contract to buy asset S with forward price K maturing at time T; buy $Ke^{-\delta T}$ units of the risk-free asset at time $t = 0$;

B: buy one unit of the asset S at the current market price S_0 at $t = 0$.

Then the only cash-flows occur at time T, where under A, the forward contract is worth $S_T - K$, and there is K risk-free; under B the asset is worth S_T, so the net cash-flows are the same, no matter what the value of S_T is, i.e. in all scenarios for the evolution of S. By LOOP, A and B must have the same price at time $t = 0$, hence $0 + Ke^{-\delta T} = S_0$, i.e. $K = S_0 e^{\delta T}$. Note that we have not made any assumptions on the distribution of S_T, other than stipulating no-arbitrage. In particular, K does not depend on such distributional assumptions on S_T, which is surprising at first sight.

What if an actual forward price K_{actual} exceeds $K = S_0 e^{\delta T}$? Arbitrageurs would buy B and sell A. Supply/demand adjustments would quickly lead to $K_{\text{actual}} = K$.

More generally, it is not necessary to assume a constant force of interest. All we needed to work out $K = S_0 e^{\delta T}$ was a discount factor from time $t = T$ to time $t = 0$. In practice, this discount factor is reflected in the time-0 price $P_0^{(T)}$ of a zero-coupon bond maturing at time T paying £1. This is our risk-free asset with which portfolio A is set up by buying K units at price $KP_0^{(T)}$. Reasoning as above, we obtain $K = S_0/P_0^{(T)}$.

Since $1/P_0^{(T)}$ is the associated accumulation factor from time $t = 0$ to time $t = T$, we can read $K = S_0/P_0^{(T)}$ as the current price S_0 of the asset accumulated using the risk-free accumulation factor.

Example 33 (Forward contract to buy a security with fixed income) Suppose that the security underlying the forward contract provides a fixed amount c_1 at time $t_1 \in (0, T)$ to the holder. With notation as in the previous example, consider

A: enter into the forward contract to buy asset S with forward price K maturing at time T; invest $Ke^{-\delta T} + c_1 e^{-\delta t_1}$ into the risk-free asset at time $t = 0$;

B: buy one unit of the asset S at the current market price S_0 at $t = 0$. At time t_1, invest the income of c_1 in the risk-free asset.

Note that both portfolios have zero net cash-flow on $(0, T)$. At time T,

A: Forward contract: $S_T - K$; risk-free holding: $K + c_1 e^{\delta(T-t_1)}$;

B: Asset S: S_T; risk-free holding from coupon: $c_1 e^{\delta(T-t_1)}$.

By LOOP, A and B have the same price at $t = 0$:

$$0 + Ke^{-\delta T} + ce^{-\delta t_1} = S_0 \quad \Rightarrow \quad K = S_0 e^{\delta T} - c_1 e^{\delta(T-t_1)}.$$

More generally, $K = (S_0 - F)e^{\delta T}$, where F is the present value at time $t = 0$ of the fixed income payments due during the term of the forward contract.

Example 34 (Forward contract to buy a security with known dividend yield) Suppose that the security underlying the forward contract pays dividend continuously at rate D. Such income is not fixed since the dividend rate is applied to the market price S_t that varies with t and is unknown for $t \in (0, T)$. If we set up portfolios as before, but now reinvesting dividend in the security, the accumulated holding at time T would be e^{DT} units of the security, since the *number of units* of the security held as t varies behaves like a bank account that accumulates interest continuously at rate D. Instead, consider

A: enter into the forward contract to buy asset S with forward price K maturing at time T; invest $Ke^{-\delta T}$ into the risk-free asset at time $t = 0$;

B: buy e^{-DT} units of the asset S at the current market price S_0 at $t = 0$. Reinvest dividend income in S immediately when it is received.

Note that both portfolios have zero net cash-flow on $(0, T)$. At time T,

A: Forward contract: $S_T - K$; risk-free holding: K;

B: Asset S: $e^{DT}e^{-DT}S_T = S_T$;

By LOOP, A and B have the same price at $t = 0$:

$$0 + Ke^{-\delta T} = S_0 e^{-DT} \quad \Rightarrow \quad K = S_0 e^{(\delta - D)T}.$$

Note, we can work out K if fixed income is reinvested in the risk-free asset and income proportional to S is reinvested in S.

Lecture 6

Arbitrage-free prices and values of securities

Reading: CT1 Unit 12

In this lecture we introduce two more ideas connected to arbitrage-free pricing. The first is that a forward contract (or other derivative security) has a no-arbitrage value at any given time t between issue and maturity (and can hence be considered itself as an underlying security for higher-order derivative securities). The second is the systematic calculation of arbitrage-free prices of options and other derivative securities.

6.1 Values of forward contracts

The standard form of the no-arbitrage pricing argument introduced in the last lecture can also be applied to assign a no-arbitrage value to certain derivative securities. Forward contracts again provide natural first examples. We use the same framework as in Example 32 of a security S and a risk-free asset accumulating at a force of interest δ.

Example 35 (Forward contract to buy a security with no income) The forward contract initially changes hands at no cost $V_0 = 0$ to either party. The sole purpose is to fix the forward price K. However, at maturity T, the contract is worth $V_T = S_T - K$ to the buyer (and $-V_T = K - S_T$ to the seller). What about intermediate times? Let us denote by V_r the value of the forward contract at time $r < T$.

A: At time r, pay V_r to enter the existing forward contract to buy asset S at time T for a forward price of K;

B: at time r, buy asset S for S_r and borrow $Ke^{-\delta(T-r)}$ risk-free.

Both portfolios have zero cash-flow in (r, T). At time T,

A: Forward contract: $S_T - K$;

B: Asset: S_T; risk-free holding $-K$.

By LOOP, A and B have the same price at $t = r$:

$$V_r = S_r - Ke^{-\delta(T-r)} = S_r - S_0 e^{\delta r} \qquad \text{since } K = S_0 e^{\delta T} \text{ by Example 32.}$$

Similar reasoning can be applied to work out values of forward contracts with fixed or dividend income.

Terminology and warning

1. A forward contract legally binds two parties to act, respectively, as seller and buyer of the underlying asset at a given future time for a given price. Since the forward contract has no money value at issue, the parties "enter" the contract, they do not "buy" the contract. It is important to clearly specify and distinguish the two parties. We adopt the usual jargon that the party committing to buy enters a "long" forward contract and the party committing to sell enters a "short" forward contract. This is because the buyer holds the asset in the long term (after the agreed sale), whereas the seller holds the asset in the short term (before the agreed sale). The similar term "short-selling" refers to selling stock not actually owned but borrowed. It is mathematically convenient to allow short-selling. In practice, short-selling is also possible, but there are some legal restrictions.

2. We have assigned arbitrage-free values to an existing contract at intermediate times. It is instructive (but neglecting some legal issues) to think of the long (or the short) contract as a piece of paper that someone else can "buy", but be aware that the value may well be negative so that the term "buying" can be misleading. It is more appropriate to "take over" the contract, meaning to take over the long (or the short) position of the contract. Again, the specification of short/long is crucial and can only be omitted if the context makes the side absolutely clear.

6.2 Arbitrage-free pricing in discrete state spaces

Let us now approach arbitrage-free pricing in a more systematic way – forward contracts are very special derivative securities, others include European call options with value $P_T = (S_T - K)^+ = \max(S_T - K, 0)$ at maturity T, when the underlying asset S may be bought at the strike price K; in certain models, there is a replicating portfolio and a unique arbitrage-free time-0 value P_0, in others there is no replicating portfolios and a whole range of possible prices P_0 that each does not lead to arbitrage. To avoid serious technical complications and to focus on the no-arbitrage reasoning, we consider discrete markets only, with a finite number of time steps as well as a finite number of scenarios.

As a basic building block, consider a one-step model, where the time-0 picture is fixed, and at time 1 it will be in one of n scenarios. Assume that

- there are m assets available at time 0, the price at time 0 of asset i is $X_0^{(i)}$;
- asset $X^{(i)}$ is worth $X_1^{(i)}(j)$ at time 1 if scenario j occurs;
- there are no arbitrage opportunities.

This last point needs to be checked in specific models. Recall that the definition of arbitrage distinguishes immediate risk-free trading profits and risk-free zero-value portfolios with positive probability of future positive value. Note that the former can be turned into the latter kind as soon as there is an asset with non-negative values in all scenarios, by investing the immediate trading profit into that asset. In that case, we assume that there is a risk-free zero-initial-value portfolio with positive value in one scenario, and deduce a contradiction by identification of a negative value in a different scenario violating the risk-free assumption.

Under what conditions can we price an asset Y paying $Y_1(j)$ at time 1 if scenario j occurs?

Definition 36 An *Arrow-Debreu security* is a security that pays one unit in a particular scenario and zero otherwise.

In our model above, let us denote by $A^{(k)}$ the Arrow-Debreu security for scenario k, i.e. the asset paying

$$A_1^{(k)}(j) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then, we can clearly represent the n-vector $X_1^{(i)}$ in the standard basis $(A_1^{(1)}, \ldots, A_1^{(n)})$ as

$$X_1^{(i)}(\cdot) = \sum_{k=1}^{n} X_1^{(i)}(k) A_1^{(k)}(\cdot), \qquad 1 \le i \le m.$$

A simple no-arbitrage argument based on portfolios

A: $X_1^{(i)}(k)$ units of $A^{(k)}$ for $1 \le k \le n$;

B: one unit of asset $X^{(i)}$;

yields that at time $t = 0$, we have

$$\sum_{k=1}^{n} X_1^{(i)}(k) A_0^{(k)} = X_0^{(i)}, \qquad 1 \le i \le m.$$

In matrix notation this is

$$X_1 A_0 = X_0, \qquad \text{i.e.} \quad \begin{pmatrix} X_1^{(1)}(1) & X_1^{(1)}(2) & \cdots & X_1^{(1)}(n) \\ \vdots & \vdots & \ddots & \vdots \\ X_1^{(m)}(1) & X_1^{(m)}(2) & \cdots & X_1^{(m)}(n) \end{pmatrix} \begin{pmatrix} A_0^{(1)} \\ A_0^{(2)} \\ \vdots \\ A_0^{(n)} \end{pmatrix} = \begin{pmatrix} X_0^{(1)} \\ \vdots \\ X_0^{(m)} \end{pmatrix}.$$

How does this help to price an asset Y? As it stands, we have shown that *if* we can price all Arrow-Debreu securities, then the asset prices X_0 can be expressed in terms of the Arrow-Debreu security prices A_0 as $X_0 = X_1 A_0$.

If $m = n$ and the matrix X_1 is invertible, then we find the unknown $A_0 = X_1^{-1} X_0$ from given X_0 and X_1. Another simple no-arbitrage argument yields for any asset Y

$$Y_0 = \begin{pmatrix} Y_1(1) & Y_1(2) & \cdots & Y_1(n) \end{pmatrix} \begin{pmatrix} A_0^{(1)} \\ A_0^{(2)} \\ \vdots \\ A_0^{(n)} \end{pmatrix} = Y_1 A_0 = Y_1 X_1^{-1} X_0.$$

If X_1 is not invertible, we apply standard linear algebra. First, if X_1 has rank n, we can drop rows to obtain an invertible $n \times n$ submatrix. This corresponds to dropping assets whose values at time 1 are linear combinations of other assets. Note here, that the assumption that there are no arbitrage opportunities implies that values at time 0 follow the same linear combinations and so $\mathrm{rk}(X_1) = \mathrm{rk}(X_1|X_0)$, where $(X_1|X_0)$ is the matrix X_1 augmented by the vector X_0 as a further column.

If $\mathrm{rk}(X_1) < n$, we cannot express all Arrow-Debreu securities as linear combinations of the m assets (we may not be able to uniquely price any Arrow-Debreu securities). We can uniquely price an asset Y if Y_1 is in the row space of X_1. Specifically, $Y_1 = CX_1$, i.e.

$$\begin{pmatrix} Y_1(1) & Y_1(2) & \cdots & Y_1(n) \end{pmatrix} = \begin{pmatrix} C^{(1)} & \cdots & C^{(m)} \end{pmatrix} \begin{pmatrix} X_1^{(1)}(1) & X_1^{(1)}(2) & \cdots & X_1^{(1)}(n) \\ \vdots & \vdots & \ddots & \vdots \\ X_1^{(m)}(1) & X_1^{(m)}(2) & \cdots & X_1^{(m)}(n) \end{pmatrix}$$

implies, by a standard no-arbitrage argument that

$$Y_0 = CX_0 = \begin{pmatrix} C^{(1)} & \cdots & C^{(m)} \end{pmatrix} \begin{pmatrix} X_0^{(1)} \\ \vdots \\ X_0^{(m)} \end{pmatrix}.$$

The vector C, or rather, the associated asset holdings, are referred to as a *portfolio*. The vector Y_1 is called the *payoff* of asset Y, while the scalar Y_0 is called the *price* of asset Y.

If Y_1 is not in the row space of X_1, there is an open interval of arbitrage-free prices.

Example 37 Suppose $m = n = 2$. We call the assets R and S, the scenarios u and d.

(a)

Asset	time-0 price	time-1 price (u)	time-1-price (d)
R	6	7	5
S	11	14	10

There is an arbitrage opportunity: buy 1 unit of S and sell 2 units of R.

(b)

Asset	time-0 price	time-1 price (u)	time-1-price (d)
R	5	7	3
S	11	14	10

There is no arbitrage opportunity, because assets have positive values, and any zero-initial-value portfolio is a multiple of either $C_- = (-11, 5)$ or $C_+ = (11, -5)$, and these produce negative values

$$V_-(u) = -11 \times 7 + 5 \times 14 = -7 \qquad \text{and} \qquad V_+(d) = 11 \times 3 - 5 \times 10 = -17.$$

The matrix X_1 of time-1 prices is invertible. We can price every asset Y paying $Y(u)$ in scenario u and $Y(d)$ in scenario d as

$$Y_0 = \begin{pmatrix} Y(u) & Y(d) \end{pmatrix} \begin{pmatrix} 7 & 3 \\ 14 & 10 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 11 \end{pmatrix} = \frac{17}{28} Y(u) + \frac{7}{28} Y(d).$$

Recall that the coefficients are also the Arrow-Debreu security prices $A_0^{(u)} = 17/28$ and $A_0^{(d)} = 7/28 = 1/4$.

Lecture 7

No-arbitrage and risk-neutral probabilities

In this lecture we give re-express the valuation formulas derived in the last lecture, as expectations in a framework of "risk-neutral probabilities".

7.1 Arrow-Debreu securities and risk-neutral probability measures

In the m-asset model with invertible X_1, suppose that one of the assets is a risk-free asset with a payoff 1 irrespective of the scenario, traded at e^{-r} at time 0 so that r can be seen as the force of interest that applies to investments into the risk-free asset ("bank account"). Note that holding one unit of every Arrow-Debreu security also gives a guaranteed payoff of 1 at time 1, so that r is determined by

$$\sum_{k=1}^{n} A_0^{(k)} = e^{-r} \qquad \Rightarrow \qquad \sum_{k=1}^{n} A_0^{(k)} e^{r} = 1.$$

If we let $q_k = A_0^{(k)} e^r$, then $\sum_{k=1}^{n} q_k = 1$ and

$$Y_0 = e^{-r} \sum_{k=1}^{n} Y_1(k) q_k = e^{-r} \mathbb{E}_q(Y_1),$$

where $\mathbb{P}_q$ is a probability measure on $\Omega = \{1, \ldots, n\}$ that assigns probabilities $\mathbb{P}_q(\{k\}) = q_k$ and $Y_1 : \Omega \to \mathbb{R}$ can be considered as a random variable of payoffs at time $t = 1$. This measure $\mathbb{P}_q$ is called the risk-neutral probability measure. Hence, the arbitrage-free price for Y is the discounted expected payoff under the risk-neutral probability measure.

Arbitrage-free pricing is often done in the context of a stochastic model for the assets $X^{(1)}, \ldots, X^{(m)}$, which assigns probabilities $\mathbb{P}_p(\{k\}) = p_k$ to the n possible scenarios (with large n, to be realistic). Note that the construction of the risk-neutral measure $\mathbb{P}_q$ was only based on the possible scenarios (independent of $\mathbb{P}_p$, of course) and is the unique measure that provides arbitrage-free prices. In general, we will therefore have $\mathbb{P}_p \neq \mathbb{P}_q$.

7.2 The binomial model

As an example, consider the situation where

- S_t is the price of a non-dividend paying stock at discrete times $t = 0$ and $t = 1$, where S_1 is random with

  $\mathbb{P}(S_1 = S_0 u) = p_u$ and $\mathbb{P}(S_1 = S_0 d) = p_d$, where $d < u$ and $p_d = 1 - p_u \in (0, 1)$;

- B_t is the amount at times $t = 0$ and $t = 1$ of risk-free asset ("bond") per unit of cash at $t = 0$; with the associated force of interest r, we have $B_t = e^{rt}$;

- there are no trading costs, no minimum or maximum units of trading, stocks and bonds are only bought and sold at discrete times $t = 0$ and $t = 1$.

Let us study arbitrage in this model. First note that by just trading the bond and the stock at time 0, we cannot make an immediate profit. By arbitrage, we therefore mean deals of zero initial cost, no risk of future loss, and a non-zero probability of a future profit. Furthermore, deals of zero initial cost can only consist of borrowing and selling bonds to buy stock or borrowing and selling stock to buy bonds. This model is arbitrage-free if and only if $d < e^r < u$, since

- for $u > d \geq e^r$, borrowing and selling the bond to buy the stock gives arbitrage,

- for $d < u \leq e^r$, borrowing and selling the stock to buy the bond gives arbitrage,

- for $d < e^r < u$, borrowing and selling the bond to buy the stock gives a profit of $S_0 u - S_0 e^r > 0$ or a loss $S_0 d - S_0 e^r < 0$, both with positive probability, and so does borrowing and selling the stock to buy the bond: $S_0 e^r - S_0 u < 0$ and $S_0 e^r - S_0 d > 0$.

