Lecture 6

Mortgages and loans

Reading: CT1 Core Reading Unit 8, McCutcheon-Scott Sections 3.7-3.8

As we indicated in the Introduction, interest-only and repayment loans are the formal inverse cash-flows of securities and annuities. Therefore, most of the last lecture can be reinterpreted for loans. We shall here only translate the most essential formulae and then pass to specific questions and features arising in (repayment) loans and mortgages, e.g. calculations of outstanding capital, proportions of interest/repayment, discount periods and rates used to compare loans/mortgages.

6.1 Loan repayment schemes

Definition 46 A *repayment scheme* for a loan of L in the model $\delta(\cdot)$ is a cash-flow

$$c = ((t_1, X_1), (t_2, X_2), \ldots, (t_n, X_n))$$

such that

$$L = \mathrm{Val}_0(c) = \sum_{k=1}^{n} v(t_k)X_k = \sum_{k=1}^{n} e^{-\int_0^{t_k} \delta(t)dt} X_k. \tag{1}$$

This ensures that, in the model given by $\delta(\cdot)$, the loan is repaid after the nth payment since it ensures that $\mathrm{Val}_0((0, -L), c) = 0$ so also $\mathrm{Val}_t((0, -L), c) = 0$ for all t.

Example 47 A bank lends you £1,000 at an effective interest rate of 8% p.a. initially, but due to rise to 9% after the first year. You repay £400 both after the first and half way through the second year and wish to repay the rest after the second year. How much is the final payment? We want

$$1,000 = 400v(1) + 400v(1.5) + Xv(2) = \frac{400}{1.08} + \frac{400}{(1.09)^{1/2}1.08} + \frac{X}{(1.09)(1.08)},$$

which gives $X = £323.59$.

Example 48 Often, the payments X_k are constant (*level payments*) and the times t_k are are regularly spaced (so we can assume $t_k = k$). So

$$L = \mathrm{Val}_0((1, X), (2, X), \ldots, (n, X)) = X\ a_{\overline{n|}} \qquad \text{in the constant-}i\text{ model.}$$

6.2 Loan outstanding, interest/capital components

The payments consist both of *interest* and *repayment of the capital.* The distinction can be important e.g. for tax reasons. Earlier in term, there is more capital outstanding, hence more interest payable, hence less capital repaid. In later payments, more capital will be repaid, less interest. Each payment pays *first* for interest due, then repayment of capital.

Example 48 (continued) Let $n = 3$, $L = 1,000$ and assume a constant-i model with $i = 7\%$. Then

$$X = 1,000/a_{\overline{3}|7\%} = 1,000/(2.624316) = 381.05.$$

Furthermore,

	time 1	time 2	time 3
interest due	$1,000 \times 0.07$ $= 70$	688.95×0.07 $= 48.22$	356.12×0.07 $= 24.93$
capital repaid	$381.05 - 70$ $= 311.05$	$381.05 - 48.22$ $= 332.83$	$381.05 - 24.93$ $= 356.12$
amount outstanding	$1,000 - 311.05$ $= 688.95$	$688.95 - 332.83$ $= 356.12$	0

Let us return to the general case. In our example, we kept track of the amount outstanding as an important quantity. In general, for a loan L in a $\delta(\cdot)$-model with payments $c_{\leq t} = ((t_1, X_1), \dots, (t_m, X_m))$, the outstanding debt at time t is L_t such that

$$\mathrm{Val}_t((0, -L), c_{\leq t}, (t, L_t)) = 0,$$

i.e. a single payment of L_t would repay the debt.

Proposition 49 (Retrospective formula) *Given* L, $\delta(\cdot)$, $c_{\leq t}$,

$$L_t = \mathrm{Val}_t((0, L)) - \mathrm{Val}_t(c_{\leq t}) = A(0, t)L - \sum_{k=1}^{m} A(t_k, t)X_k.$$

Recall here that $A(s, t) = e^{\int_s^t \delta(r)dr}$.

Alternatively, for a given repayment scheme, we can also use the following *prospective formula.*

Proposition 50 (Prospective formula) *Given* L, $\delta(\cdot)$) *and a repayment scheme* c,

$$L_t = \mathrm{Val}_t(c_{>t}) = \frac{1}{v(t)} \sum_{k:t_k>t} v(t_k)X_k. \tag{2}$$

Proof: $\mathrm{Val}_t((0, -L), c_{\leq t}, (t, L_t)) = 0$ and $\mathrm{Val}_t((0, -L), c_{\leq t}, c_{>t}) = 0$ (since c is a repayment scheme), so

$$L_t = \mathrm{Val}_t((t, L_t)) = \mathrm{Val}_t(c_{>t}).$$

□

Corollary 51 *In a repayment scheme $c = ((t_1, X_1), (t_2, X_2), \dots, (t_n, X_n))$, the jth payment consists of*

$$R_j = L_{t_{j-1}} - L_{t_j}$$

capital repayment and

$$I_j = X_j - R_j = L_{t_{j-1}}(A(t_{j-1}, t_j) - 1)$$

interest payment.

Note I_j represents the interest payable on a sum of $L_{t_{j-1}}$ over the period (t_{j-1}, t).

6.3 Fixed, capped and discount mortgages

In practice, the interest rate of a mortgage is rarely fixed for the whole term and the lender has some freedom to change their Standard Variable Rate (SVR). Usually changes are made in accordance with changes of the UK base rate fixed by the Bank of England. However, there is often a special "initial period":

Example 52 (Fixed period) For an initial 2-10 years, the interest rate is fixed, usually below the current SVR, the shorter the period, the lower the rate.

Example 53 (Capped period) For an initial 2-5 years, the interest rate can fall parallel to the base rate or the SVR, but cannot rise above the initial level.

Example 54 (Discount period) For an initial 2-5 years, a certain discount on the SVR is given. This discount may change according to a prescribed schedule.

Regular (e.g. monthly) payments are always calculated as if the current rate was valid for the whole term (even if changes are known in advance). So e.g. a discount period leads to lower initial payments. Any change in the interest rate leads to changes in the monthly payments.

Initial advantages in interest rates are usually combined with early redemption penalties that may or may not extend beyond the initial period (e.g. 6 months of interest on the amount redeemed early).

