Lecture 15

Stochastic interest rates and corporate bonds

Reading: McCutcheon-Scott Chapter 12, CT1 Unit 14

This lecture briefly discusses ways to model more realistic stochastic interest rates. We also move on to the final topic of random cash-flows, in the specific example of corporate bonds, which have a risk of default. Since these two quite separate topics appear in the same lecture, we also point out in passing that they can be related by modelling default as an interest rate of $+\infty$ for the relevant time period.

15.1 Dependent annual interest rates

In practice, interest rates do not fluctuate as strongly as in the independent interest rates model. In fact, when interest rates are high, the next period is quite likely to show another high interest rate, similarly with low rates. In fact, the Monetary Policy Committee of the Bank of England meets every month to decide on changes to the Base Rate to which many commercial bank rates are coupled. Often, the rate remains unchanged: e.g.

- in 2000, 2002 and also more recently in 2010 and 2011, there have been no changes to the base rate, at all;
- in 2001, there were 6 reductions of 0.25%, 1 reduction of 0.5% and 5 meetings with unchanged rates;
- between October 2008 and March 2009 the monthly changes were -0.5%, -1.5%, -1.0%, -0.5%, -0.5%, -0.5% to reach a hard lower boundary of a base rate of 0.5%.

This can be modelled by centering the new interest rate around the current interest rate, or between the current and a general long term mean interest rate.

Example 100 (Random walk) Let $\Delta_0 = \mu$, $\Delta_j = \log(1 + I_j) = \Delta_{j-1} + \varepsilon_j$, where ε_j, $j \geq 1$, are independent $\mathrm{N}(0, \sigma^2)$ random variables, so

$$\Delta_j = \mu + \varepsilon_1 + \cdots + \varepsilon_j \sim \mathrm{N}(\mu, j\sigma^2)$$

and

$$\Delta_1 + \cdots + \Delta_n = n\mu + n\varepsilon_1 + (n-1)\varepsilon_2 + \cdots + 2\varepsilon_{n-1} + \varepsilon_n \sim \mathrm{N}(n\mu, \sigma^2(1+4+\cdots+(n-1)^2+n^2)).$$

The effect of random shocks ε_j on the interest rate is permanent. The variance of Δ_n grows like n, the variance of $\Delta_1 + \cdots + \Delta_n$ grows like n^3 (between $n^3/8$ and n^3).

Example 101 (Autoregressive model) Let $\Delta_0 = \mu$, $\Delta_j = \theta\Delta_{j-1} + (1-\theta)\mu + \varepsilon_j$, where $\theta \in [0, 1)$ induces some "mean reversion". Conditional on Δ_{j-1}, we have

$$\Delta_j \sim \mathrm{N}(\theta\Delta_{j-1} + (1-\theta)\mu, \sigma^2).$$

This can also be expressed as $(\Delta_j - \mu) = \theta(\Delta_{j-1} - \mu) + \varepsilon_j$, say $D_j = \theta D_{j-1} + \varepsilon_j$. Now

$$\begin{array}{rcl}
D_1 &=& \varepsilon_1 \sim \mathrm{N}(0, \sigma^2) \\
D_2 &=& \theta\varepsilon_1 + \varepsilon_2 \sim \mathrm{N}(0, (1+\theta^2)\sigma^2) \\
D_3 &=& \theta^2\varepsilon_1 + \theta\varepsilon_2 + \varepsilon_3 \sim \mathrm{N}(0, (1+\theta^2+\theta^4)\sigma^2) \\
\cdots & \vdots & \cdots \\
D_n &=& \theta^{n-1}\varepsilon_1 + \theta^{n-2}\varepsilon_2 + \cdots + \theta\varepsilon_{n-1} + \varepsilon_n \sim \mathrm{N}(0, (1+\theta^2+\cdots+\theta^{2(n-1)})\sigma^2) \\
D_1 + \cdots + D_n &=& (1+\theta+\cdots+\theta^{n-1})\varepsilon_1 + (1+\theta+\cdots+\theta^{n-2})\varepsilon_2 + \cdots + \varepsilon_n \sim \mathrm{N}(0, r_n^2)
\end{array}$$

where $r_n^2 < n\sigma^2/(1-\theta^2)$, since $1+\theta^2+\cdots+\theta^{2(n-1)} = (1-\theta^{2n})/(1-\theta^2) < 1/(1-\theta^2)$. In particular, the variance of Δ_n is now of constant order and the variance of $\Delta_1 + \cdots + \Delta_n$ grows at rate n.