Let us now assume no-arbitrage, i.e. $d < e^r < u$. Consider a derivative which pays c_u if the price of the underlying goes up, c_d if down. What is V_0, the price of the derivative at time $t = 0$? We can work this out "on foot" by constructing a replicating portfolio, i.e. a portfolio (ϕ, ψ) of units of (stock, bond) at time 0. At time 1

$$V_1 = \begin{cases} \phi S_0 u + \psi e^r & \text{if stock price went up,} \\ \phi S_0 d + \psi e^r & \text{if stock price went down.} \end{cases}$$

We want (ϕ, ψ) to be a replicating portfolio, i.e.

$$\begin{array}{l} \phi S_0 u + \psi e^r = c_u \\ \phi S_0 d + \psi e^r = c_d \end{array} \qquad \Rightarrow \qquad \phi = \frac{c_u - c_d}{S_0(u - d)} \quad \text{and} \quad \psi = e^{-r} \frac{c_d u - c_u d}{u - d}.$$

Therefore, $V_0 = \phi S_0 + \psi = e^{-r}(q c_u + (1 - q) c_d)$, where $q = (e^r - d)/(u - d)$. Note that the no-arbitrage condition implies $0 < q < 1$. In terms of the payoff V_1 of the derivative at time $t = 1$, we can write

$$V_0 = e^{-r} \mathbb{E}_q(V_1),$$

where $\mathbb{P}_q(S_1 = S_0 u) = q$ and $\mathbb{P}_q(S_1 = S_0 d) = 1 - q$. Note that q depends only on u, d and r, not on the real world probabilities p_u and p_d or the derivative payoffs c_u and c_d.

Alternatively, we can apply the general machinery developed last time: we have $n = 2$,

$$V_1 = (c_d, c_u), \quad X_0 = \begin{pmatrix} 1 \\ S_0 \end{pmatrix}, \quad X_1 = \begin{pmatrix} e^r & e^r \\ S_0 d & S_0 u \end{pmatrix},$$

so that

$$X_1^{-1} = \frac{1}{e^r S_0(u-d)} \begin{pmatrix} S_0 u & -e^r \\ -S_0 d & e^r \end{pmatrix}$$

and

$$V_0 = V_1 X_1^{-1} X_0 = \frac{c_d(S_0 u - S_0 e^r) + c_u(-S_0 d + e^r S_0)}{e^r S_0(u-d)} = \frac{e^{-r}}{u-d}((e^r - d)c_u + (u - e^r)c_d)$$
$$= e^{-r}(qc_u + (1-q)c_d).$$

7.3 Futures, options and other financial products

Last term's material included a brief overview of the main derivative securities, i.e. financial products whose value depends on an underlying asset. Suppose that we have an arbitrage-free model for a stock price S_t and also a risk-free asset $B_t = e^{rt}$. Then we consider the following derivatives:

- Forward contracts: Agreement at time 0 to buy (sell) the stock at time T at the (arbitrage-free) forward price $K = S_0 e^{rT}$, see Example 32. No money changes hands at time 0. The actual value of the stock at time T, *at maturity*, is S_T, so the long position (buyer's position) in the forward contract is worth $V_T = S_T - K$ at time T. By construction, the contingent claim V_T can always be hedged and gives zero value $V_0 = 0$ at time of issue. In a *complete model* (where all Arrow-Debreu securities can be priced), this can also be confirmed by checking that the expected discounted payoff vanishes under the risk-neutral probabilities.

- A *futures contract* is a slight modification of a forward contract, where a third party (clearing house) overseas the execution of the contract and collects certain margins (payments) from the two parties. Specifically, if the stock price drops below (respectively rises above) the bond price, the buyer (respectively seller) has to deposit the difference. This ensures that both parties honour the contract. With the contracted transaction, the deposit is returned. For the buyer, it is the amount $(K - S_T)^+$ that he had to pay above the current stock price. For the seller, it is the amount $(S_T - K)^+$ that he received below the current stock price. If these deposits accumulate at the same risk-free rate as the bond and as long as they are counted towards the depositor, they do not affect the intermediate values of the contract, as long as the contract is honoured. In practice (in the presence of transaction costs), the margins do not trace each stock price movement, but are updated at a fixed frequency and only if amounts exceed certain thresholds.

- *Options* come in various flavours. Standard *vanilla* options have a fixed maturity T and payoffs V_T that only depend on the stock price S_T at maturity, $V_T = f(S_T)$. In a complete model, they can be priced as $V_0 = e^{-rT}\mathbb{E}_q(V_T)$. The prime examples are *European call options* and *European put options*, which, respectively, give the right, but not the obligation, to buy or sell the stock at maturity T at a strike price K. The payoff of the call is $V_T = (S_T - K)^+$, because

- if $S_T > K$, the option is exercised, the stock bought for K, sold again for S_T giving a payoff of $V_T = S_T - K$;
- if $S_T \leq K$, the option is not exercised and the payoff is $V_T = 0$.

The payoff of the put is $V_T = (K - S_T)^+$. American calls and puts can be exercised at any time up to and including maturity. Beyond vanilla options, the payoff can be path-dependent, i.e. dependent on $(S_t, t \leq T)$, e.g. *barrier options* that lose their value if the stock price exceeds a certain barrier b. For such an up-and-out barrier call option, we have $V_T = 1_{\{S_t \leq b, 0 \leq t \leq T\}}(S_T - K)^+$. These are more difficult to price even in complete models, because expectations are getting more difficult to evaluate.

Lecture 8

Duration, convexity and immunisation

Reading: McCutcheon-Scott Chapter 10, CT1 Unit 13

Suppose an institution holds assets of value V_A to meet liabilities of V_L and that at time 0, we have $V_A \geq V_L$. If interest rates applicable for discounting fall (rise), both V_A and V_L will increase (decrease). Under what conditions can we ensure that we still have $V_A \geq V_L$ under modified interest rates?

Obviously, full matching of our assets to our liabilities would achieve this. In practice however, full matching is difficult and it is instructive to ask in what circumstances might a partial match be sufficient?

In this lecture we introduce the notions of "volatility" and "convexity" of a cash-flow that reflect how present values of cash-flows change when interest rates change. Matching volatilities and dominating the convexity of liabilities provides a useful partial match.

8.1 Duration/volatility

For simplicity we assume a constant force of interest, i.e. a flat yield curve with $y_t = f_{t,r} = i$, $Y_t = F_{t,r} = \delta$, for all $t < r$.

Consider a discrete cash-flow $C = ((t_k, c_k), 1 \leq k \leq n)$. Let

$$V = \sum_{k=1}^{n} c_k \left(\frac{1}{1+i}\right)^{t_k} = \sum_{k=1}^{n} c_k e^{-t_k \delta}$$

be the present value of the payments in the constant-i model or equivalently in the constant-δ model. We will consider V to be a function of i or δ:

Definition 38 For a cash-flow C with present value V in the constant interest model, we introduce the notions of

(a) *volatility* or *effective duration* $\nu(i) = -\dfrac{d(\ln(V))}{di} = -\dfrac{1}{V}\dfrac{dV}{di}$.

(b) *discounted mean term (DMT)* or *MacAuley duration* $\tau(\delta) = -\dfrac{d(\ln(V))}{d\delta} = -\dfrac{1}{V}\dfrac{dV}{d\delta}$.

Note that with $1+i=e^\delta$ we have $di/d\delta=e^\delta$, or $\delta=\log(1+i)$ gives $d\delta/di=(1+i)^{-1}$, so

$$\tau(\delta)=-\frac{1}{V}\frac{dV}{d\delta}=-\frac{1}{V}\frac{dV}{di}\frac{di}{d\delta}=e^\delta\nu(e^\delta-1)=(1+i)\nu(i).$$

Furthermore, $\nu(i)=-\frac{1}{V}\frac{dV}{di}=\frac{1}{V}\sum_{k=1}^n c_k t_k\left(\frac{1}{1+i}\right)^{t_k+1}$ and $\tau(\delta)=\sum_{k=1}^n t_k\frac{c_k e^{-\delta t_k}}{\sum_{j=1}^n c_j e^{-\delta t_j}}$, where the last formula explains the name "discounted mean term", since $\tau(\delta)$ is indeed the mean term of the cash-flow, weighted by present value, provided that all weights have the same sign, i.e. provided that the cash-flow has either only inflows or only outflows. Specifically, it is a weighted average of t_k, with weights $w_k=c_k e^{-\delta t_k}/\sum_{j=1}^n c_j e^{-\delta t_j}$ that clearly sum up to 1 as we sum $k=1,\ldots,n$.

Example 39 DMT of an n-year coupon-paying bond, annual coupons of D, redemption proceeds of R, is

$$\tau=\frac{D(Ia)_{\overline{n}|}+Rnv^n}{Da_{\overline{n}|}+Rv^n},\qquad \text{where } v=(1+i)^{-1}=e^{-\delta}.$$

To see this, note that

$$V=D\sum_{k=1}^n v^k+Rv^n \quad\Rightarrow\quad \frac{dV}{d\delta}=D\sum_{k=1}^n(-k)v^k-Rnv^n=-D(Ia)_{\overline{n}|}-Rnv^n.$$

Note that for $D=0$ and $R=1$, we obtain the special case of a zero-coupon bond *of duration* n, which has DMT n. Note however, that it is the MacAuley duration, not the effective duration which equals n. The effective duration is $n(1+i)^{-1}$.

8.2 Convexity

Definition 40 For a cash-flow C with present value A in the constant interest model, we introduce the notion of *convexity* $c(i)=\frac{1}{V}\frac{d^2V}{di^2}$.

For a cash-flow $C=((t_1,c_1),\ldots,(t_n,c_n))$, this is $c(i)=\frac{1}{V}\sum_{k=1}^n c_k t_k(t_k+1)\left(\frac{1}{1+i}\right)^{t_k+2}$.
Convexity gives a measure of the change in duration when the interest rate changes.

Example 41 (a) For a zero-coupon bond of duration n, we obtain

$$V_n=(1+i)^{-n}\quad\Rightarrow\quad \nu_n(i)=n(1+i)^{-1}\quad\text{and}\quad c_n(i)=n(n+1)(1+i)^{-2}.$$

(b) For the sum of two zero-coupon bonds with $V=(1+i)^{-(n-m)}+(1+i)^{-(n+m)}$ we can derive simple expressions as weighted averages of the respective quantities for zero-coupon bonds, with notation as in (a):

$$\nu(i)=\nu_{n-m}(i)\frac{(1+i)^m}{(1+i)^m+(1+i)^{-m}}+\nu_{n+m}(i)\frac{(1+i)^{-m}}{(1+i)^m+(1+i)^{-m}}$$

and

$$c(i)=c_{n-m}(i)\frac{(1+i)^m}{(1+i)^m+(1+i)^{-m}}+c_{n+m}(i)\frac{(1+i)^{-m}}{(1+i)^m+(1+i)^{-m}}.$$

Note in particular that in the relevant case of small $i > 0$, the largest convexities among $m \in \{0, \ldots, n\}$ are found for the most spread-out cases of m close to n. Specifically, note that for fixed n and $i \downarrow 0$, the weights in the averages tend to 1/2, so $\nu(i) \to n$ and $c(i) \to n(n+1) + m^2$. In the next section, we will see that suitable mixtures of short and long term assets can provide protection against changes in interest rates.

8.3 Immunisation

Definition 42 Consider a fund with asset cash-flow A and liability cash-flow L. Let V_A and V_L be their present values. We say that at interest rate i_0 the fund is *immunised against small movements in the interest rate* if $V_A(i_0) = V_L(i_0) > 0$ and if there is $\varepsilon > 0$ such that $V_A(i) \geq V_L(i)$ for all $i \in (i_0 - \varepsilon, i_0 + \varepsilon)$.

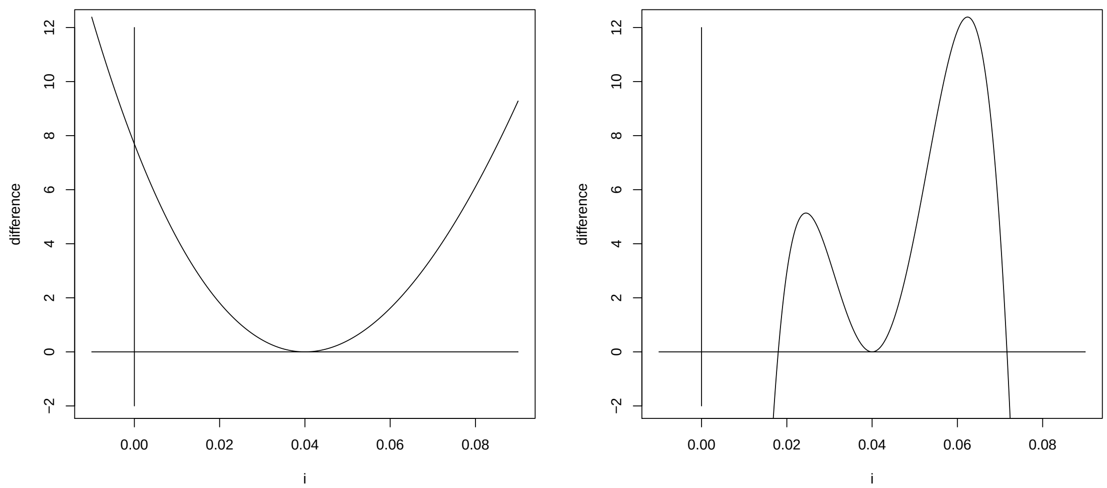

Figure 8.1: Plots of the difference of $V_A(i) - V_L(i)$ for liability cash-flows $L_1 = ((2, 10K))$ and $L_2 = ((0, 76.7M), (2, 500M), (4, 90.5M))$ and $i_0 = 4\%$, with matching asset portfolios of the form $A = ((1, a), (3, b))$, where immunisation works for $b_1 = 10K * (1.04)/2 = 5.2K$ and $a_1 = 10K/2(1.04) \approx 4.8K$; similarly $b_2 \approx 34.74M$ and $a_2 \approx 31.98M$.

Two examples of immunisation are illustrated in Figure 8.1. In the first case, where $L_1 = ((2, 10K))$ and $A_1 \approx ((1, 4.8K), (3, 5.2K))$, immunisation holds, in fact, against any change in interest rates. In the second case of $L_2 = ((0, 76.7M), (2, 500M), (4, 90.5M))$ and $A_2 \approx ((1, 31.98M), (3, 34.74M))$, immunisation holds just for small moves in interest rates ($i \in (1.8\%, 7.1\%)$, so we can choose any $\varepsilon < 2.18\%$). The following theorem explains how we can set up asset portfolios.

Theorem 43 (Redington) *If $V_A(i_0) = V_L(i_0) > 0$, $\nu_A(i_0) = \nu_L(i_0)$ and $c_A(i_0) > c_L(i_0)$, then at rate i_0, the fund is immunised against small movements in the interest rate.*

Proof: Consider the surplus $S(i) = V_A(i) - V_L(i)$. By Taylor's theorem, we have

$$\begin{aligned} S(i) &= S(i_0) + (i - i_0)S'(i_0) + \tfrac{1}{2}(i - i_0)^2 S''(i_0) + O((i - i_0)^3) \\ &= (V_A(i_0) - V_L(i_0)) - (i - i_0)(V_A(i_0)\nu_A(i_0) - V_L(i_0)\nu_L(i_0)) \\ &\qquad + \tfrac{1}{2}(i - i_0)^2 \left(V_A(i_0)c_A(i_0) - V_L(i_0)c_L(i_0)\right) + O((i - i_0)^3) \\ &= 0 - 0 + \tfrac{1}{2}(i - i_0)^2 V_A(i_0)(c_A(i_0) - c_L(i_0)) + O((i - i_0)^3) \geq 0 \end{aligned}$$

for $|i - i_0|$ sufficiently small. □

Example 44 (i) Suppose you have a liability to pay £10K in two years time and can invest into zero-coupon bonds of duration 1 and 3 years, both yielding $i_0 = 4\%$. We wish to immunise against small moves in interest rates by setting up a portfolio $A_1 = ((1, a_1), (3, b_1))$ of a_1 term-1 zero-coupon bonds and b_1 term-3 zero-coupon bonds. Redington's Theorem suggests to match $V_A(4\%) = V_L(4\%)$ and $\nu_A(4\%) = \nu_L(4\%)$, but given the first equation, the second is equivalent to $V'_A(4\%) = V'_L(4\%)$ or, if we like, $-1.04 \times V'_A(4\%) = -1.04 \times V'_L(4\%)$. Therefore, we have two linear equations

$$\left.\begin{array}{l} a_1/1.04 + b_1/(1.04)^3 = 10K/(1.04)^2 \\ a_1/1.04 + 3b_1/(1.04)^3 = 20K/(1.04)^2 \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} a_1 = 10K/2(1.04) \approx 4.8K \\ b_1 = 10K * (1.04)/2 = 5.2K. \end{array}\right.$$

Now $c_A(4\%) > c_L(4\%)$ iff $V''_A > V''_L$ iff $2a_1/(1.04)^3 + 12b_1/(1.04)^5 > 60K/(1.04)^4$. In numbers this is $59.8K > 51.3K$, so by Redington's Theorem, immunisation holds. The left-hand plot of Figure 8.1 illustrates $V_A(i) - V_L(i)$ for $i \in (-1\%, 9\%)$. In fact, in this example immunisation of the current position holds against any move in interest rates. However, some of the risks that we identify in the next section also apply to this example.

(ii) Suppose you have liabilities $L_2 = ((0, 76.7M), (2, 500M), (4, 90.5M))$. We leave to the reader to set up the linear equations to identify the portfolio $((1, a_2), (3, b_2))$ that matches present values and volatilities. The solution is $a_2 \approx 31.98M$ and $b_2 \approx 34.74M$. Again, convexities can be checked. The right-hand plot of Figure 8.1 illustrates that immunisation does not hold against larger moves in interest rates.

8.4 Limitations of classical immunisation theory

1. The theory relies on a *small* change in interest rates. The fund may not be protected against large changes. In practice, this is not usually a problem as the theory is fairly robust; only large changes and strange liabilities may lead to problems at this point; rebalancing helps when interest rates change gradually rather than abruptly.
2. The need of constant rebalancing of the portfolio is not unproblematic as this can be costly, in practice.
3. We assumed a constant interest rate now and at future time. This is rather suspect since yield curves are not flat in practice. On the problem sheet, we even point out some arbitrage problems under such model assumptions when a flat yield curve shifts up or down.
4. The theory is aimed at meeting fixed monetary liabilities, whereas in practice many liabilities are real. The theory can be adjusted to include inflation by using index-linked assets, but time lags may be a problem, also when rebalancing.
5. Assets of suitably long term may not exist.
6. There may be uncertainties in timing or amount of liability outgo.

What is done in practice? A broader risk management is based on asset-liability modelling using stochastic models with Monte-Carlo simulation, sensitivity analysis and/or scenario testing.

Lecture 9

Modelling future lifetimes

Reading: Gerber Sections 2.1, 2.2, 2.4, 3.1, 3.2, 4.1

In this lecture we introduce and apply actuarial notation for lifetime distributions.