Example 55 We continue Example 44 and consider the discount mortgage of £85,000 with interest rates of $i_1 = \text{SVR} - 2.96\% = 2.99\%$ in year 1, $i_2 = \text{SVR} - 1.76\% = 4.19\%$ in year 2 and SVR of $i_3 = 5.95\%$ for the remainder of a 20-year term; a £100 Product Fee is added to the initial loan amount, a £25 Funds Transfer Fee is deducted from the Net Amount provided to the borrower. Then the borrower receives £$84,975$, but the initial loan outstanding is $L_0 = 85,100$. With annual payments, the repayment scheme is $c = ((1, X), (2, Y), (3, Z), \dots, (20, Z))$, where

$$X = \frac{L_0}{a_{\overline{20}|2.99\%}}, \quad Y = \frac{L_1}{a_{\overline{19}|4.19\%}}, \quad Z = \frac{L_0}{a_{\overline{18}|5.95\%}}.$$

With

$$a_{\overline{20}|2.99\%} = \frac{1-(1.0299)^{-20}}{0.0299} = 14.89124 \qquad \Rightarrow \qquad X = \frac{L_0}{a_{\overline{20}|2.99\%}} = 5,714.77,$$

we will have a loan outstanding of $L_1 = L_0(1.0299) - X = 81,929.72$ and then

$$Y = \frac{L_1}{a_{\overline{19}|4.19\%}} = 6,339.11, \quad L_2 = L_1(1.0419) - Y = 79,023.47, \quad Z = \frac{L_0}{a_{\overline{18}|5.95\%}} = 7270.96.$$

With monthly payments, we can either repeat the above with $12\widetilde{X} = L_0/a^{(12)}_{\overline{20}|2.99\%}$ etc. to get monthly payments $((1/12,\widetilde{X}),(2/12,\widetilde{X}),\ldots,(11/12,\widetilde{X}),(1,\widetilde{X}))$ for the first year and then proceed as above. But, of course, L_1 and L_2 will be exactly as above, and we can in fact replace parts of the repayment scheme $c = ((1,X),(2,Y),(3,Z),\ldots,(20,Z))$ by *equivalent cash-flows*, where equivalence means same discounted value. Using in each case the appropriate interest rate in force at the time

- $((1,X))$ is equivalent to $((1/12,\widetilde{X}),\ldots,(11/12,\widetilde{X}),(1,\widetilde{X}))$ in the constant i_1-model, where $12\widetilde{X}s^{(12)}_{\overline{1}|2.99\%} = X$, so $\widetilde{X} = X/12s^{(12)}_{\overline{1}|2.99\%} = \frac{1}{12}Xi_1^{(12)}/i_1 = 469.83$.
- $((2,Y))$ is equivalent to $((1+1/12,\widetilde{Y}),\ldots,(1+11/12,\widetilde{Y}),(2,\widetilde{Y}))$ in the constant i_2-model, where $\widetilde{Y} = \frac{1}{12}Yi_2^{(12)}/i_2 = 518.38$.
- $((k,Z))$ is equivalent to $((k-11/12,\widetilde{Z}),(k-10/12,\widetilde{Z}),\ldots,(k-1/12,\widetilde{Z}),(k,\widetilde{Z}))$ in the constant i_3-model, where $\widetilde{Z} = \frac{1}{12}Zi_3^{(12)}/i_3 = 589.99$.

6.4 Comparison of mortgages

How can we compare deals, e.g. describe the "overall rate" of variable-rate mortgages?

A method that is still used sometimes, is the "flat rate"

$$F = \frac{\text{total interest}}{\text{total term} \times \text{initial loan}} = \frac{\sum_{j=1}^n I_j}{t_n L} = \frac{\sum_{j=1}^n X_j - L}{t_n L}.$$

This is *not* a good method: we should think of interest paid on *outstanding debt* L_t, not on all of L. E.g. loans of different terms but same constant rate have different flat rates.

A better method to use is the Annual Percentage Rate (APR) of Section 4.3.

Example 55 (continued) The Net Amount provided to the borrower is $L = 84,975$, so the flat rate with annual payments is

$$F_{\text{annual}} = \frac{X + Y + 18Z - L}{20 \times L} = 3.41\%,$$

while the yield is 5.445%, i.e. the APR is 5.4% – we calculated this in Example 44.

With monthly payments we obtain

$$F_{\text{monthly}} = \frac{12\widetilde{X} + 12\widetilde{Y} + 18 \times 12\widetilde{Z} - L}{20 \times L} = 3.196\%,$$

while the yield is 5.434%, i.e. the APR is still 5.4%.

The yield is also more stable under changes of payment frequency than the flat rate.

Lecture 7

Funds and weighted rates of return

Reading: CT1 Core Reading Unit 9, McCutcheon-Scott Sections 5.6-5.7

Funds are pools of money into which various people pay for various reasons, e.g. investment opportunities, reserves of pension schemes etc. A fund manager maintains a portfolio of investment products (fixed-interest securities, equities, derivative products etc.) adapting it to current market conditions, often under certain constraints, e.g. at least some fixed proportion of fixed-interest securities or only certain types of equity ("high-tech" stocks, or only "ethical" companies, etc.). In this lecture we investigate the performance of funds from several different angles.

7.1 Money-weighted rate of return

Consider a fund, that is in practice a portfolio of asset holdings whose composition changes over time. Suppose we look back at time T over the performance of the fund during $[0, T]$. Denote the value of the fund at any time t by $F(t)$ for $t \in [0, T]$. If no money is added/withdrawn between times s and t, then the value changes from $F(s)$ to $F(t)$ by an accumulation factor of $A(s, t) = F(t)/F(s)$ that reflects the rate of return $i(s, t)$ such that $A(s, t) = (1+i(s, t))^{t-s}$. In particular, the yield of the fund over the whole time interval $[0, T]$ is then $i(0, T)$, the yield of the cash-flow $((0, F(0)), (T, -F(T)))$.

Note that we do not specify the portfolio or any internal change in composition here. This is up to a fund manager, we just assess the performance of the fund as reflected by its value. If, however, there have been external changes, i.e. deposits/withdrawals in $[0, T]$, then rates such as $i(0, T)$ in terms of $F(0)$ and $F(T)$ as above, become meaningless. We record such external changes in a cash-flow c. What rate of return did the fund achieve?