Both these models were Markov chains: the distribution of Δ_j depended on previous $(\Delta_1, \ldots, \Delta_{j-1})$ only through Δ_{j-1}.

15.2 Modelling the force of interest

When reducing the time unit, one can also pass to continuous-time limits. In fact, on credit markets, where credit is traded, market interest rates fluctuate much more than e.g. the Base Rate of the Bank of England. Such market rates react directly to supply and demand.

First note that our previous models can be viewed as models with piecewise constant forces of interest. We started with a single random force of interest Δ, constant for all time. We then considered forces of interest that were constant Δ_j during each time unit $(j-1, j]$. For the last two examples, we can get interesting limits as we let our time unit tend to zero.

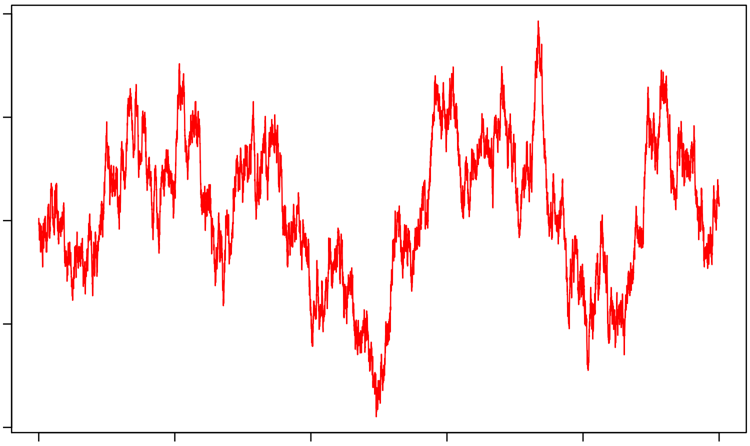

Figure 15.1: Brownian motion

E.g., in the random walk model, we had $\Delta_0 = \mu$ and $\Delta_n = \Delta_{n-1} + \varepsilon_n$ for i.i.d. $\varepsilon_n \sim \mathrm{N}(0, \sigma^2)$. We can model $p$thly changing forces by setting $\Delta_0^{(p)} = \mu$ and for $n \geq 1$

$$\Delta_{\frac{n}{p}}^{(p)} = \Delta_{\frac{n-1}{p}}^{(p)} + \varepsilon_{\frac{n}{p}}^{(p)}, \qquad \varepsilon_{\frac{n}{p}}^{(p)} \sim \mathrm{N}\left(0, \frac{\sigma^2}{p}\right) \text{ independent,}$$

where we understand that $\Delta_{n/p}^{(p)}$ is to apply during $((n-1)/p, n/p)$. Note that the models are consistent in that $\Delta_n^{(p)} \sim \Delta_n$ for all n, p. In the limit $p \to \infty$ we get Brownian motion $(\Delta_t)_{t \geq 0}$ (let time unity tend to zero). Brownian motion is a continuous Markov process such that $B(t) \sim \mathrm{N}(\mu, \sigma^2 t)$ for all $t \geq 0$. See Figure 15.1 for a simulation of Brownian motion.

Similarly, we can get continuous processes as limits of te mean-reverting walks. These are described by appropriate stochastic differential equations, e.g. we can take market instantaneous forward rates F_r and have a model that reverts to F_r instead of μ.

What can we do with these models?

- Pricing of derivative contracts (derived from interest rates);
- assessment and quantification of interest rate risk in portfolios.