9.1 Introduction to life insurance

The lectures that follow are motivated by the following problems.

1. An individual aged x would like to buy a *life annuity* (e.g. a pension) that pays him a fixed amount N p.a. for the rest of his life. How can a life insurer determine a fair price for this product?

2. An individual aged x would like to buy a *whole life insurance* that pays a fixed amount S to his dependants upon his death. How can a life insurer determine a fair single or annual premium for this product?

3. Other related products include *pure endowments* that pay an amount S at time n provided an individual is still alive, an *endowment assurance* that pays an amount S either upon an individual's death or at time n whichever is earlier, and a *term assurance* that pays an amount S upon an individual's death only if death occurs before time n.

The answer to these questions will depend on the chosen model of the future lifetime T_x of the individual.

9.2 Lives aged x

Recall our standard notation for continuous lifetime distributions. For a continuously distributed random lifetime T, we write $F_T(t) = \mathbb{P}(T \leq t)$ for the cumulative distribution function, $f_T(t) = F_T'(t)$ for the probability density function, $\overline{F}_T(t) = \mathbb{P}(T > t)$ for the survival function and $\mu_T(t) = f_T(t)/\overline{F}_T(t)$ for the force of mortality. Let us write $\omega_T = \inf\{t \geq 0 : \overline{F}_T(t) = 0\}$ for the maximal possible lifetime. Also recall

$$\overline{F}_T(t) = \exp\left(-\int_0^t \mu_T(s)ds\right).$$

We omit index T if there is no ambiguity.

Suppose now that T models the future lifetime of a new-born person. In life insurance applications, we are often interested in the future lifetime of a person aged x, or more precisely the *residual lifetime* $T - x$ given $\{T > x\}$, i.e. given survival to age x. For life annuities this determines the random number of annuity payments that are payable. For a life assurance contract, this models the time of payment of the sum assured. In practice, insurance companies perform medical tests and/or collect employment/geographical/medical data that allow more accurate modelling. However, let us here assume that no such other information is available. Then we have, for each $x \in [0, \omega)$,

$$\mathbb{P}(T - x > y | T > x) = \frac{\mathbb{P}(T > x + y)}{\mathbb{P}(T > x)} = \frac{\overline{F}(x + y)}{\overline{F}(x)}, \qquad y \geq 0,$$

by the definition of conditional probabilities $\mathbb{P}(A|B) = \mathbb{P}(A \cap B)/\mathbb{P}(B)$.

It is natural to directly model the residual lifetime T_x of an individual (a *life*) aged x as

$$\overline{F}_x(y) = \mathbb{P}(T_x > y) = \frac{\overline{F}(x + y)}{\overline{F}(x)} = \exp\left(-\int_x^{x+y} \mu(s)ds\right) = \exp\left(-\int_0^y \mu(x + t)dt\right).$$

We can read off $\mu_x(t) = \mu_{T_x}(t) = \mu(x + t)$, i.e. the force of mortality is still the same, just shifted by x to reflect the fact that the individual aged x and dying at time T_x is aged $x + T_x$ at death. We can also express cumulative distribution functions

$$F_x(y) = 1 - \overline{F}_x(y) = \frac{\overline{F}(x) - \overline{F}(x + y)}{\overline{F}(x)} = \frac{F(x + y) - F(x)}{1 - F(x)},$$

and probability density functions

$$f_x(y) = \begin{cases} F_x'(y) = \frac{f(x+y)}{1-F(x)} = \frac{f(x+y)}{\overline{F}(x)}, & 0 \leq y \leq \omega - x, \\ 0, & \text{otherwise.} \end{cases}$$

There is also actuarial lifetime notation, as follows

$${}_tq_x = F_x(t), \quad {}_tp_x = 1 - {}_tq_x = \overline{F}_x(t), \qquad q_x = {}_1q_x, \quad p_x = {}_1p_x, \qquad \mu_x = \mu(x) = \mu_x(0).$$

In this notation, we have the following *consistency condition* on lifetime distributions for different ages:

Proposition 45 *For all $x \geq 0$, $s \geq 0$ and $t \geq 0$, we have*

$${}_{s+t}p_x = {}_sp_x \times {}_tp_{x+s}.$$

By general reasoning, the probability that a life aged x survives for $s + t$ years is the same as the probability that it first survives for s years and then, aged $x + s$, survives for another t years.

Proof: Formally, we calculate the right-hand side

$$\begin{aligned} {}_sp_x \times {}_tp_{x+s} &= \mathbb{P}(T_x > s)\mathbb{P}(T_{x+s} > t) = \frac{\mathbb{P}(T > x + s)}{\mathbb{P}(T > x)} \frac{\mathbb{P}(T > x + s + t)}{\mathbb{P}(T > x + s)} = \mathbb{P}(T_x > s + t) \\ &= {}_{s+t}p_x. \end{aligned}$$

□

We can also express other formulas, that we have already established, in actuarial notation:

$$f_x(t) = {}_tp_x\mu_{x+t}, \qquad {}_tp_x = \exp\left(-\int_0^t \mu_{x+s}ds\right), \qquad {}_tq_x = \int_0^t {}_sp_x\mu_{x+s}ds.$$

The first one says that to die at t, life x must survive for time t and then die instantaneously.

9.3 Curtate lifetimes

In practice, many cash flows pay at discrete times, often at the end of each month. Let us begin here by discretising continuous lifetimes to integer-valued lifetimes. This is often done in practice, with interpolation being used for finer models.

Definition 46 Given a continuous lifetime random variable T_x, the random variable $K_x = [T_x]$, where $[\cdot]$ denotes the integer part, is called the associated *curtate lifetime.*

Of course, one can also model curtate lifetimes directly. Note that, for continuously distributed T_x

$$\mathbb{P}(K_x = k) = \mathbb{P}(k \le T_x < k+1) = \mathbb{P}(k < T_x \le k+1) = {}_kp_x \times q_{x+k}.$$

Also, by Proposition 45,

$$\mathbb{P}(K_x \ge k) = {}_kp_x = \prod_{j=0}^{k-1} p_{x+j},$$

i.e. K_x can be thought of as the number of successes before the first failure in a sequence of independent Bernoulli trials with varying success probabilities p_{x+j}, $j \ge 0$. Here, success is the survival of a year, while failure is death during the year.

Proposition 47 *We have* $\mathbb{E}(K_x) = \sum_{k=1}^{[\omega-x]} {}_kp_x$.

Proof: By definition of the expectation of a discrete random variable,

$$\begin{aligned}\sum_{k=1}^{[\omega-x]} {}_kp_x = \sum_{k=1}^{[\omega-x]} \mathbb{P}(K_x \ge k) &= \sum_{k=1}^{[\omega-x]}\sum_{m=k}^{[\omega-x]} \mathbb{P}(K_x = m) \\ &= \sum_{m=1}^{[\omega-x]}\sum_{k=1}^{m} \mathbb{P}(K_x = m) = \sum_{m=0}^{[\omega-x]} m\mathbb{P}(K_x = m) = \mathbb{E}(K_x).\end{aligned}$$

□

Example 48 If T is exponentially distributed with parameter $\mu \in (0, \infty)$, then $K = [T]$ is geometrically distributed:

$$\mathbb{P}(K = k) = \mathbb{P}(k \le T < k+1) = e^{-k\mu} - e^{-(k+1)\mu} = (e^{-\mu})^k(1 - e^{-\mu}), \qquad k \ge 0.$$

We identify the parameter of the geometric distribution as $e^{-\mu}$. Note also that here $p_x = e^{-\mu}$ and $q_x = 1 - e^{-\mu}$ for all x.

In general, we get

$$\mathbb{P}(K_x \geq k) = \prod_{j=0}^{k-1} p_{x+j} = {}_kp_x = \exp\left(-\int_0^k \mu(x+t)dt\right) = \prod_{j=0}^{k-1} \exp\left(-\int_{x+j}^{x+j+1} \mu(s)ds\right),$$

so that we read off

$$p_x = \exp\left(-\int_x^{x+1} \mu(s)ds\right), \qquad x \geq 0.$$

In practice, μ is often assumed constant between integer points (denoted $\mu_{x+0.5}$) or continuous piecewise linear between integer points.

9.4 Examples

Let us now return to our motivating problem.

Example 49 (Whole life insurance) Let K_x be a curtate future lifetime. A whole life insurance pays one unit at the end of the year of death, i.e. at time K_x+1. In the model of a constant force of interest δ, the random discounted value at time 0 is $Z = e^{-\delta(K_x+1)}$, so the fair premium for this random cash-flow is

$$A_x = \mathbb{E}(Z) = \mathbb{E}(e^{-\delta(K_x+1)}) = \sum_{k=1}^{\infty} e^{-\delta k}\mathbb{P}(K_x = k-1) = \sum_{m=0}^{\infty}(1+i)^{-m-1}{}_mp_x q_{x+m},$$

where $i = e^{\delta} - 1$ and A_x is just the actuarial notation for this expected discounted value.

Lecture 10

Lifetime distributions and life-tables

Reading: Gerber Sections 2.3, 2.5, 3.2

In this lecture we give an introduction to life-tables. We will not go into the details of constructing life-tables, which is one of the subjects of BS3b Statistical Lifetime-Models. What we are mostly interested in is the use of life-tables to price life insurance products.

10.1 Actuarial notation for life products.

Recall the motivating problems. Let us introduce the associated notation.

Example 50 (Term assurance, pure endowment and endowment) The fair premium of a term insurance is denoted by

$$A^1_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1}{}_kp_x q_{x+k}.$$

The superscript 1 above the x indicates that 1 is only paid in case of death within the period of n years.

The fair premium of a pure endowment is denoted by $A_{x:\overset{1}{\overline{n}|}} = v^n{}_np_x$. Here the superscript 1 indicates that 1 is only paid in case of survival of the period of n years.

The fair premium of an endowment is denoted by $A_{x:\overline{n}|} = A^1_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}$, where we could have put a 1 above both x and n, but this is omitted being the default, like in previous symbols.

Example 51 (Life annuities) Given a constant i interest model, the fair premium of an ordinary (respectively temporary) life annuity for a life aged x is given by

$$a_x = \sum_{k=1}^{\infty} v^k{}_kp_x \quad \text{respectively} \quad a_{x:\overline{n}|} = \sum_{k=1}^{n} v^k{}_kp_x.$$

For an ordinary (respectively temporary) life annuity-due, an additional certain payment at time 0 is made (and any payment at time n omitted):

$$\ddot{a}_x = 1 + a_x \quad \text{respectively} \quad \ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|}.$$

10.2 Simple laws of mortality

As we have seen, the theory is nicest for exponentially distributed lifetimes. However, the exponential distribution is not actually a good distribution to model human lifetimes:

- One reason that we have seen is that the curtate lifetime is geometric, i.e. each year given survival up to then, there is the same probability of dying in the next year. In practice, you would expect that this probability increases for higher ages.
- There is clearly significantly positive probability to survive up to age 70 and zero probability to survive to age 140, and yet the exponential distribution suggests that

  $$\mathbb{P}(T > 140) = e^{-140\mu} = (e^{-70\mu})^2 = (\mathbb{P}(T > 70))^2.$$

  Specifically, if we think there is at least 50% chance of a newborn to survive to age 70, there would be at least 25% chance to survive to age 140; if we think that the average lifetime is more than 70, then $\mu < 1/70$, so $\mathbb{P}(T > 140) > e^{-2} > 10\%$.
- More formally, the exponential distribution has the lack of memory property, which here says that the distribution of T_x is still exponential with the same parameter, independent of x. This would mean that there is no ageing.

These observations give some ideas for more realistic models. Generally, we would favour models with an eventually increasing force of mortality (in reliability theory such distributions are called IFR distributions – increasing failure rate).

1. Gompertz' Law: $\mu(t) = Bc^t$ for some $B > 0$ and $c > 1$.
2. Makeham's Law: $\mu(t) = A + Bc^t$ for some $A \geq 0$, $B > 0$ and $c > 1$ (or $c > 0$ to include DFR – decreasing failure rate cases).
3. Weibull: $\mu(t) = kt^\beta$ for some $k > 0$ and $\beta > 0$.

Makeham's Law actually gives a reasonable fit for ages 30-70.

10.3 The life-table

Suppose we have a population of newborn individuals (or individuals aged $\alpha > 0$, some lowest age in the table). Denote the size of the population by ℓ_α. Then let us observe each year the number ℓ_x of individuals still alive, until the age when the last individual dies reaching $\ell_\omega = 0$. Then out of ℓ_x individuals, ℓ_{x+1} survived age x, and the proportion ℓ_{x+1}/ℓ_x can be seen as the probability for each individual to survive. So, if we set

$$p_x = \ell_{x+1}/\ell_x \qquad \text{for all } \alpha \leq x \leq \omega - 1.$$

we specify a curtate lifetime distribution. The function $x \mapsto \ell_x$ is usually called the *life-table* in the strict sense. Note that also vice versa, we can specify a life-table with any given curtate lifetime distribution by choosing $\ell_0 = 100,000$, say, and setting $\ell_x = \ell_0 \times {}_xp_0$. In this case, we should think of ℓ_x as the *expected* number of individuals alive at age x. Strictly speaking, we should distinguish p_x and its estimate $\widehat{p}_x = \ell_{x+1}/\ell_x$, but this notion

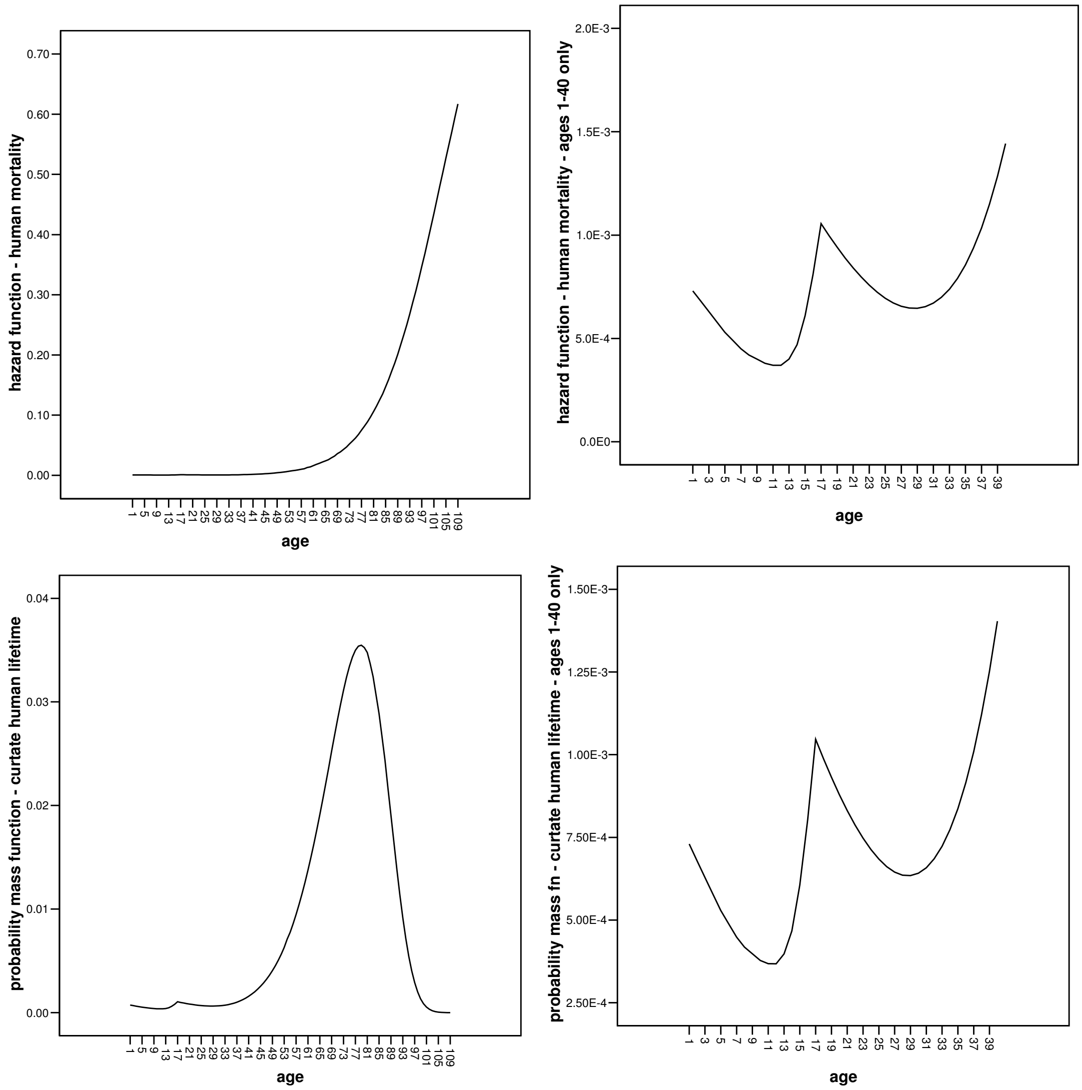

Figure 10.1: Human lifetime distribution from the 1967/70 life-table – constructed using advanced techniques including graduation/smoothing; notice in the bottom left plot, amplified in the right-hand plot, infant mortality and "accident hump" around age 20.

had not been developed when actuaries first did this, and rather leave the link to statistics and the interpretation of ℓ_x as an observed quantity, which it usually isn't anyway for real life-tables that were set up by rather more sophisticated methods.

There is a lot of notation associated with life-tables, and the 1967/70 life-table that we will work with actually consists of several pages of tables of different life functions. We work with the table for q_x, and there is also a table containing all ℓ_x, or indeed $d_x = \ell_x - \ell_{x+1}$, $\alpha \leq x \leq \omega - 1$. Observe that, with this notation, $q_x = d_x/\ell_x$.

See Figure 10.1 for plots of hazard function ($x \mapsto q_x$) and probability mass functions ($x \mapsto \mathbb{P}(K = x)$). Note that initially and up to age 35, the one-year death probabilities are below 0.001, then increase to cross 0.1 at age 80 and 0.5 at age 100.

10.4 Example

We will look at the 1967/70 life-table in more detail in the next lecture. Let us finish this lecture by giving an (artificial) example of a life-table constructed from a parametric family. A more realistic example is on the next assignment sheet.