Definition 56 Let $F(s)$ be the value of a fund at time s, $c_{(s,t]}$ the cash-flow describing its in- and outflows during the time interval $(s, t]$, and $F(t)$ the fund value at time t. The *money-weighted rate of return* $\text{MWRR}(s, t)$ of the fund between times s and t is defined to be the yield of the cash-flow

$$((s, F(s)), c_{(s,t]}, (t, -F(t))).$$

If the fund is an investment fund belonging to an investor, the money-weighted rate of return is the yield of the investor.

7.2 Time-weighted rate of return

Consider again a fund as in the last section, using the same notation. If no money is added/withdrawn between times s and t, then $i(s,t)$ reflects the yield achieved by the fund manager purely by adjusting the portfolio to current market conditions. If the value of the fund goes up or down at any time, this reflects purely the evolution of assets held in the portfolio at that time, assets which were selected by the fund manager.

If there are deposits/withdrawals, they also affect the fund value, but are not under the control of the fund manager. What rate of return did the fund manager achieve?

Definition 57 Let $F(s+)$ be the initial value of a fund at time s, $c_{(s,t]} = (t_j, c_j)_{1\le j\le n}$ the cash-flow describing its in- and outflows during the time interval $(s,t]$, $F(t-)$ the value at time t. The *time weighted rate of return* $\mathrm{TWRR}(s,t)$ is defined to to be $i \in (-1,\infty)$ such that

$$(1+i)^{t-s} = \frac{F(t_1-)}{F(s+)}\,\frac{F(t_2-)}{F(t_1+)}\cdots\frac{F(t_n-)}{F(t_{n-1}+)}\,\frac{F(t-)}{F(t_n+)},$$

i.e.

$$\log(1+i) = \sum_{j=0}^{n} \frac{t_{j+1}-t_j}{t-s}\log(1+i(t_j,t_{j+1})),$$

where $F(t_j-)$ and $F(t_j+)$ are the fund values just before and just after time t_j, so that $c_j = F(t_j+) - F(t_j-)$, and where $t_0 = s$ and $t_{n+1} = t$.

The time-weighted rate of return is a time-weighted (geometric) average of the yields achieved by the fund manager between external cash-flows. Compared to the TWRR, the MWRR gives more weight to periods where the fund is big.

7.3 Units in investment funds

When two or more investors invest into the same fund, we want to keep track of the value of each investor's money in the fund.

Example 58 Suppose a fund is composed of holdings of two investors, as follows.

- Investor A invests £100 at time 0 and withdraws his holdings of £130 at time 3.
- Investor B invests £290 at time 2 and withdraws his holdings of £270 at time 4.

The yields of the two investors are $y_A = 9.14\%$ and $y_B = -3.51\%$, based respectively on cash-flows $((0,100),(3,-130))$ and $((2,290),(4,-270))$.

The cash-flow of the fund is

$$c = ((0,100),(2,290),(3,-130),(4,-270)).$$

Its yield is $\mathrm{MWRR}(0,4) = 1.16\%$.

To calculate the TWRR, we need to know fund values. Clearly $F(0+) = 100$ and $F(4-) = 270$. Suppose furthermore, that $F(2-) = 145$, then $F(2+) = 145 + 290 = 435$.

During the time interval $(2,3)$, both investors achieve the same rate of return, so $145 \to 130$ means $F(2+) = 435 \to 390 = F(3-)$, then $F(3+) = 390 - 130 = 260$. Now

$$(1+\text{TWRR})^4 = \frac{F(2-)}{F(0+)}\frac{F(3-)}{F(2+)}\frac{F(4-)}{F(3+)} = \frac{145}{100}\frac{390}{435}\frac{270}{260} = 1.35 \quad \Rightarrow \quad \text{TWRR} = 7.79\%.$$

We can see that the yield of Investor B pulls MWRR down because Investor B put higher money weights than Investor A. On the other hand the TWRR is high, because three good years outweigh the single bad year – it does not matter for TWRR that the bad year was when the fund was biggest, but this is again what pulls MWRR down.

A convenient way to keep track of the value of each investor's money is to assign units to investors and to track prices $P(t)$ per unit. There is always a normalisation choice, typically but not necessarily fixed by setting £1 per unit at time 0. The TWRR is now easily calculated from the unit prices:

Proposition 59 *For a fund with unit prices $P(s)$ and $P(t)$ at times s and t, we have*

$$(1+\text{TWRR}(s,t))^{t-s} = \frac{P(t)}{P(s)}.$$

Proof: Consider the external cash-flow $c_{(s,t]} = (t_j, c_j)_{1\le j\le n}$. Denote by $N(t_j-)$ and $N(t_j+)$ the total number of units in the fund just before and just after t_j. Since the number of units stays constant on (t_{j-1}, t_j), we have $N(t_{j-1}+) = N(t_j-)$. With $F(t_j\pm) = N(t_j\pm)P(t_j)$, we obtain

$$\begin{aligned}(1+\text{TWRR}(s,t))^{t-s} &= \frac{F(t_1-)}{F(s+)}\frac{F(t_2-)}{F(t_1+)}\cdots\frac{F(t_n-)}{F(t_{n-1}+)}\frac{F(t-)}{F(t_n+)}\\ &= \frac{N(t_1-)P(t_1)}{N(s+)P(s)}\frac{N(t_2-)P(t_2)}{N(t_1+)P(t_1)}\cdots\frac{N(t_n-)P(t_n)}{N(t_{n-1}+)P(t_{n-1})}\frac{N(t-)P(t)}{N(t_n+)P(t_n)}\\ &= \frac{P(t)}{P(s)}.\end{aligned}$$

□

Cash-flows of investors can now be conveniently described in terms of units. The yield achieved by an investor I making a single investment buying N_I units for $N_IP(s)$ and selling N_I units for $N_IP(t)$ is the TWRR of the fund between these times, because the number of units cancels in the yield equation $N_IP(s)(1+i)^{t-s} = N_IP(t)$.

Example 58 (continued) With $P(0) = 1$, Investor A buys 100 units, we obtain $P(2) = 1.45$ (from $F(2-) = 145$), $P(3) = 1.30$ (from the sale proceeds of 130 for Investor A's 100 units). With $P(2) = 1.45$, Investor B receives 200 units for £290, and so $P(4) = 1.35$(from the sale proceeds of 270 for Investor B's 200 units).