15.3 Example: corporate bonds

Let us return to the independent interest rate model, where $I_n > -1$, $n \geq 1$, are independent and identically distributed. It will be useful to recall that *expected* discounted values in this model (cf. Proposition 97) involve expected discount factors $w = \mathbb{E}((1 + I_1)^{-1}) > (1 + \mathbb{E}(I_1))^{-1}$, and for a cash-flow $c = ((t_1, c_1), \ldots, (t_n, c_n))$ with $t_m \in \mathbb{N}$ for all m, we get

$$\mathbb{E}\left(\sum_{m=1}^{n} c_m (1 + I_1)^{-1} \times \cdots \times (1 + I_{t_m})^{-1}\right) = \sum_{m=1}^{n} c_m w^{t_m} = \mathrm{NPV}(k),$$

where $w = (1+k)^{-1}$. We will now discuss a very special (almost degenerate) case for the distribution of I_n. To make further use of this example in Lecture 16, we develop this here already in a more natural way as a random cash-flow model.

A *corporate bond* pays a fixed coupon rate j. In usual notation, purchase price P and yield i are related by

$$P = ja_{\overline{n}|i} + R(1+i)^{-n} \qquad \text{if we ignore the risk of default/bankruptcy.}$$

However, there is uncertainty about whether payment actually happens. The simplest model to reflect this uncertainty is if there is a random default time T at which all payments stop: the actual cash-flow is $C = c_{<T}$.

More specifically, suppose there is an annual default probability of $1-p$ (conditionally given that default has not yet happened), in the sense that m years without default happen with probability p^m, so that T has a geometric distribution geom(p), i.e.

$$\mathbb{P}(T = m+1) = p^m(1-p), \qquad m \geq 0,$$

and a payment at time m will happen if and only if $T > m$, i.e. with probability $\mathbb{P}(T > m) = p^m$. A potential payment (t_m, c_m) is then modelled by a random cash-flow $(t_m, B_m c_m)$, where $\mathbb{P}(B_m = 1) = p^m$ and $\mathbb{P}(B_m = 0) = 1 - p^m$, so that also $\mathbb{E}(B_m) = p^m$. For an interest rate i, the expected discounted value of the cash-flow $C = ((t_1, B_1c_1), \ldots, (t_n, B_nc_n))$ is

$$\mathbb{E}\left(\mathrm{NPV}_C(i)\right) = \mathbb{E}\left(\sum_{m=1}^{n} B_m c_m (1+i)^{-t_m}\right) = \sum_{m=1}^{n} \mathbb{E}(B_m) c_m (1+i)^{-t_m} = \sum_{m=1}^{n} p^m c_m (1+i)^{-m}.$$

If we take k such that $p(1+i)^{-1} = (1+k)^{-1}$, we get

$$\mathbb{E}\left(\mathrm{NPV}_C(i)\right) = \sum_{m=1}^{n} c_m (1+k)^{-m} = \mathrm{NPV}_c(k).$$

We have proved the following result:

Proposition 102 *Let c be a discrete cash-flow with integer payment times and $T \sim$ geom(p), i.e. $\mathbb{P}(T = m) = p^{m-1}(1-p)$, $m \geq 1$. Let $C = c_{<T}$. Then for any $i > -1$,*

$$\mathbb{E}\left(\mathrm{NPV}_C(i)\right) = \mathrm{NPV}_c(k),$$

where $k = (1+i-p)/p$.

In particular, if we are given the yield i of a corporate bond (which is only achieved if default does not occur!) and the yield k of a comparable default-free bond, there is an implied annual default probability of $1 - p = 1 - (1+i)/(1+k) = (k-i)/(1+k)$.

Note that discount factors $(1+i)^{-1} \to 0$ as $i \to \infty$, so at $i = \infty$, any future cash-flow has zero discounted value. With $\mathbb{P}(I_m = i) = p$ and $\mathbb{P}(I_m = \infty) = 1 - p$, we have

$$\mathbb{E}((1+I_m)^{-1}) = p(1+i)^{-1} + (1-p) \times 0 = p(1+i)^{-1}.$$

Therefore, we can think of the geometric default time model as a model of stochastic interest rates.

Lecture 16

Random cash-flows and risk

Reading: CT1 Core Reading Unit 7.2, 10.2-10.3, 11.2, McCutcheon-Scott Sections 7.3

In this final lecture, we continue the study of random cash-flows, of which corporate bonds are an important example.

16.1 Net prices

Let us start by noting the principle that we used for pricing corporate bonds.