Example 52 We find records about a group of 10,000 from when they were all aged 20, which reveal that 8,948 were still alive 5 years later and 7,813 another 5 years later. This means, that we are given

$$\ell_{20} = 10,000, \qquad \ell_{25} = 8,948 \qquad \text{and} \qquad \ell_{30} = 7,813.$$

Of course, this is not enough to set up a life-table, since all we would be able to calculate is

$${}_5p_{20} = \frac{\ell_{25}}{\ell_{20}} = 0.895 \qquad \text{and} \qquad {}_5p_{25} = \frac{\ell_{30}}{\ell_{25}} = 0.873.$$

However, if we assume that the population has survival function

$$\overline{F}_T(t) = \exp(-Ax - Bx^2), \qquad \text{for some } A \geq 0 \text{ and } B \geq 0,$$

we can solve two equations to identify the parameters that exactly replicate these two probabilities:

$$\begin{aligned} {}_5p_{20} &= \mathbb{P}(T > 25|T > 20) = \frac{\exp(-25A - 25^2B)}{\exp(-20A - 20^2B)} = \exp(-5A - 225B) \\ {}_pp_{25} &= \exp(-5A - 275B). \end{aligned}$$

(More sophisticated statistical techniques are beyond the scope of this course.) Here, we obtain that

$$B = \frac{\ln(0.895) - \ln(0.873)}{50} = 0.00049 \qquad \text{and} \qquad A = \frac{-\ln(0.895) - 225B}{5} = 0.00018,$$

and we can then construct the whole associated lifetable as

$$\ell_x = \ell_{20} \times {}_{x-20}p_{20} = 10,000 \exp(-(x-20)A - (x^2 - 20^2)B).$$

The reliability away from $x \in [20, 30]$ must be questioned, but we can compute the implied $\ell_{100} = 89.29$.

Lecture 11

Select life-tables and applications

Reading: Gerber 2.5, 2.6

11.1 Select life-tables

Some life-tables are select tables, meaning that mortality rates depend on the duration from joining the population. This is relevant if there is e.g. a medical check at the policy start date, which can reject applicants with poor health and only offer policies to applicants with better than average health and lower than average mortality, at least for the next few years. We denote by

- $[x]$ the age at the date of joining the population;
- $q_{[x]}$ the mortality rate for a life aged x having joined the population at age x;
- $q_{[x]+j}$ the mortality rate for a life aged $x+j$ having joined the population at age x;
- q_{x+r} the ultimate mortality for a life aged x having joined the population at age x;

It is supposed that after some select period of r years, mortality no longer varies significantly with duration since joining, but only with age: this is the *ultimate* portion of the table.

Example 53 The 1967/70 life-table has three columns, two select columns for $q_{[x]}$ and $q_{[x]+1}$ and one ultimate column for q_{x+2}. To calculate probabilities for $K_{[x]}$, the relevant entries are the three entries in row $[x]$ and all ultimate entries below this row. E.g.

$$\mathbb{P}(K_{[x]} = 4) = (1 - q_{[x]})(1 - q_{[x]+1}(1 - q_{x+2})(1 - q_{x+3})q_{x+4}.$$

Using the excerpt of Figure 11.1 of the table for an age [55], we obtain specifically

$$\mathbb{P}(K_{[55]} = 4) = (1 - 0.00447)(1 - 0.00625)(1 - 0.01050)(1 - 0.01169)0.01299 \approx 0.01257.$$

Example 54 Consider a 4-year temporary life assurance issued to a checked life [55]. Assuming a constant interest rate of $i = 4\%$, we calculate

$$A^{\,1}_{[55]:\overline{4}|} = vq_{[55]} + v^2 p_{[55]} q_{[55]+1} + v^3 p_{[55]} p_{[55]+1} q_{57} + v^4 p_{[55]} p_{[55]+1} p_{57} q_{58} = 0.029067.$$

Therefore, the fair price of an assurance with sum assured $N = £100,000$ is $£2,906.66$, payable as a single up-front premium.

Age $[x]$	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	Age $x+2$
53	.00376288	.00519413	.00844128	55
54	.00410654	.00570271	.00941902	56
55	.00447362	.00625190	.01049742	57
56	.00486517	.00684424	.01168566	58
57	.00528231	.00748245	.01299373	59
58	.00572620	.00816938	.01443246	60
59	.00619802	.00890805	.01601356	61
60	.00669904	.00970168	.01774972	62
61	.00723057	.01055365	.01965464	63
62	.00779397	.01146756	.02174310	64

Figure 11.1: Extract from mortality tables of assured lives based on 1967-70 data.

11.2 Multiple premiums

We will return to a more complete discussion of multiple premiums later, but for the purpose of this section, let us define:

Definition 55 Given a constant i interest model, let C be the cash flow of insurance benefits, the annual fair level premium of C is defined to be

$$P_x = \frac{E(DVal_0(C))}{\ddot{a}_x}.$$

This definition is based on the following principles:

Proposition 56 *In the setting of Definition 55, the expected dicounted benefits equal the expected discounted premium payments. Premium payments stop upon death.*

Proof: The expected discounted value of premium payments $((0, P_x), \ldots, (K_x, P_x))$ is $P_x \ddot{a}_x = E(DVal_0(C))$. □

Example 57 We calculate the fair annual premium in Example 54. Since there should not be any premium payments beyond when the policy ends either by death or by reaching the end of term, the premium payments form a effectively a temporary life annuity-due, i.e. cash flow $((0, 1), (1, 1), (2, 1), (3, 1))$ restricted to the lifetime $K_{[55]}$. First

$$\ddot{a}_{[55]:\overline{4}|} = 1 + vp_{[55]} + v^2 p_{[55]} p_{[55]+1} + v^3 p_{[55]} p_{[55]+1} p_{57} \approx 3.742157$$

and the annual premium is therefore calculated from the life table in Figure 11.1 as

$$P = \frac{NA^{1}_{[55]:\overline{4}|}}{\ddot{a}_{[55]:\overline{4}|}} = 776.73.$$

11.3 Interpolation for non-integer ages $x+u$, $x \in \mathbb{N}$, $u \in (0,1)$

There are two popular models. Model 1 is to assume that the force of mortality μ_t is constant between each $(x, x+1)$, $x \in \mathbb{N}$. This implies

$$p_x = \exp\left(-\int_x^{x+1} \mu_t dt\right) = \exp\left(-\mu_{x+0.5}\right) \qquad \Rightarrow \qquad \mu_{x+\frac{1}{2}} = -\ln(p_x).$$

Also, for $0 \le u \le 1$,

$$\mathbb{P}(T > x+u | T > x) = \exp\left\{-\int_x^{x+u} \mu_t dt\right\} = \exp\left\{-u\mu_{x+0.5}\right\} = (1-q_x)^u,$$

and with notation $T = K + S$, where $K = [T]$ is the integer part and $S = \{T\} = T - [T] = T - K$ the fractional part of T, this means that

$$\mathbb{P}(S \le u | K = x) = \frac{\mathbb{P}(x \le T \le x+u)/\mathbb{P}(T > x)}{\mathbb{P}(x \le T < x+1)/\mathbb{P}(T > x)} = \frac{1 - \exp\left\{-u\mu_{x+0.5}\right\}}{1 - \exp\left\{-\mu_{x+0.5}\right\}}, \qquad 0 \le u \le 1,$$

a distribution that is in fact the exponential distribution with parameter $\mu_{x+0.5}$, truncated at $\omega = 1$: for exponentially distributed E

$$\mathbb{P}(E \le u | E \le 1) = \frac{\mathbb{P}(E \le u)}{\mathbb{P}(E \le 1)} = \frac{1 - \exp\left\{-u\mu_{x+0.5}\right\}}{1 - \exp\left\{-\mu_{x+0.5}\right\}}, \qquad 0 \le u \le 1.$$

Since the parameter depends on x, S is not independent of K here.

Model 2 is convenient e.g. when calculating variances of continuous lifetimes: we assume that S and K are independent, and that S has a uniform distribution on $[0,1]$. Mathematically speaking, these models are not compatible: in Model 2, we have, instead, for $0 \le u \le 1$,

$$\begin{aligned} \bar{F}_{T_x}(u) &= \mathbb{P}(T > x+u | T > x) \\ &= \mathbb{P}(K \ge x+1 | K \ge x) + \mathbb{P}(S > u | K = x)\mathbb{P}(K = x | T > x) \\ &= (1-q_x) + (1-u)q_x = 1 - uq_x. \end{aligned}$$

We then calculate the force of mortality at $x+u$ as

$$\mu_{x+u} = -\frac{\bar{F}'_{T_x}(u)}{\bar{F}_{T_x}(u)} = \frac{q_x}{1-uq_x}$$

and this is increasing in u. Note that μ is discontinuous at (some if not all) integer times. The only exception is very artificial, as we require $q_{x+1} = q_x/(1-q_x)$, and in order for this to not exceed 1 at some point, we need $q_0 = \alpha = 1/n$, and then $q_k = \frac{\alpha}{1-k\alpha}$, $k = 1, \ldots, n-1$, with $\omega = n$ maximal age. Usually, one accepts discontinuities.

If one of the two assumptions is satisfied, the above formulas allow to reconstruct the full distribution of a lifetime T from the entries $(q_x)_{x\in\mathbb{N}}$ of a life-table: from the definition of conditional probabilities

$$\mathbb{P}(S \le t - [t] | K = [t]) = \frac{\mathbb{P}(K = [t], S \le t - [t])}{\mathbb{P}(K = [t])} = \begin{cases} \dfrac{1 - e^{-(t-[t])\mu_{x+0.5}}}{1 - e^{-\mu_{x+0.5}}} & \text{in Model 1,} \\ t - [t] & \text{in Model 2,} \end{cases}$$

we deduce that

$$\mathbb{P}(T \leq t) = \mathbb{P}(K \leq [t] - 1) + \mathbb{P}(K = [t])\mathbb{P}(S \leq t - [t] | K = [t]),$$

and we have already expressed the distribution of K in terms of $(q_x)_{x \in \mathbb{N}}$.

11.4 Practical concerns

- Mortality depends on individual characteristics (rich, athletic, adventurous). Effects on mortality can be studied. Models include scaling or shifting already existing tables as a function of the characteristics
- We need to estimate future mortality, which may not be the same as current or past mortality. Note that this is inherently two-dimensional: a 60-year old in 2060 is likely to have a different mortality from a 60-year old in 2010. A two-dimensional life-table should be indexed separately by calender year of birth and age, or current calender year and age. Prediction of future mortality is a major actuarial problem. Extrapolating substantially into the future is subject to considerable uncertainty.
- There are other risks, such as possible effects of pills to halt ageing, or major influenza outbreak. How can such risks be hedged?

Lecture 12

Evaluation of life insurance products

Reading: Gerber Sections 3.2, 3.3, 3.4, 3.5, 3.6, 4.2

12.1 Life assurances

Recall that the fair single premium for a whole life assurance is

$$A_x = \mathbb{E}(v^{K_x+1}) = \sum_{k=0}^{\infty} v^{k+1} {}_kp_x q_{x+k},$$

where $v = e^{-\delta} = (1+i)^{-1}$ is the discount factor in the constant-i model. Note also that higher moments of the present value are easily calculated. Specifically, the second moment

$${}^2A_x = \mathbb{E}(v^{2(K_x+1)}),$$

associated with discount factor v^2, is the same as A_x calculated at a rate of interest $i' = (1+i)^2 - 1$, and hence

$$\mathrm{Var}(v^{K_x+1}) = \mathbb{E}(v^{2(K_x+1)}) - \left(\mathbb{E}(v^{K_x+1})\right)^2 = {}^2A_x - (A_x)^2.$$

Remember that the variance as expected *squared* deviation from the mean is a quadratic quantity and mean and variance of a whole life assurance of sum assured S are

$$\mathbb{E}(Sv^{K_x+1}) = SA_x \qquad \text{and} \qquad \mathrm{Var}(Sv^{K_x+1}) = S^2\mathrm{Var}(v^{K_x+1}) = S^2({}^2A_x - (A_x)^2).$$

Similarly for a term assurance,

$$\mathbb{E}\left(Sv^{K_x+1}1_{\{K_x<n\}}\right) = SA^1_{x:\overline{n}|} \quad \text{and} \quad \mathrm{Var}\left(Sv^{K_x+1}1_{\{K_x<n\}}\right) = S^2\left({}^2A^1_{x:\overline{n}|} - (A^1_{x:\overline{n}|})^2\right)$$

for a pure endowment

$$\mathbb{E}\left(Sv^n1_{\{K_x\geq n\}}\right) = SA_{x:\overline{n}|}^{\;\,1} = Sv^n{}_np_x \quad \text{and} \quad \mathrm{Var}(Sv^n1_{\{K_x\geq n\}}) = S^2\left({}^2A_{x:\overline{n}|}^{\;\,1} - (A_{x:\overline{n}|}^{\;\,1})^2\right)$$

and for an endowment assurance

$$\mathbb{E}(Sv^{\min(K_x+1,n)}) = SA_{x:\overline{n}|} \quad \text{and} \quad \mathrm{Var}(Sv^{\min(K_x+1,n)}) = S^2\left({}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2\right)$$

Note that

$$S^2\left({}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2\right) \neq S^2\left({}^2A^1_{x:\overline{n}|} - (A^1_{x:\overline{n}|})^2\right) + S^2\left({}^2A_{x:\overline{n}|}^{\ \ 1} - (A_{x:\overline{n}|}^{\ \ 1})^2\right),$$

because term assurance and pure endowment are *not* independent, quite the contrary, the product of their discounted values always vanishes, so that their covariance $-S^2A^1_{x:\overline{n}|}A_{x:\overline{n}|}^{\ \ 1}$ is maximally negative. In other notation, from the variance formula for sums of dependent random variables,

$$\begin{aligned}
&\mathrm{Var}(Sv^{\min(K_x+1,n)}) = \mathrm{Var}(Sv^{K_x+1}1_{\{K_x<n\}} + Sv^n1_{\{K_x\geq n\}})\\
&\quad= \mathrm{Var}(Sv^{K_x+1}1_{\{K_x<n\}}) + \mathrm{Var}(Sv^n1_{\{K_x\geq n\}}) + 2\mathrm{Cov}(Sv^{K_x+1}1_{\{K_x<n\}}, Sv^n1_{\{K_x\geq n\}})\\
&\quad= \mathrm{Var}(Sv^{K_x+1}1_{\{K_x<n\}}) + \mathrm{Var}(Sv^n1_{\{K_x\geq n\}}) - 2\mathbb{E}(Sv^{K_x+1}1_{\{K_x<n\}})\mathbb{E}(Sv^n1_{\{K_x\geq n\}})\\
&\quad= S^2\left({}^2A^1_{x:\overline{n}|} - (A^1_{x:\overline{n}|})^2\right) + S^2\left({}^2A_{x:\overline{n}|}^{\ \ 1} - (A_{x:\overline{n}|}^{\ \ 1})^2\right) - S^2A^1_{x:\overline{n}|}A_{x:\overline{n}|}^{\ \ 1}.
\end{aligned}$$

12.2 Life annuities and premium conversion relations

Recall present values of whole-life annuities, temporary annuities and their due versions

$$a_x = \sum_{k=1}^{\infty} v^k\,{}_kp_x, \quad a_{x:\overline{n}|} = \sum_{k=1}^{n} v^k\,{}_kp_x, \qquad \ddot{a}_x = 1 + a_x \quad \text{and} \quad \ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|}.$$

Also note the simple relationships (that are easily proved algebraically)

$$a_x = vp_x\ddot{a}_{x+1} \qquad \text{and} \qquad a_{x:\overline{n}|} = vp_x\ddot{a}_{x+1:\overline{n}|}.$$

By general reasoning, they can be justified by saying that the expected discounted value of regular payments in arrears for up to n years contingent on a life x is the same as the expected discounted value of up to n payments in advance contingent on a life $x+1$, discounted by a further year, and given survival of x for one year (which happens with probability p_x).

To calculate variances of discounted life annuity values, we use premium conversion relations:

Proposition 58 *$A_x = 1 - d\ddot{a}_x$ and $A_{x:\overline{n}|} = 1 - d\ddot{a}_{x:\overline{n}|}$, where $d = 1 - v$.*

Proof: The quickest proof is based on the formula $\ddot{a}_{\overline{n}|} = (1-v^n)/d$ from last term

$$\ddot{a}_x = \mathbb{E}(\ddot{a}_{\overline{K_x+1}|}) = \mathbb{E}\left(\frac{1-v^{K_x+1}}{d}\right) = \frac{1-\mathbb{E}(v^{K_x+1})}{d} = \frac{1-A_x}{d}.$$

The other formula is similar, with K_x+1 replaced by $\min(K_x+1,n)$. □

Now for a whole life annuity,

$$\mathrm{Var}(a_{\overline{K_x}|}) = \mathrm{Var}\left(\frac{1-v^{K_x}}{i}\right) = \mathrm{Var}\left(\frac{v^{K_x+1}}{d}\right) = \frac{1}{d^2}\mathrm{Var}(v^{K_x+1}) = \frac{1}{d^2}\left({}^2A_x - (A_x)^2\right).$$

For a whole life annuity-due,

$$\mathrm{Var}(\ddot{a}_{\overline{K_x+1}|}) = \mathrm{Var}(1 + a_{\overline{K_x}|}) = \frac{1}{d^2}\left({}^2A_x - (A_x)^2\right).$$

Similarly,

$$\begin{aligned}\mathrm{Var}(a_{\overline{\min(K_x,n)}|}) &= \mathrm{Var}\left(\frac{1 - v^{\min(K_x,n)}}{i}\right) = \mathrm{Var}\left(\frac{v^{\min(K_x+1,n+1)}}{d}\right)\\ &= \frac{1}{d^2}\left({}^2A_{x:\overline{n+1}|} - (A_{x:\overline{n+1}|})^2\right)\end{aligned}$$

and

$$\mathrm{Var}(\ddot{a}_{\overline{\min(K_x+1,n)}|}) = \mathrm{Var}\left(1 + a_{\overline{\min(K_x,n-1)}|}\right) = \frac{1}{d^2}\left({}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2\right).$$

12.3 Continuous life assurance and annuity functions

A whole of life assurance with payment exactly at date of death has expected present value

$$\overline{A}_x = \mathbb{E}(v^{T_x}) = \int_0^\infty v^t f_x(t)dt = \int_0^\infty v^t {}_tp_x \mu_{x+t} dt.$$

An annuity payable continuously until the time of death has expected present value

$$\overline{a}_x = \mathbb{E}\left(\frac{1 - v^{T_x}}{\delta}\right) = \int_0^\infty v^t {}_tp_x dt.$$

Note also the premium conversion relation $\overline{A}_x = 1 - \delta\overline{a}_x$.