7.4 Fees

In practice, there are fees payable to the fund manager. Two types of fees are common.

The first is a fixed rate f on the fund value, e.g. if unit prices would be $\widetilde{P}(t)$ without the fee deducted, then the actual unit price is reduced to $P(t) = \widetilde{P}(t)/(1+f)^t$. Or, for an associated force $\varphi = \log(1+f)$, with the portfolio accumulating according to $\widetilde{A}(s,t) = \exp(\int_s^t \delta(r)dr)$, i.e. $\widetilde{P}(t) = P(0)\widetilde{A}(0,t)$ satisfies $\widetilde{P}'(t) = \delta(t)P(t)$, then $P'(t) = (\delta(t) - \varphi)P(t)$, i.e.

$$P(t) = \exp(-\int_0^t (\delta(s) - \varphi)ds)P(0) = e^{-\varphi t}\widetilde{P}(t).$$

This fee is to cover costs associated with portfolio changes between external cash-flows. It is incorporated in the unit price.

A second type of fee is often charged when adding/withdrawing money, e.g. a 2% fee could mean that the purchase price per unit is $1.02P(t)$ and/or the sale price is $0.98P(t)$.

7.5 Fund types

Investment funds offer a way to invest indirectly into a wide variety of assets. Some of these assets, such as equity shares, can be risky with prices fluctuating heavily over time, but with thousands of investors investing into the same fund and the fund manager spreading the combined money over many different assets (*diversification*), funds form a less risky investment and give access to further advantages such as lower transaction costs for larger volumes traded.

Funds are extremely popular as shown by the fact that the Financial Times has 7 pages devoted to their prices each day (there are only 2 pages for share prices on the London Stock Exchange). There is a wide spectrum of funds. Apart from different constraints on the portfolio, we distinguish

- active strategy: a fund manager takes active decisions to beat the market;
- passive strategy: investments are chosen according to a simple rule.

A simple rule can be to track an *investment index*. E.g., the FTSE 100 Share Index consists of the 100 largest quoted companies by market capitalisation (number of shares times share price), accounting for about 80% of the total UK equity market capitalisation. The index is calculated on a weighted arithmetic average basis with the market capitalisation as the weights.

Lecture 8

Inflation

Reading: CT1 Core Reading Units 4 and 11.3, McCutcheon-Scott Sections 5.5, 7.11
Further reading: http://www.statistics.gov.uk/hub/economy/prices-output-and-productivity/price-indices-and-inflation

Inflation means goods become more expensive over time, the purchasing power of money falls. In this lecture we develop a model for inflation.

8.1 Inflation indices

An *inflation index* records the price at successive times of some "basket of goods". The most commonly used indices are the Retail Price Index (RPI) and the Consumer Price Index (CPI). Their baskets contain food, clothing, cars, electricity, insurance, council tax etc., and in the case of the RPI (but not the CPI) also housing-related costs such as mortgage payments.

Year	2006	2007	2008	2009	2010	2011
Retail Price Index in January	193.4	201.6	209.8	210.1	217.9	229.0
Annual Inflation rate	4.2%	4.1%	0.1%	3.7%	5.1%	

Here the inflation rates are calculated e.g. as $e_{2010} = 229.0/217.9 - 1 = 5.1\%$. Theoretically, e_k is the interest rate earned by buying the basket at time k and selling it at time $k+1$. (In practice, many goods in the basket don't allow this.)

Although we will work with "one inflation index" at a time, often RPI, it should be noted that there are other important indices that are worth mentioning. Also, prices for any specific good are likely to behave in a completely different way from the RPI. When buying a house, you may wish to consult the House Price Index (house prices increased much faster than general inflation, up to 20% in some years, for about 15 years before reaching a plateau; now there is some evidence that they have started sinking - house price deflation, negative inflation rates). As a pensioner you have different needs and there is an inflation index that takes this into account (no salaries, no mortgage rates, more weight on medical expenses etc.).

Let's use RPI for the sake of argument. To track "real value", we can work in *units of purchasing power*, not units of currency.

Example 60 In January 2009 an investor put £1,000 in a savings account at an effective rate of 4% interest. His balance in January 2010 was £1,040. The RPI basket cost $\pounds_{2009}210.10$ in 2009 and $\pounds_{2010}217.90$ in 2010, so

$$\pounds_{2009}210.10 = \pounds_{2001}217.90$$

and hence

$$\pounds_{2010}1,040 = \pounds_{2009}1,040 \times 210.10/217.90 = \pounds_{2009}1002.77 = \pounds_{2009}1,000(1 + i_R),$$

and we say that the *real effective interest rate* was $i_R = 0.277\%$. Alternatively, we can use the RPI table to express every payment in multiples of RPI, i.e. *relative to the index.* Then i_R satisfies

$$\frac{1,000}{210.1}(1 + i_R) = \frac{1,030}{217.9}.$$

We regard an inflation index $Q(t)$ as a function of time. In practice, inflation indices typically give *monthly* values. $Q(m/12)$. If more frequency is required, use interpolation, either linear interpolation of index value $Q(t)$, or (normally more appropriate) linear interpolation of $\log(Q(t))$ (since inflation acts as a multiplier).

Only the ratio of indices at different times matters. So we can fix $Q = 1$ or $Q = 100$ at a particular time.

8.2 Modelling inflation

Since $Q(t)$ represents the "accumulated value" of some basket of goods, we can give $Q(t)$ the same structure as the accumulation factors $A(0, t)$. Recall $A(0, t) = \exp(\int_0^t \delta(s)ds)$.

Definition 61 If Q has the form

$$Q(t) = Q(0) \exp\left(\int_0^t \gamma(s)ds\right)$$

for a function $\gamma\colon [0, \infty) \to \mathbb{R}$, then γ is called the (time-dependent) *force of inflation.*

In the long term, we would expect $A(0, t) > Q(t)/Q(0)$ (interest above inflation), i.e. "real interest rates" should be positive, but this may easily fail over short intervals.