Definition 103 Given an interest rate i, the *net premium/net price* for a random cash-flow, e.g. $C = ((T_1, C_1), \ldots, (T_n, C_n))$, of benefits/investment proceeds $C_j \geq 0$, is the expected present value

$$A = \mathbb{E}\left(\mathrm{NPV}(i)\right) = \sum_{m=1}^{n} \mathbb{E}\left(C_m (1+i)^{-T_m}\right).$$

The term "net price" is appropriate for investment products, while "net premium" is used for insurance products (life assurance, home insurance, etc.). In both frameworks, actual prices/premiums will be higher, because of administrative cost and/or risk, e.g. an insurer's risk that actual benefits in a portfolio of policies exceed average benefits.

If the times of C are deterministic $T_m = t_m$ and only the amounts C_m random, the net price is

$$A = \sum_{m=1}^{n} \mathbb{E}(C_m)(1+i)^{-t_m}$$

and depends only on the mean amounts, since the deterministic $(1+i)^{-t_m}$ can be taken out of the expectation. Such a situation arises for share dividends.

Example 104 (Discounted Dividend Model) Consider a share which has just paid a dividend of d_0. Suppose that each year (or half-year), the dividend increases by an independent random factor F_m, $m \geq 1$, with $\mathbb{E}(F_m) = f$ (could also decrease for $F_m < 1$). Then the mth dividend will be

$$D_m = d_0 \times F_1 \times \cdots \times F_m, \qquad \text{with } \mathbb{E}(D_m) = d_0 \times \mathbb{E}(F_1) \times \cdots \times \mathbb{E}(F_m) = d_0 f^m.$$

What is the fair price for this share? We assume that annual dividends continue indefinitely, so the random cash-flow is $C = ((1, D_1), (2, D_2), \ldots) = ((m, D_m), m \geq 1)$ with

$$\mathbb{E}(\mathrm{NPV}(i)) = \sum_{m\geq 1} \mathbb{E}(D_m)(1+i)^{-m} = \sum_{m\geq 1} d_0 f^m (1+i)^{-m} = \frac{d_0 f}{1+i} \frac{1}{1-f(1+i)^{-1}} = \frac{d_0 f}{1+i-f},$$

provided that $f(1+i)^{-1} < 1$, i.e. $f < 1+i$, for the geometric series to converge.

If more specifically

$$C_m = \begin{cases} c_m & \text{with probability } p_m, \\ 0 & \text{with probability } 1-p_m, \end{cases}$$

we can say $C_m = B_m c_m$, where B_m is a Bernoulli random variable with parameter p_m, i.e.

$$B_m = \begin{cases} 1 & \text{with probability } p_m, \\ 0 & \text{with probability } 1-p_m. \end{cases}$$

For the random cash-flow $C = ((t_1, B_1 c_1), \ldots, (t_n, B_n c_n))$, we have

$$A = \sum_{m=1}^{n} \mathbb{E}(B_m c_m (1+i)^{-t_m}) = \sum_{m=1}^{n} c_m (1+i)^{-t_m} \mathbb{E}(B_m) = \sum_{m=1}^{n} p_m c_m (1+i)^{-t_m}.$$

Note that we have not required the B_m to be independent (nor assumed anything at all about their dependence structure).

Example 105 A corporate bond is of this form, with $B_m = 1$ if $T > m$ for a default time T. A life annuity is also of this form, again with $B_m = 1$ if $T > m$ for a lifetime T.

If the amounts of a random cash-flow are fixed $C_j = c_j$ and only the times T_j are random, the net premium is

$$A = \sum_{m=1}^{n} c_m \mathbb{E}((1+i)^{-T_m}) = \sum_{m=1}^{n} c_m \mathbb{E}(e^{-\delta T_m}),$$

where $\delta = \log(1+i)$. These expectations are generating functions of T_m. Such a situation arises for life insurance payments, where a single payment is made at the time of death.

16.2 Expected yield

For deterministic cash-flows that can be interpreted as investment deals (or loan schemes), we defined the yield as an intrinsic rate of return. For a random cash-flow, this notion gives a random yield which is usually difficult to use in practice. Instead, we define:

Definition 106 Let C be a random cash-flow. The *expected yield* of C is the interest rate $i \in (-1, \infty)$, if it exists and is unique such that

$$\mathbb{E}(\mathrm{NPV}(i)) = 0,$$

where $\mathrm{NVP}(i) = i\text{-}\mathrm{Val}_0(C)$ denotes the net present value of C at time 0 discounted at interest rate i.