For a term assurance with payment exactly at the time of death, we obtain

$$\overline{A}^1_{x:\overline{n}|} = \mathbb{E}(v^{T_x} 1_{\{T_x \le n\}}) = \int_0^n v^t {}_tp_x \mu_{x+t} dt.$$

Similarly, variances can be expressed, as before, e.g.

$$\mathrm{Var}(v^{T_x} 1_{\{T_x \le n\}}) = {}^2\overline{A}^1_{x:\overline{n}|} - (\overline{A}^1_{x:\overline{n}|})^2.$$

12.4 More general types of life insurance

In principle, we can find appropriate premiums for any cash-flow of benefits that depend on T_x, by just taking expected discounted values. An example of this was on Assignment 4, where an increasing whole life assurance was considered that pays $K_x + 1$ at time $K_x + 1$. The purpose of the exercise was to establish the premium conversion relation

$$(IA)_x = \ddot{a}_x - d(I\ddot{a})_x$$

that relates the premium $(IA)_x$ to the increasing life annuity-due that pays $k + 1$ at time k for $0 \le k \le K_x$. It is natural to combine such an assurance with a regular savings plan and pay annual premiums. The principle that the total expected discounted premium payments coincide with the total expected discounted benefits yield a level annual premium $(IP)_x$ that satisfies

$$(IP)_x\ddot{a}_x = (IA)_x = \ddot{a}_x - d(I\ddot{a})_x \quad \Rightarrow \quad (IP)_x = 1 - d\frac{(I\ddot{a})_x}{\ddot{a}_x}.$$

Similarly, there are decreasing life assurances. A regular decreasing life assurance is useful to secure mortgage payments. The (simplest) standard case is where a payment of $n-K_x$ is due at time K_x+1 provided $K_x < n$. This is a term assurance. We denote its single premium by

$$(DA)^1_{x:\overline{n}|} = \sum_{k=0}^{n-1}(n-k)v^{k+1}{}_kp_xq_{x+k}$$

and note that

$$(DA)^1_{x:\overline{n}|} = A^1_{x:\overline{n}|} + A^1_{x:\overline{n-1}|} + \cdots + A^1_{x:\overline{1}|} = nA^1_{x:\overline{n}|} - (IA)^1_{x:\overline{n}|},$$

where $(IA)^1_{x:\overline{n}|}$ denotes the present value of an increasing term-assurance.

Lecture 13

Premiums

Reading: Gerber Sections 5.1, 5.3, 6.2, 6.5, 10.1, 10.2

In this lecture we incorporate expenses into premium calculations. On average, such expenses are to cover the insurer's administration cost, contain some risk loading and a profit margin for the insurer. We assume here that expenses are incurred for each policy separately. In practice, actual expenses per policy also vary with the total number of policies underwritten. There are also strategic variations due to market forces.

13.1 Different types of premiums

Consider the future benefits payable under an insurance contract, modelled by a random cash flow C. Recall that typically payment for the benefits are either made by a single lump sum premium payment at the time the contract is effected (a *single-premium contract*) or by a regular annual (or monthly) premium payments of a level amount for a specified term (a *regular premium contract*). Note that we will be assuming that all premiums are paid *in advance*, so the first payment is always due at the time the policy is effected.

Definition 59

- The *net premium* (or pure premium) is the premium amount required to meet the expected benefits under a contract, given mortality and interest assumptions.
- The *office premium* (or gross premium) is the premium required to meet all the costs under an insurance contract, usually including expected benefit cost, expenses and profit margin. This is the premium which the policy holder pays.

In this terminology, the *net premium for a single premium contract* is the expected cost of benefits $\mathbb{E}(\mathrm{Val}_0(C))$. E.g., the net premium for a single premium whole life assurance policy of sum assured 1 issued to a life aged x is A_x.

In general, recall the principle that the expected present value of *net* premium payment equals the expected present value of benefit payments. For office premiums, and later premium reserves, it is more natural to write this from the insurer's perspective as

$$\text{expected present value of net premium income} = \text{expected present value of benefit outgo.}$$

Then, we can say similarly

$$\begin{aligned}&\text{expected present value of office premium income}\\&\qquad= \text{expected present value of benefit outgo}\\&\qquad\quad+ \text{expected present value of outgo on expenses}\\&\qquad\quad+ \text{expected present value of required profit loading.}\end{aligned}$$

Definition 60 A *(policy) basis* is a set of assumptions regarding future mortality, investment returns, expenses etc.

The basis used for calculating premiums will usually be more cautious than the best estimate for a number of reasons, including to allow for a contingency margin (the insurer does not want to go bust) and to allow for uncertainty in the estimates themselves.

13.2 Net premiums

We will use the following notation for the regular net premium payable annually throughout the duration of the contract:

$P_{x:\overline{n}\rvert}$	for an endowment assurance
$P^1_{x:\overline{n}\rvert}$	for a term assurance
P_x	for a whole life assurance.

In each case, the understanding is that we apply a second principle which stipulates that the premium payments end upon death, making the premium payment cash-flow a random cash-flow. We also introduce

$${}_nP_x \quad \text{as regular net premium payable for a maximum of } n \text{ years.}$$

To calculate net premiums, recall that premium payments form a life annuity (temporary or whole-life), so we obtain net annual premiums from the first principle of equal expected discounted values for premiums and benefits, e.g.

$$\ddot{a}_{x:\overline{n}\rvert} P_{x:\overline{n}\rvert} = A_{x:\overline{n}\rvert} \qquad \Rightarrow \qquad P_{x:\overline{n}\rvert} = \frac{A_{x:\overline{n}\rvert}}{\ddot{a}_{x:\overline{n}\rvert}}.$$

Similarly, $P^1_{x:\overline{n}\rvert} = \dfrac{A^1_{x:\overline{n}\rvert}}{\ddot{a}_{x:\overline{n}\rvert}}$, $P_x = \dfrac{A_x}{\ddot{a}_x}$, ${}_nP_x = \dfrac{A_x}{\ddot{a}_{x:\overline{n}\rvert}}$.

13.3 Office premiums

For office premiums, the *basis* for their calculation is crucial. We are already used to making assumptions about an interest rate model and about mortality. Expenses can be set in a variety of ways, and often it is a combination of several expenses that are charged differently. It is of little value to categorize expenses by producing a list of possibilities, because whatever their form, they describe nothing else than a cash flow, sometimes involving the premium to be determined. Finding the premium is solving an equation of value, which is usually a linear equation in the unknown. We give an example.

Example 61 Calculate the premium for a whole-life assurance for a sum assured of £10,000 to a life aged 40, where we have

Expenses: £100 to set up the policy,
30% of the first premium as a commission,
1.5% of subsequent premiums as renewal commission,
£10 per annum maintenance expenses (after first year).

If we denote the gross premium by P, then the equation of value that sets expected discounted premium payments equal to expected discounted benefits plus expected discounted expenses is

$$P\ddot{a}_{40} = 10,000A_{40} + 100 + 0.3P + 0.015Pa_{40} + 10a_{40}.$$

Therefore, we obtain

$$\text{gross premium } P = \frac{10,000A_{40} + 100 + 10a_{40}}{\ddot{a}_{40} - 0.3 - 0.015a_{40}}.$$

In particular, we see that this exceeds the net premium

$$\frac{10,000A_{40}}{\ddot{a}_{40}}.$$

13.4 Prospective policy values

Consider the benefit and premium payments under a life insurance contract. Given a policy basis and given survival to time t, we can specify the expected present value of the contract (for the insured) at a time t during the term of the contract as

$$\begin{aligned}\text{Prospective policy value} \quad &= \quad \text{expected time-}t\text{ value of future benefits} \\ &\quad - \text{expected time-}t\text{ value of future premiums.}\end{aligned}$$

We call *net premium policy value* the prospective policy value when no allowance is made for future expenses and where the premium used in the calculation is a notional premium, using the policy value basis. For the net premium policy values of the standard products at time t we write

$${}_tV_{x:\overline{n}|}, \quad {}_tV^1_{x:\overline{n}|}, \quad {}_tV_x, \quad {}_t\overline{V}_{x:\overline{n}|}, \quad {}_t\overline{V}_x, \quad \text{etc.}$$

Example 62 (n-year endowment assurance) The contract has term $\max\{K_x+1, n\}$. We assume annual level premiums. When we calculate the net premium policy value at time $k = 1, \ldots, n-1$, this is for a life aged $x+k$, i.e. a life aged x at time 0 that survived to time k. The residual term of the policy is up to $n-k$ years, and premium payments are still at rate $P_{x:\overline{n}|}$. Therefore,

$${}_kV_{x:\overline{n}|} = A_{x-k:\overline{n-k}|} - P_{x:\overline{n}|}\ddot{a}_{x+k:\overline{n-k}|}$$

But from earlier calculations of premiums and associated premium conversion relations, we have

$$P_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} \quad \text{and} \quad A_{y:\overline{m}|} = 1 - d\ddot{a}_{y:\overline{m}|},$$

so that the value of the endowment assurance contract at time k is

$$\begin{aligned} {}_kV_{x:\overline{n}|} &= A_{x+k:\overline{n-k}|} - A_{x:\overline{n}|}\frac{\ddot{a}_{x+k:\overline{n-k}|}}{\ddot{a}_{x:\overline{n}|}} \\ &= 1 - d\ddot{a}_{x+k:\overline{n-k}|} - (1 - d\ddot{a}_{x:\overline{n}|})\frac{\ddot{a}_{x+k:\overline{n-k}|}}{\ddot{a}_{x:\overline{n}|}} = 1 - \frac{\ddot{a}_{x+k:\overline{n-k}|}}{\ddot{a}_{x:\overline{n}|}}, \end{aligned}$$

where we recall that this quantity refers to a surviving life, while the prospective value for a non-surviving life is zero, since the contract will have ended.

As an aside, for a life aged x at time 0 that did not survive to time k, there are no future premium or benefit payments, so the prospective value of such an (expired) policy at such time k is zero. The insurer may have made a loss on this individual policy, such loss is paid for by parts of premiums under other policy contracts (in the same portfolio).

Example 63 (Whole-life policies) Similarly, for whole-life policies with payment at the end of the year of death, for $k = 1, 2, \ldots$,

$${}_kV_x = A_{x+k} - P_x\ddot{a}_{x+k} = A_{x+k} - \frac{A_x}{\ddot{a}_x}\ddot{a}_{x+k} = (1 - d\ddot{a}_{x+k}) - (1 - d\ddot{a}_x)\frac{\ddot{a}_{x+k}}{\ddot{a}_x} = 1 - \frac{\ddot{a}_{x+k}}{\ddot{a}_x},$$

or with payment at death for any real $t \geq 0$.

$${}_t\overline{V}_x = \overline{A}_{x+t} - \overline{P}_x\overline{a}_{x+t} = \overline{A}_{x+t} - \frac{\overline{A}_x}{\overline{a}_x}\overline{a}_{x+t} = (1 - \delta\overline{a}_{x+t}) - (1 - \delta\overline{a}_x)\frac{\overline{a}_{x+t}}{\overline{a}_x} = 1 - \frac{\overline{a}_{x+t}}{\overline{a}_x}.$$

While for whole-life and endowment policies the prospective policy values are increasing (because there will be a benefit payment at death, and death is more likely to happen soon, as the policyholder ages), the behaviour is quite different for temporary assurances (because it is also getting more and more likely that no benefit payment is made):

Example 64 (Term assurance policy) Consider a 40-year policy issued to a life aged 25 subject to A1967/70 mortality. For a sum assured of £100,000 and $i = 4\%$, the net premium of this policy can best be worked out by a computer: $£100,000P_{25:\overline{40}|} = £310.53$. The prospective policy values ${}_kV^1_{25:\overline{40}|} = A^1_{25+k,\overline{40-k}|} - P^1_{25:\overline{40}|}\ddot{a}_{25+k:\overline{40-k}|}$ per unit sum assured give policy values as in Figure 13.1, plotted against age, rising up to age 53, then falling.

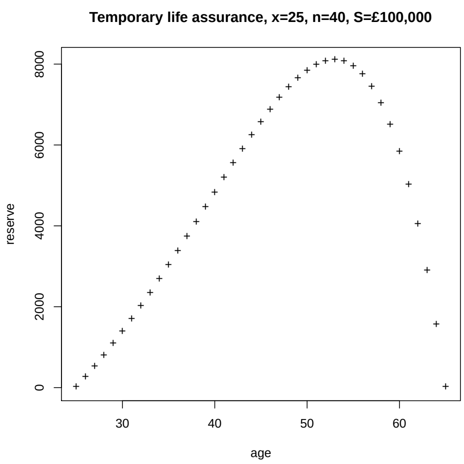

Figure 13.1: Prospective policy values for a temporary assurance.

Lecture 14

Reserves

Reading: Gerber Sections 6.1, 6.3, 6.11

In many long-term life insurance contracts the cost of benefits is increasing over the term but premiums are level (or single). Therefore, the insurer needs to set aside part of early premium payments to fund a shortfall in later years of contracts. In this lecture we calculate such reserves.

14.1 Reserves and random policy values

Recall

$$\begin{aligned}\text{Prospective policy value} &= \text{expected time-}t\text{ value of future benefits}\\ &\quad - \text{expected time-}t\text{ value of future premiums.}\end{aligned}$$

If this prospective policy value is positive, the life office needs a *reserve* for that policy, i.e. an amount of funds held by the life office at time t in respect of that policy. Apart possibly from an initial reserve that the insurer provides for solvency reasons, such a reserve typically consists of parts of earlier premium payments.

If the reserve exactly matches the prospective policy value and if experience is exactly as expected in the policy basis then reserve plus future premiums will exactly meet future liabilities. Note, however, that the mortality assumptions in the policy basis usually build a stochastic model that for a single policy will produce some spread around expected values. If life offices hold reserves for portfolios of policies, where premiums for each policy are set to match expected values, randomness will mean that some policies will generate surplus that is needed to pay for the shortfall of other policies. In practice insurers will usually reserve on a prudent (slightly cautious) basis so that aggregate future experience would have to be very bad for reserves plus future income not to cover future liabilities.

If part of the risk of very bad future experience (e.g. untimely death of a large number of policy holders) is due to specific circumstances (such as acts of terrorism or war) the insurer may be able to eliminate such risk by exclusions in the policy contract or by appropriate re-insurance, where the life office passes on such risk to a re-insurance company by paying an appropriate premium from their premium income. We will get back to the notion of pooling of insurance policies, which is one of the essential reasons why insurance works. On a superficial level, re-insurance can be seen as a pooling of portfolios of insurance policies.

When calculating present values of insurance policies and annuity contracts in the first place, it was convenient to work with expectations of present values of random cashflows depending on a lifetime random variable T_x or $K_x = [T_x]$.

We can formalise prospective policy values in terms of an underlying stochastic lifetime model, and a constant-i (or constant-δ) interest model. A life insurance contract issued to a life aged x gives rise to a random cash flow $C = C^B - C^P$ of benefit inflows C^B and premium outflows $-C^P$. The associated prospective policy value is

$$\mathbb{E}(L_t | T_x > t), \qquad \text{where } L_t = \mathrm{Val}_t(C^B_{|(t,\infty)} - C^P_{|[t,\infty)}),$$

where the subtlety of restricting to times (t,∞) and $[t,\infty)$, respectively, arises naturally (and was implicit in calculations for Examples 62 and 63), because premiums are paid in advance and benefits in arrears in the discrete model, so for $t = k \in \mathbb{N}$, a premium payment at time k is in advance (e.g. for the year $(k, k+1]$), while a benefit payment at time k is in arrears (e.g. for death in $[k-1,k)$). Note, in particular, that if death occurs during $[k-1,k)$, then $L_k = 0$ and $L_{k-1} = v - P_x$, where $v = (1+i)^{-1}$.

Proposition 65 (Recursive calculation of policy values) *For a whole-life assurance, we have*

$$({}_kV_x + P_x)(1+i) = q_{x+k} + p_{x+k}\ {}_{k+1}V_x.$$

By general reasoning, the value of the policy at time k plus the annual premium for year k payable in advance, all accumulated to time $k+1$ will give the death benefit of 1 for death in year $k+1$, i.e., given survival to time k, a payment with expected value q_{x+k} and, for survival, the policy value ${}_{k+1}V_x$ at time $k+1$, a value with expectation $p_{x+k}\ {}_{k+1}V_x$.

Proof: The most explicit actuarial proof exploits the relationships between both assurance and annuity values for consecutive ages (which are obtained by partitioning according to one-year death and survival)

$$A_{x+k} = vq_{x+k} + vp_{x+k}A_{x+k+1} \quad \text{and} \quad \ddot{a}_{x+k} = 1 + vp_{x+k}\ddot{a}_{x+k+1}.$$

Now we obtain

$$\begin{aligned} {}_kV_x + P_x &= A_{x+k} - P_x\ddot{a}_{x+k} + P_x \\ &= vq_{x+k} + vp_{x+k}A_{x+k+1} - (1 + vp_{x+k}\ddot{a}_{x+k+1})P_x + P_x \\ &= v(q_{x+k} + p_{x+k}(A_{x+k+1} - P_x\ddot{a}_{x+k+1}) \\ &= v(q_{x+k} + p_{x+k}\ {}_{k+1}V_x). \end{aligned}$$

□

An alternative proof can be obtained by exploiting the premium conversion relationship, which reduces the recursive formula to $\ddot{a}_{x+k} = 1 + vp_{x+k}\ddot{a}_{x+k+1}$. However, such a proof does not use insight into the cash-flows underlying the insurance policy.

A probabilistic proof can be obtained using the underlying stochastic model: by definition, ${}_kV_x = \mathbb{E}(L_k | T_x > k)$, but we can split the cash-flow underlying L_k as

$$C^B_{|(k,\infty)} - C^P_{|[k,\infty)} = ((0, -P_x), (1, 1_{\{k \le T_k < k+1\}}), C^B_{|(k+1,\infty)} - C^P_{|[k+1,\infty)}).$$

Taking Val_k and $\mathbb{E}(\cdot|T_x > k)$ then using $\mathbb{E}(1_{\{k \le T_x < k+1\}}|T_x > k) = \mathbb{P}(T_x < k+1|T_k > k) = q_{x+k}$, we get

$$_kV_x = -P_x + vq_{x+k} + v\mathbb{E}(L_{k+1}|T_x > k) = -P_x + vq_{x+k} + vp_{x+k}\ {}_{k+1}V_x,$$

where the last equality uses $\mathbb{E}(X|A) = \mathbb{P}(B|A)\mathbb{E}(X|A \cap B) + \mathbb{P}(B^c|A)\mathbb{E}(X|A \cap B^c)$, the partition theorem, where here $X = L_{k+1} = 0$ on $B^c = \{T_x < k+1\}$.

There are similar recursive formulas for the other types of life insurance contracts.