Definition 62 Given an interest rate model $\delta(\cdot)$ and an inflation model $\gamma(\cdot)$, we call $\delta(\cdot) - \gamma(\cdot)$ the (time-dependent) *real force of interest.*

The *real accumulation factor* $A^*(s, t) = \exp(\int_s^t (\delta(r) - \gamma(r))dr)$ captures *real* investment return, over and above inflation. With $A(s, t) = A^*(s, t)Q(t)/Q(s)$, we split the accumulation factor $A(s, t)$ into a component in line with inflation $Q(t)/Q(s)$, and the remaining $A^*(s, t)$.

The real force of interest measures interest in purchasing power.

Definition 63 For a cash-flow $c = ((t_1, c_1), \ldots, (t_n, c_n))$ "*on paper*", the cash-flow *net of inflation* or *in real terms* is given by

$$c_Q = \left(\left(t_1, \frac{Q(t_0)}{Q(t_1)} c_1 \right), \ldots, \left(t_n, \frac{Q(t_0)}{Q(t_n)} c_n \right) \right).$$

Everything is now expressed in time-t_0 money units (units of purchasing power).

We define the *real yield* $y_Q(c) = y(c_Q)$.

The value of t_0 or $Q(t_0)$ is not important: $Q(t_0)$ is just a constant in the yield equation.

Example 64 In January 1980, a bank issued a loan of £25,000. The loan was repayable after 3 years, with 10% interest payable annually in arrears. What is the *real rate of return* of the deal? With

	Jan 1980	Jan 1981	Jan 1982	Jan 1983
RPI	245.3	277.3	310.6	325.9

With money units of £2,500, the cash-flow on paper is $((0, -10), (1, 1), (2, 1), (3, 11))$, while the real cash-flow in time-0 money units is

$$\left((0, -10), \left(1, \frac{245.3}{277.3} \right), \left(2, \frac{245.3}{310.6} \right), \left(3, 11 \frac{245.3}{325.9} \right) \right)$$
$$= ((0, -10), (1, 0.8846), (2, 0.7898), (3, 8.2795)).$$

We solve the equation for the real yield

$$f(i) = -10 + 0.8846(1+i)^{-1} + 0.7898(1+i)^{-2} + 8.2795(1+i)^{-3} = 0$$

in an approximate way guessing two values and using a linear interpolation

$$f(0\%) = -0.04611, \quad f(-0.5\%) = 0.09174, \qquad i \approx -0.17\%.$$

8.3 Constant inflation rate

If the force of inflation $\gamma(\cdot)$ is constant equal to γ, then $e = e^\gamma - 1$ is the *rate of increase in the value of goods per year*. We have $Q(t+1) = (1+e)Q(t)$ and $Q(t) = Q(0)(1+e)^t$.

If also the force of interest is constant, δ, and $i = e^\delta - 1$, then so is the real force of interest $\delta - \gamma$, and we can define the real rate of interest

$$j = \exp(\delta - \gamma) - 1 = \frac{1+i}{1+e} - 1 = \frac{i-e}{1+e}.$$

Under a constant inflation rate e, real yields and yields satisfy the same relationship as real interest rates and interest rates:

Proposition 65 *Consider a constant inflation rate e. Let c be a cash-flow with yield $y(c)$. Then the real yield of c exists and is given by*

$$y_e(c) = \frac{y(c) - e}{1 + e}.$$

Proof: Write $c = ((t_1, c_1), \dots, (t_n, c_n))$ and $Q(t) = (1 + e)^t$. The real yield corresponds to the yield (if it exists) of

$$c_Q = ((t_1, (1 + e)^{-t_1} c_1), \dots, (t_n, (1 + e)^{-t_n} c_n).$$

We are looking for j, the unique solution of the real yield equation

$$\sum_{k=1}^{n} (1 + e)^{-t_k} c_k (1 + j)^{-t_k} = 0 \iff \sum_{k=1}^{n} ((1 + j)(1 + e))^{-t_k} = 0.$$

We know that the yield equation

$$\sum_{k=1}^{n} (1 + i)^{-t_k} c_k = 0$$

has a solution $i = y(c)$ that is unique in $(-1, \infty)$. But we now see that j solves the real yield equation if and only if i with $(1 + i) = (1 + e)(1 + j)$ satisfies the yield equation. Also $j > -1$ if and only if $i := (1 + e)(1 + j) - 1 > -1$, so the real yield equation has the unique solution

$$j = \frac{1 + y(c)}{1 + e} - 1 = \frac{y(c) - e}{1 + e}.$$

Therefore, this is the real yield $y_e(c)$. □

8.4 Index-linking

Suppose we wish to realise a cash-flow "in real terms", "in time-t_0 units", say $c = ((t_1, c_1), \dots, (t_n, c_n))$, with reference to some inflation index $R(t)$. Then the corresponding cash-flow "on paper" is

$$c^R = \left(\left(t_1, \frac{R(t_1)}{R(t_0)} c_1 \right), \dots, \left(t_n, \frac{R(t_n)}{R(t_0)} c_n \right) \right).$$

This is the principle of index-linked securities (and pensions, benefits, etc.), which are supposed to produce a reliable income *in real terms*.

Mathematically, this concept is the inverse of c_Q. The ratios $R(t_j)/R(t_0)$ now create time-t_j money units, whereas for c_Q, the ratio $Q(t_0)/Q(t_j)$ removed time-t_j money units expressing everything in terms of time-t_0 units. In fact, $(c_Q)^Q = c$ and $(c^Q)_Q = c$.

We will provide an example in Lecture 9 in the context of fixed-interest securities.

Lecture 9

Taxation

Reading: CT1 Core Reading Unit 11.1, 11.4-11.5, McCutcheon-Scott 2.10, 7.4-7.10, 8
Further reading: http://www.hmrc.gov.uk/cgt

Detailed taxation legislation is beyond this course, but we do address a distinction that is commonly made between (regular) *income* and *capital gain* from asset sales. In practice, both may be subject to taxation, but often at different rates. We will discuss the impact of inflation in this context, and possible inflation-adjustments.

9.1 Fixed-interest securities and running yields

Lecture 5 dealt with calculating the price of a fixed-interest security given an interest rate model and "the yield" given a price. In the context of securities, the use precise terminology is essential. The following definition introduces different notions of "yield".