14.2 Thiele's differential equation

In the case of continuous time, the analogue of the recursive formula of Proposition 65 is a differential equation. In fact, if we write that formula as

$$_{k+1}V_x - {}_kV_x = \frac{1}{p_{x+k}} \left(i\ {}_kV_x + (1+i)P_x - q_{x+k}(1 - {}_kV_x) \right),$$

it describes the variation of the prospective policy value with time. The term $i\ {}_kV_x + (1+i)P_x$ just reflects interest and premium payments, while the remainder is an adjustment for the fact that the increase is between two conditional expectations with different conditions; indeed the denominator $p_{x+k} = \mathbb{P}(T \ge x+k+1|T \ge x+k)$ is for the extra conditioning needed for ${}_{k+1}V_x$, while $q_{x+k}(1 - {}_kV_x)$ is the expected change in policy value with death occurring in $[k, k+1)$.

With such interpretations in mind, we expect that for a policy basis

- force of interest δ,
- premiums payable continuously at rate $\overline{P}_x$ per annum
- sum assured of 1 payable immediately on death

we have ${}_hp_x \approx 1$ for small $h > 0$, and hence

$$_{t+h}\overline{V}_x - {}_t\overline{V}_x \approx \delta h\ {}_t\overline{V}_x + h\overline{P}_x - (1 - {}_t\overline{V}_x)(1 - e^{-h\mu_{x+t}}).$$

Dividing by h and letting $h \downarrow 0$, we should (and, by the following theorem do) get

$$\frac{\partial}{\partial t}{}_t\overline{V}_x = \delta\ {}_t\overline{V}_x + \overline{P}_x - \mu_{x+t}(1 - {}_t\overline{V}_x).$$

Theorem 66 (Thiele) *Given a policy basis for an assurance contract for a life aged x:*

- *a force of interest $\delta(t)$ at time $t \ge 0$,*
- *force of mortality μ_{x+t} at age $x+t$,*
- *a single benefit payment of $c(t)$ payable on death at time $t \ge 0$,*
- *continuous premium payments at rate $\pi(t)$ at times $t \ge 0$,*

the premiums are net premiums if

$$\int_0^\infty \pi(t)v(t)_t p_x dt = \int_0^\infty c(t)v(t)\mu_{x+t\ t}p_x dt, \quad \textit{where } v(t) = \exp\left\{-\int_0^t \delta(s)ds\right\}.$$

Furthermore, the prospective policy value $\overline{V}(t)$ *satisfies*

$$\overline{V}'(t) = \delta(t)\overline{V}(t) + \pi(t) - \mu_{x+t}(c(t) - \overline{V}(t)). \tag{1}$$

Proof: The first assertion follows straight from the principle of net premiums to be such that expected discounted premium payments equal expected discounted benefit payments; here, the former is

$$\mathbb{E}\left(\int_0^{T_x} \pi(t)v(t)dt\right) = \int_0^\infty \pi(t)v(t)\mathbb{E}(1_{\{T_x>t\}})dt = \int_0^\infty \pi(t)v(t)_t p_x dt,$$

where we used $\mathbb{E}(1_{\{T_x>t\}}) = \mathbb{P}(T_x > t) = \overline{F}_x(t) = {}_tp_x$, and the latter is

$$\mathbb{E}\left(c(T_x)v(T_x)\right) = \int_0^\infty c(t)v(t)f_x(t)dt = \int_0^\infty c(t)v(t)\mu_{x+t\ t}p_x dt,$$

where we used the probability density function $f_x(t) = \mu_{x+t\ t}p_x$ of T_x.

For the second assertion, we note that $\overline{F}_{x+t}(r) = \overline{F}_x(t+r)/\overline{F}_x(t)$ and differentiate

$$\overline{V}(t) = \frac{1}{v(t)\overline{F}_x(t)}\int_t^\infty c(s)v(s)\mu_{x+s}\,\overline{F}_x(s)ds - \frac{1}{v(t)\overline{F}_x(t)}\int_t^\infty \pi(s)v(s)\overline{F}_x(s)ds$$

using elementary differentiation

$$\frac{1}{v(t)} = \exp\left(\int_0^t \delta(s)ds\right) \quad \Rightarrow \quad \left(\frac{1}{v}\right)'(t) = \frac{\delta(t)}{v(t)},$$

and similarly $(1/\overline{F}_x)'(t) = \mu_{x+t}/\overline{F}_x(t)$, then applying the three-factor product rule $(fgh)' = (fg)'h + fgh' = f'gh + fg'h + fgh'$ and the Fundamental Theorem of Calculus, to get

$$\overline{V}'(t) = (\delta(t) + \mu_{x+t})\overline{V}(t) - c(t)\mu_{x+t} + \pi(t).$$

□

We can also interpret Thiele's differential equation 1, as follows, in the language of a large group of policies: the change in expected reserve $\overline{V}(t)$ for a given surviving policy holder at time t consists of an infinitesimal increase due to the force of interest $\delta(t)$ acting on the current reserve volume $\overline{V}(t)$, a premium inflow at rate $\pi(t)$ and an outflow due to the force of mortality μ_{x+t} acting on the shortfall $c(t) - \overline{V}(t)$ that must be met and is hence deducted from the reserve of all surviving policy holders including our given surviving policy holder. As in the model of discrete premium/benefit cash-flows, actual reserves per policy here will fluctuate around expected reserves discussed here because the mortality part will not act continuously, but discretely (at continuously distributed random times) when the policyholders in the group of policies die; similarly the reserve of our given policyholder will actually drop to zero at death, the shortfall for the benefit payment being recovered from the reserves of surviving policyholders in the group (but remember that $\overline{V}(t)$ is a conditional expectation given survival to t, so $\overline{V}(t)$ itself will never jump to zero in a model with continuously distributed lifetimes).

Lecture 15

Risk pooling

Reading: Gerber Sections 6.3

In this lecture we discuss the role of risk as the driver of all insurance.

15.1 Risk

What is risk? We attempt answers at two levels, real life and stochastic models. In real life, the following provide typical examples of risk and how insurance products respond to the endeavour of passing on risk to somebody else, typically to an insurance company.

- Risk of death. Particularly untimely death (beyond the emotional dimension) often cuts income streams that dependants of the deceased rely on. A life assurance provides a lump sum to replace any/some lost income in the case of premature death, against regular premium payments while alive. Such products are sold because people prefer regular premium payments (for no benefits in the likely case of survival) to serious financial problems of dependants (in the unlikely case of death).

before retirement	no death before age $x+n$	death before age $x+n$
no life assurance	no premium	serious financial problems
life assurance	small premium	no financial problems

- Risk of long life. Even the most basic retirement life costs money, any higher standard of living costs more. Given such cost, typical savings only last for a certain number of years. A pension pays a regular income for life.

after retirement	no death before age $x+n$	death before age $x+n$
savings	run out of money	leave behind some money
pension	regular pension income	regular pension income

- Risk of poor health. In some countries, health insurance is voluntary. Without insurance, every visit to a doctor/hospital is expensive, the cost over a lifetime highly unpredictable. With insurance, premium payments are regular and predictable.

	good health	poor health
without insurance	cheap	expensive/unaffordable
with insurance	regular premium	regular premium

We could similarly list home insurance, car insurance, personal liability insurance, disability insurance, valuable items insurance etc. The principle is usually the same. Most people prefer regular premium payments (for no benefits in the likely case of no claim) to "large" personal expenditure (in the unlikely case of a claim). But scale matters; should you insure your washing machine or even smaller items if you can easily afford to buy a new one when the old one fails? Since actual premiums exceed net premiums, you end up paying more premiums than you get benefits, not just on average, but also in the long term, by the Law of Large Numbers, which applies either for a large number of small items or for a large number of years. We can/must keep some (financial and other) risks. Note however that joint failure of several items due to burglary, fire, etc. is covered by a (useful) home/contents insurance .

It is instructive to consider stochastic models. We have seen stochastic models for lifetimes, based on lifetables estimated from actual mortality observations. We can similarly set up stochastic models for health expenditure, based on suitable observations of actual health expenditure, etc. In the end, we can specify, for each type of insurance, a random variable X of expenditure without insurance, and a random variable Y of expenditure with insurance. Often, Y is a deterministic premium payment or a premium payment plus a small excess payment when a claim is made. If it is deterministic, the risk of X has been completely eliminated for payment of Y. In general, $\mathbb{E}(Y) > \mathbb{E}(X)$, since the net premium principle ignores expenses etc., but Y has other preferable features, often the following

- $\mathbb{P}(Y > c) = 0$, while $\mathbb{P}(X > c) > 0$, for some threshold c where hardship starts;
- $\mathrm{Var}(Y) < \mathrm{Var}(X)$, or maybe more appropriately $\mathbb{E}((Y-\mathbb{E}(X))^2) \leq \mathbb{E}((X-\mathbb{E}(X))^2)$.

The term "hardship" needs defining. It certainly covers "bankruptcy", and this is why some insurances are compulsory (health, car, professional liability insurance for certain professions, personal liability, etc. – there are, however, differences between countries). More appropriate would be "loss of lifestyle".

So, risk is financial uncertainty. In a scenario-based analysis of the real world, we should try to pass on risk for which there are scenarios that cause financial hardship. In a stochastic model of the real world, we may decide to pass on risk for which there is positive probability of financial hardship.

Why would an insurer want to take on the risk? One incentive clearly is $\mathbb{E}(Y) > \mathbb{E}(X)$, but there is more.

15.2 Pooling

An individual threshold for financial hardship is smaller than an insurer's threshold, so if $\mathbb{P}(X > c_P) > 0$ was a problem for a policy holder, we can still have $\mathbb{P}(X > c_I) = 0$ for the insurer. But beyond this, the main reason lies in the original idea of insurance that many pay a little for the big needs of a few, particularly when it is not known at the time of payment (or agreement to pay) who the few will be. Let us now formalise this pooling effect.

An insurer takes on many risks $X_1, \ldots, X_n$, say. In the simplest case, these can be assumed to be independent and identically distributed. This is a realistic assumption for a portfolio of identical life assurance policies issued to a homogeneous population. This is not a reasonable assumption for a flood insurance, because if one property is flooded due to extreme weather that must be expected to also affect other properties.

The random total claim amount $S = X_1 + \cdots + X_n$ must be ensured by premium payments $A = Y_1 + \cdots + Y_n$, say, to be determined. Usually, the fair premium $\mathbb{E}(S)$ leaves too much risk to the insurer. E.g. the loss probabilities $\mathbb{P}(S > \mathbb{E}(S))$ is usually too high. The following result suggests to set (deterministic) premiums Y such that the probability of a loss for the insurer does not exceed $\varepsilon > 0$.

Proposition 67 *Given a random variable X_1 with mean μ and variance σ^2, representing the benefits from an insurance policy, we have*

$$\mathbb{P}\left(X_1 \geq \mu + \frac{\sigma}{\sqrt{\varepsilon}}\right) \leq \varepsilon,$$

and $A_1(\varepsilon) = \mu + \sigma/\sqrt{\varepsilon}$ is the premium to be charged to achieve a loss probability below ε.

Given independent and identically distributed $X_1, \ldots, X_n$ from n independent policies, we obtain

$$\mathbb{P}\left(\sum_{j=1}^{n} X_j \geq n\left(\mu + \frac{\sigma}{\sqrt{n\varepsilon}}\right)\right) \leq \varepsilon,$$

i.e. $A_n(\varepsilon) = \mu + \sigma/\sqrt{n\varepsilon}$ suffices if the risk of n policies is pooled.

Proof: The statements follow as consequences of Tchebychev's inequality:

$$\begin{aligned}\mathbb{P}\left(\sum_{j=1}^{n} X_j \geq n\left(\mu + \frac{\sigma}{\sqrt{n\varepsilon}}\right)\right) &\leq \mathbb{P}\left(\left|\frac{1}{n}\sum_{j=1}^{n} X_j - \mu\right| \geq \frac{\sigma}{\sqrt{n\varepsilon}}\right) \\ &\leq \frac{\mathrm{Var}\left(\frac{1}{n}\sum_{j=1}^{n} X_j\right)}{\left(\frac{\sigma}{\sqrt{n\varepsilon}}\right)^2} = \frac{\sigma^2}{n\frac{\sigma^2}{n\varepsilon}} = \varepsilon.\end{aligned}$$

□

The estimates used in this proposition are rather weak, and the premiums suggested require some modifications in practice, but adding a multiple of the standard deviation is one important method, also since often the variance, and hence the standard deviation, can be easily calculated. However, for large n, so-called safety loadings $A_n(\varepsilon) - \mu$ proportional to $n^{-1/2}$ are of the right order, e.g. for normally distributed risks, or in general by the Central Limit Theorem for large n, when

$$\mathbb{P}\left(\frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n\sigma^2}} \geq c\right) \approx \mathbb{P}(Z > c), \qquad \text{with } Z \text{ standard normally distributed.}$$

The important observation in these result is that the premiums $A_n(\varepsilon)$ decrease with n. This means, that the more policies an insurer can sell the smaller gets the (relative) risk, allowing him to reduce the premium. The proposition indicates this for identical policies, but in fact, this is a general rule about risks with sufficient independence and no unduly large risks.

15.3 Reserves for life assurances – mortality profit

Let us look more specifically at the risk of an insurer who has underwritten a portfolio of identical whole life assurances of independent lives aged x at the start, with annual net premium P and sum assured S. Remember that the reserve $V(k)$ for each policy satisfies

$$(V(k) + P)(1 + i) = q_{x+k}S + p_{x+k}V(k + 1) = V(k + 1) + q_{x+k}(S - V(k + 1)).$$

Actuaries consider

- the *death strain* $\Delta_k = 0$ if the life survives, $\Delta_k = S - V(k+1)$ otherwise; the *death strain at risk* or *net amount at risk* is $S - V(k + 1)$;
- the *expected death strain* $\mathbb{E}(\Delta_k) = q_{x+k}(S - V(k+1))$, or *risk premium* $v\mathbb{E}(\Delta_k)$; the remaining premium is called the *savings premium* $P_s = P - P_r = vV(k+1) - V(k)$;
- the *mortality profit* $\mathbb{E}(\Delta_k) - \Delta_k$.

Note that the mortality profit is positive if the life survives, and then just accumulated from the part P_r of the premium P which is not needed to save for $V(k+1)$ – it is hence a profit for the insurer on that policy in that year. On the other hand, the mortality "profit" is negative if the life dies. Specifically, note that the payment of the sum assured

$$S = (V(k) + P)(1 + i) + (\Delta_k - \mathbb{E}(\Delta_k))$$

is in two parts, first the funds held from that policy and then topped up to S by the amount $\Delta_k - \mathbb{E}(\Delta_k)$ causing the insurer's loss on that policy in that year.

Suppose that there are n survivors to age $x + k$ and N survivors to age $x + k + 1$. Then we have

- a *total death strain at risk* of $n(S - V(k + 1))$,
- a total expected death strain at risk of $nq_{x+k}(S - V(k + 1))$,
- a total actual death strain at risk of $(n - N)(S - V(k + 1))$,
- a total (net) mortality profit of $M = (nq_{x+k} - (n - N))(S - V(k + 1))$.

From the mortality profit discussion of individual policies, we see how the premium payments of survivors are now naturally used to help top up to the sum assured the benefits payable to the deceased, if we offset profits and losses within the portfolio.

If the insurer charges a premium $Y > P$ exceeding the net premium P, this increases the mortality profit by $(Y - P)(1 + i)$. We can relate the insurer's loss probability ε and premium Y using the Central Limit Theorem for the binomially distributed random variable $n - N$, as follows:

$$\begin{aligned}
\varepsilon &\geq \mathbb{P}(M + n(Y - P)(1 + i) \leq 0) \\
&= \mathbb{P}\left(\frac{(n - N) - nq_{x+k}}{\sqrt{nq_{x+k}(1 - q_{x+k})}} \geq \frac{n(Y - P)(1 + i)}{(S - V(k + 1))\sqrt{nq_{x+k}(1 - q_{x+k})}}\right) \\
&\approx \mathbb{P}\left(Z \geq \frac{n(Y - P)(1 + i)}{(S - V(k + 1))\sqrt{nq_{x+k}(1 - q_{x+k})}}\right)
\end{aligned}$$

where Z is a standard normally distributed random variable. For $\varepsilon = 1\%$, this gives

$$Y - P \geq 2.33\, v\, (S - V(k + 1))\sqrt{q_{x+k}(1 - q_{x+k})/n}.$$

Lecture 16

Pricing risks – premium principles

Reading: Gerber Section 5.2
Further reading: Gerber "An Introduction to Mathematical Risk Theory"
Huebner Foundation 1979

In this lecture we collect from previous lectures several ways to incorporate risk considerations into premium calculations and embed these into a more general framework.

16.1 Safety loading

In the sequel, we think of random variables S as modelling risks, e.g. the total claim amount from an insurance policy or a portfolio of policies.

Definition 68 A *premium principle* is a rule H that assigns with every random variable S a real number or ∞, denoted $H(S)$. We call $H(S)-\mathbb{E}(S)$ *safety loading*, often expressed as a percentage of $\mathbb{E}(S)$. If $H(S)=\infty$, the risk S is called *uninsurable*.

We have already come across several natural ways of setting premiums, starting from and adding to net premiums. Here is a list. We will study some of their properties.

Example 69 (Net premium principle) The fair premium from Definition 4, in the context of other premiums called the *net premium* is the case $H=\mathbb{E}$, i.e. $H(S)=\mathbb{E}(S)$.

Example 70 (Expected value principle) $H(S)=(1+\lambda)\mathbb{E}(S)$ suggests a safety loading proportional to $\mathbb{E}(S)$.

Example 71 (Variance principle) $H(S)=\mathbb{E}(S)+\alpha\mathrm{Var}(S)$ suggests a safety loading proportional to $\mathrm{Var}(S)$.

Example 72 (Standard deviation principle) $H(S)=\mathbb{E}(S)+\beta\sqrt{\mathrm{Var}(S)}$ suggests a safety loading proportional to the standard deviation of S as in Proposition 67.

Example 73 (Percentile principle) $H(S)=\min\{A:\mathbb{P}(S>A)\le\varepsilon\}$ suggests the minimal premium that bounds the probability of loss by a given level. In fact, Proposition 67 relates this Percentile principle to the Standard deviation principle.

Example 74 (Exponential principle) $H(S)=\log(\mathbb{E}(\exp\{aS\}))/a$.