Definition 66 Given a security, the yield $y(c)$ of the underlying cash-flow

$$c = ((0, -NP_0), (1, Nj), \ldots, (n-1, Nj), (n, Nj + NR))$$

is called the *yield to redemption.*

If the security is traded for P_k per unit nominal at time k, then the ratio j/P_k of dividend (coupon) rate and price per unit nominal is called the *running yield* of the security at time k.

For equities the definition of the running yield applies with price P_k per share and dividend D_k instead of coupon rate j. The price P_k determines the current capital value of the security/share, and the running yield then expresses the rate at which interest is paid on the capital value. This distinction of yield to redemption and running yield is related to the notions of interest income and capital gains relevant for taxation.

- *Income Tax* may be payable on income (including interest, coupon payments, dividend payments);
- *Capital Gains Tax (CGT)* may be payable when goods (including shares, securities etc.) are sold for a profit.

Suppose a good was bought at a purchase price C and sold at a sale price S, this gives a *capital gain* $S-C$ (a capital loss, if negative). If $S>C$, then CGT may by payable on $S-C$.

Tax is payable if neither the investor nor the asset are exempt from tax. Tax rates may vary between different investors (e.g. income tax bands of 20%, 40% and 50%).

The yield to redemption reflects both income and capital gains (or losses), whereas the running yield only reflects the income part.

Example 67 Given a security with semi-annual coupons at 6%, with redemption date three years from now that is currently traded above par at 105%, the running yield is $6/105 \approx 5.7\%$. The yield to redemption is the solution of

$$\begin{aligned} 0 &= i\text{-}Val_0((0,-105),(0.5,3),(1,3),(1.5,3),(2,3),(2.5,3),(3,103)) \\ &= -105 + 6a^{(2)}_{\overline{3}|i} + (1+i)^{-3}100 \end{aligned}$$

and numerically, we calculate a yield to redemption of $\approx 4.3\%$. The difference is due to the capital loss that is ignored by the running yield.

9.2 Income tax and capital gains tax

For simplicity, we will assume that tax is always due at the time of the relevant cash-flow, i.e. if an investor is to receive a payment c_k, but is subject to income tax at rate r_1, then the cash-flow is reduced to $c_k(1-r_1)$, and if an investor is to receive a payment $c_n = C + (S-C)$ consisting of capital C and capital gain/loss $S-C$, but is subject to CGT at rate r_2, then the cash-flow is reduced to $S - r_2(S-C)_+$, where $(S-C)_+ = \max(S-C, 0)$ denotes the positive part.

Example 68 The holder of a savings account paying interest at 2.5% *gross*, subject to income tax at 20%, receives *net* interest payments at rate 2.5%(1-20%)=2% net.

Example 69 An investor who bought equities for $C = £80$ and sells for $S = £100$ within a year and is subject to 40% capital gains tax, only receives $S-(S-C)40\% = £92$.

A key example where usually both income tax and CGT apply is a fixed-interest security.

Example 70 If the holder of a fixed-interest security is liable to income tax at rate r_1 and capital gains tax at rate r_2, in principle, and if the fixed-interest security is not exempt from any of these taxes, then the liabilities are as follows.

After purchase at a price A, the *gross cash-flow* (ignoring tax, or assuming no liability to tax) would be

$$c = ((1/2, Nj/2), (1, Nj/2), \ldots, (n-1/2, Nj/2), (n, Nj/2 + NR)),$$

but income tax reduces coupon payments $Nj/2$ to $Nj(1-r_1)/2$ and CGT reduces the redemption payment NR to $NR - r_2(NR-A)_+$.

If the security is not held for the whole term but is sold at time k for P_k per unit nominal, capital gains tax reduces the sales proceeds P_kN to $P_kN - r_2(P_kN - A)_+$.

9.3 Offsetting

If capital gains and capital losses occur due to assets sold in the same tax year, a taxpayer may (under certain restrictions) *offset* the losses L agains the gains G, to pay CGT only on $(G - L)_+$.

Example 71 Asset A (silverware) is sold for $£1,865$ (previously bought for $£1,300$). Asset B (a painting) is sold for $£500$ (previously bought for $£900$).

Under liability to CGT at 40%, the tax due for Asset A on its own would be

$$40\% \times (£1,865 - £1,300) = 40\% \times £565 = £226.$$

Offsetting agains losses from Asset B, tax is only due

$$40\% \times ((£1,865 - £1,300) - (£900 - £500)) = 40\% \times £165 = £66.$$

9.4 Indexation of CGT

If purchase and sale are far apart, paying CGT on $(S - C)_+$ may not be fair, since no allowance has been made for inflation. The government may decide to tax only on *real capital gain*, for which the purchase price C is revalued to account for inflation.

Example 72 Suppose shares were bought in March 1990 for £1,000 and sold in April 1996 for £1,800, with CGT charged at 40%. Then without indexation, CGT would be

$$40\% \times (£1,800 - £1,000) = £320.$$

With RPI in March 1990 at 131.4 and in April 1996 at 152.6, we have

$$£_{\text{Mar1990}}1,000 = £_{\text{Apr1996}}\frac{152.6}{131.4}1,000 = £_{\text{Apr1996}}1,161.34.$$

So, CGT due on sale is

$$40\% \times (£1,800 - £1,161.34) = £255.45.$$

9.5 Inflation adjustments

We return to a general inflation index $R(\cdot)$. Fixed-interest securities are useful to provide a regular income stream. But with inflation reducing the real value of the coupon payments, this stream is decreasing in real terms. If we want to achieve a cash-flow

$$c = ((1/2, jN/2), (1, jN/2), \dots, (n - 1/2, jN/2), (n, jN/2 + RN))$$

in real terms, we need to boost the coupon payments by $R(\cdot)$:

$$c^R = \left(\left(1/2, \frac{R(1/2)}{R(0)} jN/2\right), \left(1, \frac{R(1)}{R(0)} jN/2\right), \dots, \left(n-1/2, \frac{R(n-1/2)}{R(0)} jN/2\right), \left(n, \frac{R(n)}{R(0)} (jN/2 + RN)\right)\right)$$

to achieve $(c^R)_R = c$. Since payment of $R(k/2)/R(0)jN/2$ is made at time $k/2$, $R(k/2)$ must be known at time $k/2$, so we cannot take $R(t) = \mathrm{RPI}(t)$, since data collection and aggregation to form index values is not instantaneous. In practice, the lag is often significant, e.g. 8 months, i.e. $R(t) = \mathrm{RPI}(t - 8/12)$. An advantage is that both parties know the sizes of payments well in advance. A disadvantage, is that the real yield $y_Q(c^R) = y((c^R)_Q)$ for $Q(t) = \mathrm{RPI}(t)$ will not exactly equal $y(c)$, but it will normally be similar: with normalisation $R(0) = Q(0) = 1$, we obtain

$$(c^R)_Q = \left(\left(1/2, \frac{R(1/2)}{Q(1/2)} jN/2\right), \left(1, \frac{R(1)}{Q(1)} jN/2\right), \dots, \left(n-1/2, \frac{R(n-1/2)}{Q(n-1/2)} jN/2\right), \left(n, \frac{R(n)}{Q(n)} (jN/2 + RN)\right)\right).$$