Note that a simple Taylor expansion of the moment generating function and the log series gives

$$\frac{\log(\mathbb{E}(\exp\{aS\}))}{a} \approx \mathbb{E}(S) + \frac{a}{2}\mathrm{Var}(S),$$

if S is approximately normally distributed, with equality for normal S. Hence, the Exponential principle coincides with the Variance principle for normal risks, but if higher moments exceed the normal higher moments (an indication of heavy tails, larger probabilities of extreme events), there is an important further risk loading in the Exponential premium principle leading to higher premiums than the Variance principle for such risks.

There are other and more general principles using utility functions that we do not discuss here. The Exponential principle plays an important role. All principles have advantages and disadvantages.

16.2 Desirable properties

Some desirable properties that we would like a premium principle H to have

1. Nonnegative safety loading: $H(S) \geq \mathbb{E}(S)$ for all S.
2. No ripoff: $\mathbb{P}(H(S) \geq S) < 1$.
3. Consistency: $H(S+c) = H(S) + c$.
4. Additivity: If S_1 and S_2 are *independent*, then $H(S_1 + S_2) = H(S_1) + H(S_2)$.

One can then establish the following table

Property	Ex. 69	Ex. 70	Ex. 71	Ex. 72	Ex. 73	Ex. 74
1.	yes	no	yes	yes	no	yes
2.	yes	no	no	no	yes	yes
3.	yes	no	yes	yes	yes	yes
4.	yes	yes	yes	no	no	yes

Under certain assumptions, it can be shown that not only among these examples, but much more generally, only the net premium principle and the exponential principle have all these properties. Since the net premium principle does not take account of any risk, this makes a strong case for the exponential principle.

16.3 An example for the exponential principle

Example 75 Consider a 10-year temporary assurance for a life aged 40, with sum assured C. Consider a constant interest rate of $i = 4\%$ and a residual lifetime distribution that is uniform with terminal age 100.

- We calculate

$$A^{1}_{40:\overline{10}|} = \frac{1}{60}v + \frac{1}{60}v^2 + \cdots + \frac{1}{60}v^{10} = \frac{1}{60}a_{\overline{10}|} = 0.1352$$

  to find the net single premium $0.1352C$.

- For the net annual premium, consider pure endowment and whole-life assurance

$$A_{40:\overline{10|}}^{\ \ \ 1} = \frac{50}{60}v^{10} = 0.5630 \quad \text{and} \quad A_{40:\overline{10|}} = A^{1}_{40:\overline{10|}} + A_{40:\overline{10|}}^{\ \ \ 1} = 0.6982.$$

The latter is useful to apply the premium conversion relation

$$\ddot{a}_{40:\overline{10|}} = \frac{1 - A_{40:\overline{10|}}}{d} = 7.8476$$

to get the net annual premium for the temporary life assurance

$$P_0 = C\frac{A^{1}_{40:\overline{10|}}}{\ddot{a}_{40:\overline{10|}}} = 0.0172C.$$

- The single (gross) premium under the Exponential principle with $a = 10^{-6}$ depends non-linearly on the sum assured C. The risk is $S = Cv^{K_{40}+1}1_{\{K_{40}<10\}}$, so

$$\begin{aligned} H(S) &= \frac{\log(\mathbb{E}(\exp\{aS\}))}{a} = \frac{\log(\mathbb{E}(\exp\{aCv^{K_{40}+1}1_{\{K_{40}<10\}}\}))}{a} \\ &= \frac{1}{a}\log\left(\frac{1}{60}\exp\{aCv\} + \frac{1}{60}\exp\{aCv^2\} + \cdots + \frac{1}{60}\exp\{aCv^{10}\} + \frac{50}{60}\right). \end{aligned}$$

We can calculate the following table.

Sum assured	100,000	300,000	1,000,000	3,000,000	10,000,000
Net premium	13,518	40,554	135,182	405,545	1,351,816
Gross premium	13,991	44,968	190,590	1,031,584	6,719,547
Safety loading	3%	11%	41%	154%	397%

Loosely speaking, $1/a$ is a typical sum assured. It appears as factor of S and $H(S)$.

- To turn the single gross premium into annual gross premiums, we should not just spread $H(S)$ over a temporary life annuity – this reasoning is based on expected values as in the Net premium principle. In fact, there is also risk associated with annual premium payments. We can use Property 3., or directly reformulate the Exponential premium principle for single premiums as

$$\begin{aligned} P_s = H(S) = \frac{\log(\mathbb{E}(\exp\{aS\}))}{a} \quad &\Longleftrightarrow \quad \mathbb{E}(e^{aS}) = e^{aP_s} \Longleftrightarrow \mathbb{E}(e^{a(S-P_s)}) = 1 \\ &\Longleftrightarrow \quad 0 = \frac{\log(\mathbb{E}(\exp\{a(S-P_s)\}))}{a}, \end{aligned}$$

where the risk $S - P_s$ is premium-free. Similarly, we now look for an annual premium P_a such that the associated risk $S - P_a\ddot{a}_{\overline{\min(K_{40}+1,10)|}}$ is premium-free, i.e. we solve

$$\begin{aligned} 1 &= \mathbb{E}\left(\exp\left\{a\left(Cv^{K_{40}+1}1_{\{K_{40}<10\}} - P_a\ddot{a}_{\overline{\min(K_{40}+1,10)|}}\right)\right\}\right) \\ &= \frac{1}{60}e^{a(Cv-P_a)} + \frac{1}{60}e^{a(Cv^2-P_a(1+v))} + \cdots + \frac{1}{60}e^{a(Cv^{10}-P_a(1+v+\cdots+v^9))} + \frac{50}{60}e^{-P_a(1+\cdots+v^9)} \end{aligned}$$

numerically to obtain the following table.

Sum assured	100,000	300,000	1,000,000	3,000,000	10,000,000
Net annual premium	1,722	5,167	17,225	51,675	172,248
Gross annual premium	1,793	5,843	26,450	221,887	5,524,456
Safety loading	4%	13%	54%	329%	3,107%

16.4 Reinsurance

In Example 75, the annual premiums become quite low for small sums assured, and an insurer will add a significant expense loading and profit margin. More importantly, the annual premiums become very large for large sums assured. This makes sense, because a large sum assured in a portfolio of smaller sums assured presents an unduly large risk. Any risk reduction due to pooling breaks down (cf. Proposition 67 and the discussion thereafter). It should either not be insured or the insurer should seek *reinsurance* for this risk (whatever the premium charged).

There are several types of reinsurance. The simplest is proportional reinsurance. An insurer passes on a fixed percentage of the risk to a reinsurance company.

Example 76 (Proportional reinsurance) If an insurer passes on 50% of each risk to a reinsurance company, for a corresponding share of the premium, the insurer's risk is reduced by a factor of 2. Under the Exponential premium principle H^a with typical sum assured $1/a$, this means that a risk S (e.g. sum assured of C in case of death) is split in halves and the premium charged is

$$H^a_{50\%-\mathrm{re}}(S) = 2H^a(S/2) = H^{2a}(S) \leq H^a(S).$$

The effect is that the insurer can offer a higher typical sum assured which leads to lower premiums overall (by Jensen's inequality). The risk passed on to the reinsurer is identical to the risk of the insurer.

Another type of reinsurance is excess of loss reinsurance. An insurer can effectively cap their loss per policy (or their total loss, depending on the reinsurance contract) for a premium payment to a reinsurer. This means that the risk of a large loss is passed on to a reinsurer. This type of reinsurance only works if there is a sufficient number of sufficiently independent risks of similar scale for the pooling effect to take place for the reinsurer's portfolio of large risks.

Example 77 (Individual excess of loss reinsurance) Fix $1/a$ as the typical sum assured in each reasonably homogeneous portfolio of risks, and to apply the Exponential premium principle with this value of a. If a sum assured exceeds $1/a$ by more than a factor of 2 or 3, the safety loading becomes unattractive. Rather than charging the unattractive premium and deterring the customer, an insurer can insure a capped sum assured as part of the existing portfolio and seek to reinsure the excess at a lower premium:

$$P_{\mathrm{tot}} = P_{\mathrm{in}} + P_{\mathrm{re}}, \quad \text{insurer: } P_{\mathrm{in}} = H^a(\max\{S, 3/a\}), \quad \text{reinsurer: } P_{\mathrm{re}} = H^{a_0}((S - 3/a)^+),$$

where this is often only attractive to reinsurance companies that collect large risks; for a portfolio of large risks, a large $1/a_0 >> 1/a$ leads to affordable premiums. The customer is then charged a combined premium P_{tot}, consisting of P_{in} for the capped risk and P_{re} for the excess.

Appendix A

Assignments

Assignment sheets are issued on Mondays of weeks 2-7. They are made available on the website of the course at

http://www.stats.ox.ac.uk/~winkel/bs4b.html.

For undergraduates, seven sets of six one-hour classes take place in room 104 of 1 South Parks Road (Seminar Room, Department of Statistics) in weeks 3 to 8, on Wednesdays and Thursdays, as follows:

- Wednesdays 9.45-10.45am
- Wednesdays 10.45-11.45am
- Wednesdays 2.30-3.30pm
- Wednesdays 3.30-4.30pm
- Wednesdays 4.30-5.30pm
- Thursdays 10.00-11.00am
- Thursdays 11.00am-12.00pm

The class allocation can be accessed from the course website shortly after the first lecture, as soon as I have evaluated your requests. Only undergraduates can sign up for classes. For students on the M.Sc. in Applied Statistics, there will be two classes at times to be announced. All others should talk to me after one of the first two lectures.

Scripts are to be handed in at the Department of Statistics, 1 South Parks Road.

Exercises on the problem sheets vary in style and difficulty. If you find an exercise difficult, please do not deduce that you cannot solve the following exercises, but aim at giving each exercise a serious try. **Solutions will be provided on the course website.**

There are lecture notes available. Please print these so that we can make best use of the lecture time. Please let me know of any typos or other comments that you may have, e.g. by sending an email to winkel@stats.ox.ac.uk.

Below are some comments on the recommended Reading and Further Reading literature.

All publications below are available from the

Publications Unit
Institute of Actuaries
4 Worcester Street
Oxford OX1 2AW.

between Worcester College and Gloucester Green.

Faculty & Institute of Actuaries: CT1 Financial mathematics 2004-2011

This is the Core Reading for the Professional Examination corresponding to BS4/OBS4. About two thirds of the BS4/OBS4 material can be found here, mostly Michaelmas Term material, though. In some places, the presentation is more practically oriented than the lectures.

McCutcheon and Scott: An Introduction to the Mathematics of Finance 1986

This is *the* book on the subject, but again mostly on the Michaelmas Term material. The Core Reading above is still largely based on this book. Although the book is getting old, while the Core Reading is kept up to date, I recommend it as it is a more detailed account of these two thirds of this unit. Also, the presentation is more mathematical. There are many exercises with solutions. This book is also available from the Institute of Actuaries.

Zima and Brown: Mathematics of Finance 1993

This book has been written for Canadian (and US) actuarial students. The approach is very elementary with many worked-out examples, practical methods and exercises, but possibly the right source to find clarification in some fundamental points of the course. It also covers some of the life insurance material, which will occupy the second half of this term.

Gerber: Life Insurance Mathematics 1990

Only the introductory chapter (14 pages) concern the 'two thirds' of the BS4/OBS4 course, the essence in a nutshell, but don't be tempted to exclusively rely on this as it does not go into details of applications. The focus of this book is on life insurance, and it is our main reference for the second half of this term. The approach is mathematical.

Bowers et al.: Actuarial mathematics 1997

This compendium takes a different approach, in a sense starting with the 'missing third' and mentioning our 'two thirds' only in passing, but addressing much more, for instance utility and risk theory, models for pensions and general insurance. I recommend this only for further reading beyond the scope of this unit, since I find it difficult to extract those parts relevant for us.

A.1 Random cash-flows and stochastic interest rates

Please hand in scripts by Monday 31 January 2011, 11.15am, Department of Statistics (or by Tuesday 1 February 2011, 11.15am, if your class is on Thursday).
For some of the exercises on this sheet, you will need values of the standard normal distribution function. If $Z \sim \mathrm{N}(0,1)$*, then:*

z	-3	-2.3263	-2.4778	-1.8349	-0.039	0.233
$\mathbb{P}(Z \le z)$	0.00135	0.01000	0.00661	0.03326	0.48445	0.59211

1. Mr Strauss dies and leaves an estate valued at £50,000 in a bank account earning interest at rate $i^{(12)} = 9\%$. She has three children: Jim, aged 7, Fred, aged 5, and Sandra, just turned 4. The estate will be divided among the surviving children 14 years from now, on Sandra's 18th birthday. Find the expected value of the inheritance for each child given the following, and assuming independence.

Age today	Probability of Survival for 14 years
7	0.95
5	0.97
4	0.98

2. A £1000 20-year bond has coupons at $j^{(2)} = 12\%$ and is redeemable at par. Find the purchase price which provides an expected annual yield of $i^{(2)} = 14\%$, under the assumption of a semi-annual default probability of 2%. After default, no more payments take place. (You may find it convenient to work with 6 months as the basic time unit).

3. Show that for an interest-rate model $\delta(\cdot)$, a continuous cash-flow $c = (c(t), t \ge 0)$ and a random variable T with hazard rate $\mu(\cdot)$, we have

$$\mathbb{E}(\delta\text{-Val}_0(c_{[0,T)})) = \widetilde{\delta}\text{-Val}_0(c),$$

where $\widetilde{\delta}(\cdot) = \delta(\cdot) + \mu(\cdot)$.

4. (a) Show that for all random cash flows C_1 and C_2 with the same expected yield, $C_1 + C_2$ has also the same expected yield, provided that the expected yield of $C_1 + C_2$ exists.

   (b) Give an example of two random cash-flows C_1 and C_2 with random yields Y_1 and Y_2, and with $\mathbb{E}(Y_1) = \mathbb{E}(Y_2)$, for which the random yield of $C_1 + C_2$ exists and has a different expected value.

5. Suppose that the interest rate for the next six months is known to be 5.5% (effective rate per annum), while the rate for the six months after that is unknown and assumed to be uniformly distributed on the interval $(4\%, 6\%)$. Under this assumption, find the expectations of:

   (a) the accumulated value after one year of £100 invested now;

   (b) the discounted present value of a payment of £100 in a year's time.

6. Let I_j denote the effective rate of interest in the year $j-1$ to j. Suppose that, for $j \geq 1$

$$I_{j+1} = \begin{cases} I_j + 0.02 & & 0.25 \\ I_j & \text{with probability} & 0.5 \\ I_j - 0.02 & & 0.25 \end{cases}$$

Given that $I_1 \equiv 0.06$, calculate the probability that an investment of 1 at time 0 accumulates to more than 1.2 at time 3.

7. Let $1+I$ be a lognormal random variable with parameters μ and σ^2, mean $1+j$ and variance s^2. Show that

$$\sigma^2 = \log\left(1 + \left(\frac{s}{1+j}\right)^2\right) \qquad \text{and} \qquad \mu = \log\left(\frac{1+j}{\sqrt{1+\left(\frac{s}{1+j}\right)^2}}\right)$$

8. The rate of return on an investment in a given year is denoted by Y. Suppose $1+Y$ is lognormally distributed. The expected value of the rate of return is 5% and its standard deviation is 11%.

   (a) Calculate the parameters of the lognormal distribution of $1+Y$.

   (b) Calculate the probability that the rate of return for the year lies between 4% and 7%.

9. A company is adopting a particular investment strategy such that the expected annual effective rate of return from investments is 7% and the standard deviation of annual returns is 9%. Annual returns are independent and $(1+I_j)$ is lognormally distributed, where I_j is the return in the jth year.

   (a) Calculate the expected value and standard deviation of an investment of 1 unit over 10 years, deriving all formulae that you use.

   (b) Calculate the probability that the accumulation of such an investment will be less than 50% of its expected value in ten years' time.

   (c) The company has an outstanding debt and must make a payment of £140,000 in 10 years time. Calculate the probability that an investment of £120,000 now will provide sufficient funds to meet this liability.

10. Suppose the force of interest Δ_j during the year from $j-1$ to j is given by

$$\Delta_j = \mu + \frac{1}{\sqrt{2}}\epsilon_{j-1} + \frac{1}{\sqrt{2}}\epsilon_j,$$

where $\epsilon_0, \epsilon_1, \epsilon_2, \ldots$ are i.i.d. random variables with distribution $\mathrm{N}(0,\sigma^2)$.

   (a) Show that $\Delta_j \sim \mathrm{N}(\mu,\sigma^2)$ for all j.

   (b) Write an expression for $\Delta_1 + \Delta_2 + \cdots + \Delta_n$ in terms of the random variables ϵ_j. Hence show that the accumulated value at time n of 1 unit invested at time 0 has a lognormal distribution, and find its parameters.

Course website: http://www.stats.ox.ac.uk/~winkel/bs4b.html

A.2 No-arbitrage pricing

Please hand in scripts by Monday 7 February 2011, 11.15am, Department of Statistics (or by Tuesday 8 February 2011, 11.15am, if your class is on Thursday).

1. (a) Suppose that there exists a risk-free asset with constant force of interest r. Using the no arbitrage assumption, show from first principles that the forward price F, agreed at time 0 and to be paid at time T, for an asset S, with no income and with value S_0 at time 0, is given by $F = S_0 e^{rT}$. Assume that there are no transaction costs.

   (b) Extend the argument in (a) to derive the forward price where the asset provides a fixed known (cash-flow of) income.

   (c) In the setting of (a) and (b), calculate the value V_t of the contract to buy asset S at time T at intermediate times $t \in (0, T)$.

2. The forward rate from time t to time $t + 1$, $f_{t,1}$, has the following values

$$f_{0,1} = 4.0\%, \quad f_{1,1} = 4.5\%, \quad f_{2,1} = 4.8\%.$$

   (a) Assuming no arbitrage, calculate

      i. the price per £100 nominal of a 3-year bond paying an annual coupon in arrears of 5%, redeemed at par in exactly three years, and

      ii. the gross redemption yield from the bond.

   (b) Explain why a bond with a higher coupon would have a lower gross redemption yield, for the same term to redemption.

3. An asset has a current price of $100p$. It will pay an income of $5p$ in 20 days' time. Given a risk-free rate of interest of 6% per annum convertible half-yearly and assuming no arbitrage, calculate the forward price to be paid in 40 days.

4. $f_{t,r}$ is the forward rate applicable over the period t to $t + r$. i_t is the spot rate over the period 0 to t. The gross redemption yield from a one-year bond with a 6% annual coupon is 6% per annum effective; the gross redemption yield from a two-year bond with a 6% annual coupon is 6.3%; and the gross redemption yield from a three year bond with a 6% annual coupon is 6.6% per annum effective. All the bonds are redeemed at par and are exactly one year from the next coupon payment.