Lecture 10

Project appraisal

Reading: CT1 Core Reading Unit 9, McCutcheon-Scott Sections 5.1-5.4

For a given investment project, the notion of yield (of the underlying cash-flow) provides a way to assess the project by identifying an internal rate of return (interest rate). We develop this further in this lecture, focussing on profitability as the main criterion, and we also discuss problems that arise when comparing two investment projects or business ventures.

10.1 Net cash-flows and a first example

Recall that the yield $y(c)$ of a cash-flow c as unique solution of $\text{NPV}(i) = i\text{-Val}(c) = 0$ represents the boundary between profitability ($\text{NPV}(i) > 0$) and unprofitability ($\text{NPV}(i) < 0$) as a function of the interest rate i. Specifically, each investment project (with net outflows before net inflows) is profitable if its yield exceeds the market interest rate ($y(c) > i$). Here, we use the word "net" to refer to the fact that both in- and outflows have been incorporated in c and, where they happen at the same time, just the combined flow at any time is considered.

Example 73 You purchase party furniture for £10,000 to run a small hire business. You expect continuous income from hiring fees at rate £1,200 p.a., but also expect expenditure for wear and tear of £400 p.a. So, the *net* rental income will be £800 p.a. After 20 years we expect to sell the party furniture for £6,000. The only net outflow precedes the inflows, so the yield exists. We solve the yield equation

$$0 = f(i) = -10,000 + 800\overline{a}_{\overline{20}|i} + 5,000(1+i)^{-20} = 800\frac{1-(1+i)^{-20}}{\log(1+i)} + 5,000(1+i)^{-20}$$

numerically. Since £800 interest on 10,000 is 8% and we have 40% capital loss, 2% p.a. (this is a very rough estimate as we ignore compounding over 20 years), we guess $i = 6\%$

$$f(6\%) = 1,319.37 > 0, \quad f(7\%) = 319.01 > 0, \quad f(7.5\%) = -129.78 < 0,$$

so

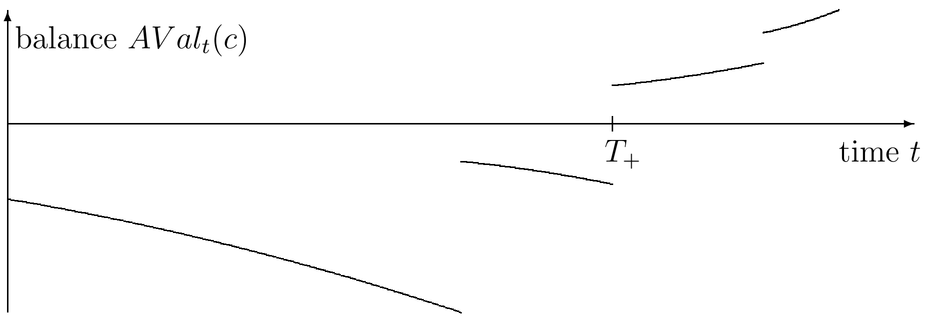
$$i \approx 7\%\frac{-f(7.5\%)}{f(7\%) - f(7.5\%)} + 7.5\%\frac{f(7\%)}{f(7\%) - f(7.5\%)} = 7.36\%.$$

10.2 Payback periods and a second example

For a profitable investment project in a given interest rate model $\delta(\cdot)$ we can study the evolution of the accumulated value $\mathrm{AVal}_t(c_{\leq t})$. With outflows preceding inflows, this accumulated value will first be negative, but reach a positive terminal value at or before the end of the project (positive because the project is profitable). We are interested in the time when it becomes first positive.

Definition 74 Given a model $\delta(\cdot)$ and a profitable cash-flow c with outflows preceding inflows. We define the *discounted payback period*

$$T_+ = \inf\{t \geq 0 : \mathrm{AVal}_t(c_{\leq t}) \geq 0\} = \inf\{t \geq 0 : \mathrm{DVal}_t(c_{\leq t}) \geq 0\}.$$



If rather than investing existing capital, the investment project is financed by taking out loans and then using any inflows for repayment, then T_+ is the time when the account balance changes from negative to positive, i.e. when the debt has been repaid. All remaining inflows after T_+ contribute fully to the profit of the project.

Since the periods of borrowing and saving are well-separated by T_+, we can here deal with interest models that have different borrowing rates $\delta_-(\cdot)$ and savings rates $\delta_+(\cdot)$. The definition of T_+ will then be based only on $\delta_-(\cdot)$ and the final accumulated profit can then be calculated from $\delta_-(\cdot)$-$\mathrm{AVal}_{T_+}(c_{\leq T_+})$ and $\delta_+(\cdot)$-$\mathrm{DVal}_{T_+}(c_{>T_+})$.

Example 75 A home-owner is considering to invest in a solar energy project. Purchasing and installing solar panels on his roof costs £10,000 in three months' time. Energy is then generated continuously at rate £1,000 p.a. The expected lifetime of the panels is 20 years. With an interest rate of $i = 8\%$ on the loan account, is the investment profitable? If so, what is the discounted payback period?

With money units of £1,000 and a time origin "in three months' time", the outflow of 10 happens at time 0 and the continuous inflow, c, at unit rate starts at time 0:

$$\mathrm{AVal}_t((0,-10), c_{\leq t}) = -10(1+i)^t + \overline{s}_{\overline{t}|} = (1+i)^t \left(\frac{1}{\log(1+i)} - 10 \right) - \frac{1}{\log(1+i)}.$$

Then for $i = 8\%$ and and $t = 20$, we have

$$\mathrm{AVal}_{20}((0,-10), c_{\leq 20}) = 0.95939,$$

so we expect an accumulated profit of £959.39 > 0 making the project profitable.