   (a) i. Calculate i_1, i_2 and i_3 assuming no arbitrage.

      ii. Calculate $f_{0,1}$, $f_{1,1}$ and $f_{2,1}$ assuming no arbitrage.

   (b) Explain why the forward rates increase more rapidly with term than the spot rates.

5. The current price of a security is $S_0 = £2$. In 6 months' time the security will pay a dividend equal to $10\text{p} + 0.04 S_{\frac{1}{2}}$ (where $S_{\frac{1}{2}}$ is the price of the security in 6 months' time). A riskless asset is available with constant interest rate 5% pa. What forward price should be agreed now to buy the security in 9 months' time?

6. There are two securities S^1 and S^2 in a market. The values of both securities at $t = 0$ is 1. The value of S^1 at $t = 1$ is $1 + i$. The value of S^2 at $t = 1$ is either u or d, each with positive probability. Assume $0 < d < u$. Show that an arbitrage opportunity exists in this model if and only if either $1 + i \geq u$ or $1 + i \leq d$.

7. Consider an idealised one-period securities market model consisting of two assets. At the end of the period, the market will be either "up" or "down". The first asset pays £5 if the market is up and £1.25 if the market is down. The second asset pays £1.25 if the market is up and £5 if the market is down. At $t = 0$, the first asset sells for £2.75 and the second for £1.90. An investor would like to receive £1.25 if the market is up and nothing if the market is down. Find portolio holdings of the two assets which have the payoff the investor would like. What is the current price of such a portfolio?

8. Consider a two-asset model (A, S) with three states of the world (d, s, u) where

state of the world	A_0	A_1	S_0	S_1
u	1	1	1	2
s	1	1	1	1
d	1	1	1	0

(a) Show that this model is arbitrage-free.

(b) Use no-arbitrage arguments to price the contingent claim that pays $P(u) = 3$, $P(s) = 2$ and $P(d) = 1$, respectively.

(c) Consider the European call option C to buy the risky asset S at time 1 at a strike price 1.

(i) Show that C cannot be uniquely priced using no-arbitrage arguments.

(ii) Show that each of the suggested prices $C_0 = 1$, $C_0 = 0.5$ and $C_0 = 0$ would lead to arbitrage in the three-asset model (A, S, C).

(iii) Show that none of the suggested prices $C_0 = \alpha \in (0, 0.5)$ leads to arbitrage in the three-asset model (A, S, C), and show that every contingent claim Y can be hedged in this model.

A.3 Duration, convexity and immunisation

Please hand in scripts by Monday 14 February 2011, 11.15am, Department of Statistics (or by Tuesday 15 February 2011, 11.15am, if your class is on Thursday).

1. Recall from Question 8 of Assignment 2 the three-asset model

state of the world	A_0	S_0	C_0	A_1	S_1	C_1
u	1	1	α	1	2	1
s	1	1	α	1	1	0
d	1	1	α	1	0	0

for $\alpha \in \mathbb{R}$.

(a) Let $\alpha \in \mathbb{R}$. Show that every contingent claim can be hedged. Comment.

(b) Let $\alpha \in \mathbb{R}$. Try to calculate risk-neutral probabilities. How is the no-arbitrage condition $\alpha \in (0, 0.5)$ reflected in your risk-neutral probabilities?

(c) Let $\alpha \in (0, 0.5)$. Calculate the arbitrage-free price of a European call option D to buy the risky asset at time 1 at strike price $K \in (0, 2)$. For each $K \in (0, 2)$, deduce the range of arbitrage-free prices for D in the two-asset model (A, S).

2. An investment project involves payments of £1,000,000 now and £500,000 two years from now. The proceeds from the project are £2,000,000, payable in exactly four years' time.

Calculate the volatility and the convexity of this project, as a function of the underlying interest rate i (assumed constant).

Calculate the yield and compare volatility and convexity at $i = 4\%$ and $i = 8\%$.

3. Consider a portfolio of n fixed income bonds (any positive cash flows) with present values $A_1, \ldots, A_n$. For $j = 1, \ldots, n$, let the discounted mean term and the convexity of the jth bond be denoted by τ_j and c_j. Let $p_j = A_j/A$ be the proportion of the jth security in the portfolio, $A = A_1 + \cdots + A_n$.

Show that the discounted mean term and the convexity of the portfolio are

$$\sum_{j=1}^{n} p_j \tau_j \qquad \text{and} \qquad \sum_{j=1}^{n} p_j c_j.$$

4. A fund has to provide an annuity of £50,000 p.a. payable yearly in arrears for the next 9 years followed by a final payment of £625,000 in 10 years' time.

The fund has earmarked cash assets equal to the present value of the payments and the fund manager wants to invest these in two zero coupon bonds, A, repayable after 5 years, and B, repayable after 20 years.

How much should the manager invest in A and B to have the same volatility in the assets and liabilities, assuming an effective rate of interest of 7% p.a.?

5. A fund will need to make payments of £10,000 at the end of each of the next five years. It wishes to immunise using two zero coupon bonds, one maturing in 5 years and one in 1 year. The rate of interest is 5% p.a.

   (a) Calculate the present value of the liabilities.

   (b) Find the discounted mean term.

   (c) Calculate the nominal amounts of the zero-coupon bonds needed to equate the present value and duration of assets and liabilities.

   (d) Calculate the convexity of the assets, and comment on immunisation (Redington).

6. A government bond pays a coupon half-yearly in arrears of 10% per annum. It is to be redeemed at par in exactly ten years. The gross redemption yield from the bond is 6% per annum convertible half-yearly. Calculate the discounted mean term of the bond in years.

   Explain why the duration of the bond would be longer if the coupon rate were 8% per annum instead of 10% per annum.

7. Redington's immunisation assumes a flat yield curve that may shift up or down over time. Show that such yield curves are inconsistent with an assumption of no arbitrage. You may assume that at any future time instant the flat yield is different to the initial flat yield with some positive probability.

   [Hint: Consider a flat yield curve with flat yield at time 0 of i_0 *and at time 1 of* i_1. *Consider two portfolios A and B at times 0 and 1. Portfolio A comprises one unit of zero coupon bond with term 1, and* $(1+i_0)^2$ *units of zero coupon bond with term 3. Portfolio B comprises* $2\times(1+i_0)$ *units of zero coupon bond with term 2.]*

A.4 Life tables and force of mortality

Please hand in scripts by Monday 21 February 2011, 11.15am, Department of Statistics (or by Tuesday 22 February 2011, 11.15am, if your class is on Thursday).

1. Let μ_x be the force of mortality, and ℓ_x the corresponding life table.

   (i) Show that ${}_{n|m}q_x = \int_n^{n+m} {}_tp_x \mu_{x+t} dt$
   [*Here* ${}_{n|m}q_x$ *means the probability of death of a life currently aged* x *between times* $x+n$ *and* $x+n+m$.]

   (ii) Show that $\mu_{x+0.5} \approx -\log p_x$ and $\mu_x \approx -0.5\,(\log p_x + \log p_{x-1})$.

   (iii) If $\ell_x = 100(100-x)^{1/2}$, find μ_{84} exactly.

2. (i) Gompertz's Law has $\mu_x^{(1)} = Bc^x$. Show that the corresponding survival function is given by ${}_tp_x^{(1)} = g^{c^x(c^t-1)}$ where $\log g = -B/\log c$.

   (ii) Makeham's Law has $\mu_x^{(2)} = A + Bc^x$. Show that ${}_tp_x^{(2)} = s^t {}_tp_x^{(1)}$ where $s = e^{-A}$.

   (iii) If $\mu_x = A \log x$, find an expression for ℓ_x/ℓ_0.

   (iv) If it is assumed that A1967-70 table follows Makeham's Law, use ℓ_{30}, ℓ_{40}, ℓ_{50} and ℓ_{60} to find A, B and c.
   ($\ell_{30} = 33839$, $\ell_{40} = 33542$, and $\ell_{50} = 32670$, $\ell_{60} = 30040$).

3. The force of mortality for table 2 has twice the force of mortality for table 1.

   (i) Show that the probability of survival for n years under table 2 is the square of that under table 1.

   (ii) Suppose that table 1 follows the Gompertz Law from the previous question. Show that the probability of survival for n years for a life aged x under table 2 is the same as that under table 1 for a life aged $x+a$, for some $a > 0$. Find a. Comment on the result.

4. (i) Show algebraically that $A_{x:\overline{n}|} = v\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n-1}|}$. Also demonstrate the result verbally.

   (ii) Show algebraically that $(IA)_x = \ddot{a}_x - d(I\ddot{a})_x$. Demonstrate the result verbally.
   [*Here* $(IA)_x$ *is the expected time-0 value (assuming an age of* x *at time 0) of a single payment of size* $K+1$ *made at time* $K+1$*, where death occurs in the year* $(K, K+1)$. $(I\ddot{a})_x$ *is the expected value of the stream of payments of 1 at time 0, 2 at time 1, ...,* $K+1$ *at time* K*, with the last payment made at the beginning of the year of death.*]

5. (a) Show that at age x if $0 \le a < b \le 1$ then ${}_{b-a}q_{x+a} = 1 - \frac{{}_bp_x}{{}_ap_x}$.

   (b) Hence, or otherwise, show that if deaths occurring in the year of age $(x, x+1)$ are uniformly distributed that year, then ${}_{b-a}q_{x+a} = \frac{(b-a)q_x}{1-aq_x}$.

6. Suppose that the future lifetime of a life aged x ($x > 0$) is represented by a random variable T_x distributed on the interval $(0, \omega - x)$, where ω is some maximum age.

   (i) For $0 \leq x < y < \omega$, state a consistency condition between the distributions of T_x and of T_y.

   (ii) Suppose that $0 \leq x < y < \omega$, and let $t = y - x$. Define the force of mortality at age y: (a) in terms of T_0; and (b) in terms of T_x, and show that the two definitions are equivalent.

   (iii) Prove that ${}_tp_x = \exp\left\{ -\int_0^t \mu_{x+s} ds \right\}$.

A.5 Standard life insurance products

Please hand in scripts by Monday 28 February 2011, 11.15am, Department of Statistics (or by Tuesday 1 March 2011, 11.15am, if your class is on Thursday).

1. Given a curtate lifetime $K = [T]$ and a constant-δ interest model, consider the following insurance products:

   (i) pure endowment with term n;

   (ii) whole life assurance;

   (iii) term assurance with term n;

   (iv) endowment assurance with term n (under this product, there is a payment of one unit at the end of the year of death or at end of term whichever is earlier).

   (a) Write each one as a random cash-flow of the form $C = \big((t_k, c_k B_k)\big)_{k=1,2,\ldots}$, where the B_k are Bernoulli random variables defined in terms of K.

   (b) Find expressions for the net premiums and variances of these products.

   (c) Relate the products, premiums and variances of (iv) and (i) and (iii).

   (d) Comment on (a) and (b) for the corresponding products where payment is made at death rather than at the end of the year of death (and n is not necessarily an integer).

2. A cash-flow is payable continuously at a rate of $\rho(t)$ per annum at time t provided a life who is aged x at time 0 is still alive. T_x is a random variable which models the residual lifetime in years of a life aged x.

   (a) Write down an expression, in terms of T_x, for the (random) present value at time 0 of this cash-flow, at a constant force of interest δ p.a., and show that the expected present value at time 0 of the cash-flow is equal to

   $$\int_0^\infty e^{-\delta s}\rho(s)\mathbb{P}(T_x > s)ds.$$

   (b) An annuity is payable continuously during the lifetime of a life now aged 30, but for at most 10 years. The rate of payment at all times t during the first 5 years is £5,000 p.a., and thereafter £10,000 p.a. The force of mortality to which this life is subject is assumed to be 0.01 p.a. at all ages between 30 and 35, and 0.02 p.a. between 35 and 40. Find the expected present value of this annuity at a force of interest of 0.05 p.a.

   (c) If the mortality and interest assumptions are as in (b), find the expected present value of the benefits of a term assurance, issued to the life in (ii), which pays £40,000 immediately on death within 10 years.

For the questions below, use the A 1967-70 Mortality table whenever table data is needed (all three columns, assuming medical checks were successful).

3. Find the present value of £5000 due in 5 years' time at $i = 4\%$ if

   a) the payment is certain to be made;

   b) the payment is contingent upon a life aged 35 now surviving to age 40.

4. How large a pure endowment, payable at age 65 can a life aged 60 buy with £1000 cash if $i = 7\%$?

5. a) Show that $\ddot{a}_x = 1 + (1+i)^{-1} p_x \ddot{a}_{x+1}$

   b) Show that $\ddot{a}_{x:\overline{n}|} = a_{x:\overline{n}|} + 1 - A_{x:\overline{n}|}^{\;\;1}$

   c) Show that $A_x = (1+i)^{-1}\ddot{a}_x - a_x$

6. Find the net single premium for a 5-year temporary life annuity issued to a life aged 65 if $i = 8\%$ for the first 3 years and $i = 6\%$ for the next 2 years.

7. Find the net annual premium for a £40000, 4-year term assurance policy issued to a life aged 26 if $i = 10\%$.

8. A deferred (temporary) life annuity is a deferred perpetuity (annuity-certain) restricted to a lifetime. For a life aged x, deferred period of m years (and a term of n years, from $m+1$ to $m+n$), the notation for the single premium is ${}_{m|}a_x$ (respectively ${}_{m|}a_{x:\overline{n}|}$).

   a) Give expressions for ${}_{m|}a_x$ and ${}_{m|}a_{x:\overline{n}|}$ in terms of the life table probabilities q_k.

   b) Show that ${}_{m|}a_x = A_{x:\overline{m}|}^{\;\;1} a_{x+m}$.

9. Describe the benefit which has the present value random variable function given by Z below; T denotes the future lifetime of a life aged x.

$$Z = \begin{cases} \bar{a}_{\overline{n}|} & T \leq n \\ \bar{a}_{\overline{T}|} & T > n \end{cases}$$

10. A special deferred annuity provides as benefits for a life aged 60:

    - on survival to age 65 an annuity of £2,000 p.a. payable in advance for two years certain and for life thereafter

    - on death between ages 63 and 65, £5,000 payable at the end of the year of death

    - on death between ages 60 and 63, £10,000 payable at the end of the year of death.

    Annual premiums are payable in advance until age 65 or earlier death. Determine the level annual premium based on an effective interest rate of 4% p.a. For the life annuity part, you may approximate all one-year death probabilities for age greater than 65 by the constant 0.05 (which leads close to the correct numerical answer).

Course website: http://www.stats.ox.ac.uk/~winkel/bs4b.html

A.6 Life insurance premiums and reserves

Please hand in scripts by Monday 7 March 2011, 11.15am, Department of Statistics (or by Tuesday 8 March 2011, 11.15am, if your class is on Thursday).

1. Give an expression (in terms of standard actuarial functions) for the annual premium for a 25 year endowment assurance on a life aged 40. The initial expenses are £2 per £100 sum insured, renewal expenses are 5% of each premium and 30p each year per £100 sum insured, and there is an initial commission of 50% of the first year's gross premium.

2. Some time ago, a life office issued an assurance policy to a life now aged exactly 55. Premiums are payable annually in advance, and death benefits are paid at the end of the year of death. The office calculates reserves using gross premium policy values. The following information gives the reserve assumptions for the policy year just completed. Expenses are assumed to be incurred at the start of the policy year.

   Reserve brought forward at the start of the policy year: £12,500

   Annual premium: £1,150

   Annual expenses: £75

   Death benefit: £50,000

   Mortality: A1967/70

   Interest 5.5% per annum

   Calculate the reserve at the end of the policy year.

3. A deferred annuity is purchased by 20 annual payments payable by a life aged 40 for a year annuity in advance of £2,500 a year, commencing in 20 years, for life. Find an expression for the premium on the basis of 4% pa interest with expenses of 5% of each premium and £5 at each annuity payment.

4. Suppose that $l_x = 100,000(100 - x)$, where $0 \leq x \leq 100$, and the interest rate is 5%.

   (a) Calculate $A_{50:\overline{10|}}$ and $\ddot{a}_{50:\overline{10|}}$.

   (b) Calculate the net annual premium for a 10 year endowment assurance for £10,000 to someone aged 50 and the policy values of years 3 and 4 using the values above.

   (c) Suppose that expenses are as follows

Commission:	50% of First Premium
	2% of Subsequent Premiums
General Expenses:	£150 Initially
	£10 in each subsequent year.

   Calculate the office premium for the policy in (b).

5. Show algebraically that the product of

   - the reserve after t years for an annual premium n-year pure endowment issued to a life aged x

   and

   - the annual premium for pure endowment of like amount issued at the same age but maturing in t years

   is constant for all values of t. (Ignore expenses.)

   Try to find an argument for this by general reasoning also.

6. (i) Show that $\overline{A}_x = 1 - \delta \overline{a}_x$.

   (ii) Consider a whole life assurance with sum assured 1 payable at the point of death, with a constant premium paid continuously. Show that the reserve at time t satisfies

$$ {}_t\overline{V}_x = 1 - \frac{\overline{a}_{x+t}}{\overline{a}_x} $$

7. Thiele's equation for the policy value at duration t, ${}_t\overline{V}_x$, of a life annuity payable continuously at rate 1 from age x is:

$$ \frac{\partial}{\partial t} {}_t\overline{V}_x = \mu_{x+t} \, {}_t\overline{V}_x - 1 + \delta \, {}_t\overline{V}_x. $$

   Derive this result algebraically, and also explain it by general reasoning.

8. Let ${}_tV_x$ be the reserve at time t on an insurance policy issued to a life aged x that pays a benefit of $S_x(t)$ on death at time t, under which premiums are payable of $P(t)$ at time t. Show by general reasoning that

$$ \frac{\partial}{\partial t} {}_tV_x = P(t) + \delta \, {}_tV_x - \mu_{x+t}(S_x(t) - {}_tV_x) $$

   where δ is the force of interest.

9. An insurer issues n identical policies. Let Y_j be the claim amount from the jth policy, and suppose that the random variables Y_j, $j = 1, \ldots, n$ are i.i.d. with mean $\mu > 0$ and variance σ^2. The insurer charges a premium of A for each policy.

   (a) Show that if $A = \mu + 10\sigma n^{-1/2}$, then the probability that total claims exceed total premiums is no more than 1%, for any value of n.

   (b) Use the Central Limit Theorem to show that if instead $A = \mu + 3\sigma n^{-1/2}$, then this probability is still less than 1%, provided n is large enough.

Course website: http://www.stats.ox.ac.uk/~winkel/bs4b.html