We calculate $T_+ = \inf\{t \geq 0 : \mathrm{AVal}_t((0,-10), c_{\leq t}) \geq 0\}$ by solving

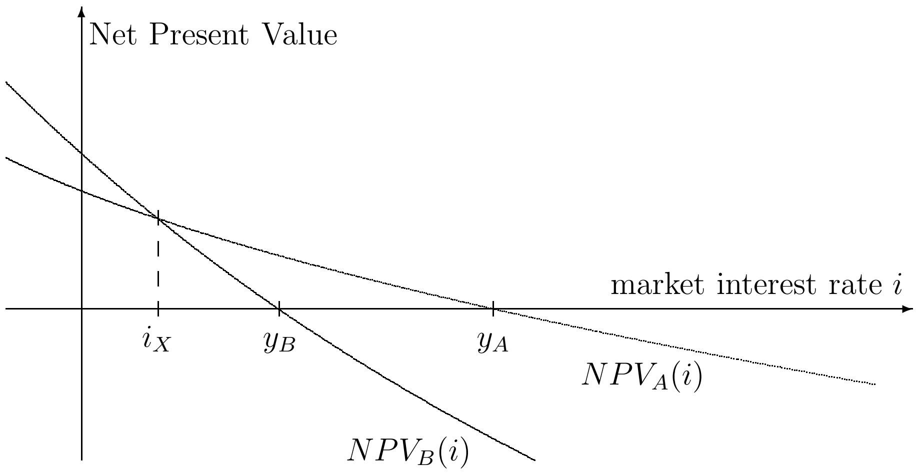
$$\mathrm{AVal}_t((0,-10), c_{\leq t}) = 0 \iff t = -\frac{\log(1 - 10\log(1+i))}{\log(1+i)} = 19.0744 \qquad \text{(19y, 27d)}$$

so seen from now, three months before the installation is complete, the discounted payback period is 19 years, 3 months and 27 days.

If loan repayment is not made continuously, but quarterly as an equivalent cash-flow (at $i = 8\%$), then the bank balance at the end of each quarter will be as with continuous payment, but only an inflow (at the end of a quarter) can make the balance nonnegative, so the discounted payback period will then be 19 years and 6 months.

10.3 Profitability, comparison and cross-over rates

Recall that a project is *profitable* at rate i if $\mathrm{NPV}(i) > 0$. Now consider cash-flows c_A and c_B representing two investment projects, and their Net Present Values $\mathrm{NPV}_A(i)$ and $\mathrm{NPV}_B(i)$. To compare projects A and B, each with all outflows before all inflows, we can calculate their yields y_A and y_B. Suppose $y_A < y_B$. If the market interest rate is in (y_A, y_B), then project B is profitable, project A is not. But this does not mean that project B is more profitable than project A (i.e. $NPV_B(i) > NPV_A(i)$) for all lower interest rates as the following figure shows.



i_X is called a *cross-over rate* and can be calculated as solution to the yield equation of $c_A - c_B$. For interest rates below i_X, project B is more profitable than project A, although its yield is smaller. A decision for one or the other project (or against both) now clearly depends on the expectations on interest rate changes.

Example 76 Compare Project A from Example 73 and Project B from Example 75. For simplicity, let us delay Project A by 3 months, so that the expenditure of £10,000 happens at the same time. This does not affect the yield. So far, we have seen that Project A has yield $y_A \approx 7.36\%$ and Project B is profitable at $i = 8\%$ and therefore has a yield $y_B > 8\%$.

To investigate potential cross-over rates, consider $c_B - c_A$, a continuous cash-flow at rate £800 − £1,000 = £ − 200 p.a., i.e. net outflow, and an inflow of £5,000 at time 20. We solve the yield equation (which has a unique solution, by Proposition 40)

$$-200\overline{a}_{\overline{20}|i} + 5,000(1+i)^{-20} = 0 \Rightarrow \cdots \Rightarrow i_X \approx 3.89\%.$$

We conclude, that Project A is more profitable for $i < 3.89\%$, while Project B is more profitable than Project A for $3.89\% < i < 7.36\% \approx y_A$, and that Project B is profitable while Project A is not for $7.36\% \approx y_A < i < y_B$. [$y_B \approx 8.29\%$.]

10.4 Reasons for different yields/profitability curves

For much of the interest rate modelling developments, it is instructive to think that at least the most secure savings/investment opportunities in the real world are consistent with an underlying time-varying force of interest. Examples in this lecture seemed to suggest otherwise. In this section we collect a variety of remarks, some of which motivate further developments of this course and should be considered again after relevant concepts have been introduced.

- There are many government bonds with a variety of redemption dates. We will indeed extract from such prices a consistent system, an implied *term structure of interest rates*. Even though this provides an interest rate model for a few decades, this model is subject to uncertainty that manifests itself e.g. in that the implied rate for a fixed future year will vary from day to day, month to month, between now and in a year's time, say. A more appropriate model that captures this uncertainty, is a *stochastic interest rate model.*

- In an efficient market with participants optimizing their profit, any definitely "profitable" investment project should involve something that is as yet unaccounted for. Otherwise, many market participants would engage in it and market forces would push up prices (or affect other parts of underlying cash-flow) and remove the profit. We will investigate related reasoning in a lecture on *arbitrage-free pricing*.

- Unlike our toy examples, assessment of genuine business ventures will have to include allowances for personal labour, administration costs etc. Such expenditure and indeed many other positions in the cash-flows will be subject to some uncertainty. A higher yield of a business venture may be due to a higher risk. As an example of such risk, we will investigate the risk of default, i.e. the risk that an income stream stops due to bankruptcy of the relevant party.

- The reason why there is a cross-over rate in Example 76 is that the cash-flow of Project A is more weighted towards later times, due to the big inflow after 20 years. As such, its NPV reacts more strongly to changes in interest rates. We will investigate related notions of *Discounted Mean Term* and *volatility* of a cash-flow, also referred to as *duration*.