This corresponds to the yield of the "average cash-flow". Note that this terminology may be misleading – this is *not* the expectation of the yield of C, even if that were to exist.

Example 107 An investment of £500, 000 provides

- a continuous income stream of £50, 000 per year, starting at an unknown time S and ending in 6 years' time;
- a payment of unknown size A in 6 years' time.

What is the expected yield under the following assumptions?

- S is uniformly distributed between [2years, 3years] (time from now);
- the mean of A is £700, 000.

We use units of £10, 000 and 1 year. The time-0 value at rate y is

$$-50 + \int_{s=S}^{6} 5(1+y)^{-s} ds + (1+y)^{-6} A.$$

The expected time-0 value is

$$\begin{aligned} &-50 + \int_{s=2}^{6} \mathbb{P}(S < s) 5(1+y)^{-s} ds + (1+y)^{-6}\mathbb{E}(A) \\ &= -50 + \int_{s=2}^{3} (s-2) 5(1+y)^{-s} ds + \int_{s=3}^{6} 5(1+y)^{-s} ds + (1+y)^{-6} 70 =: f(y). \end{aligned}$$

Set $f(y) = 0$ and find $f(10.45\%) = 0.1016$ and $f(10.55\%) = -0.1502$. So the expected yield is 10.5% to 1d.p. (note that we only need to use the *mean* of A).

As a consequence of Proposition 102, we note:

Corollary 108 *If $T \sim \text{geom}(p)$ and c is a cash-flow at integer times with yield $y(c)$, then the expected yield of $C = c_{<T}$ is $p(1 + y(c)) - 1$.*

Proof: Note that $\text{NPV}_c(y(c)) = 0$, and by Proposition 102, we have $\mathbb{E}(\text{NPV}_C(i)) = 0$, if $y(c) = (1+i-p)/p$, i.e. $1+i = p(1+y(c))$, as required. Also, this is the unique solution to the expected yield equation as otherwise the relationship $1 + i = p(1 + k)$ would give more solutions to the yield equation. $\square$

16.3 Pooling of risk

What is risk? In real life, there is risk of death, risk of exhausting money reserves, risk of large expenditure etc. The insurance industry takes on certain risks against premium payment, e.g. as fire insurance or temporary life assurance. Mathematically, we can capture many of these risks in stochastic models as events with small probability and large cost – each risk can then be expressed as net present value of a random cash-flow.

Consider a random variable X of expenditure without insurance, and a random variable Y of expenditure with insurance. Often, Y is a deterministic premium payment completely eliminating the policy holder's risk, who is prepared to pay $Y > \mathbb{E}(X)$ for this elimination of risk.

Why would an insurer want to take on the risk? One incentive clearly is $\mathbb{E}(Y) > \mathbb{E}(X)$. The main reason, however, lies in the original idea of insurance that many pay a little for the big needs of a few, particularly when it is not known at the time of payment (or agreement to pay) who the few will be. Let us now formalise this pooling effect.

An insurer takes on many risks $X_1, \ldots, X_n$, say. In the simplest case, these can be assumed to be independent and identically distributed. This is a realistic assumption for a portfolio of identical life assurance policies issued to a homogeneous population. This is not a reasonable assumption for a flood insurance, because if one property is flooded due to extreme weather that must be expected to also affect other properties.

The random total claim amount $S = X_1 + \cdots + X_n$ must be met by premium payments $A = Y_1 + \cdots + Y_n$, say. The net premium $\mathbb{E}(S)$ leaves too much risk to the insurer, but we set premiums so that the probability of a loss for the insurer does not exceed $\varepsilon > 0$:

Proposition 109 *Given a random variable X_1 with mean μ and variance σ^2, representing the benefits from an insurance policy, we have*

$$\mathbb{P}\left(X_1 \geq \mu + \frac{\sigma}{\sqrt{\varepsilon}}\right) \leq \varepsilon,$$

and $A_1(\varepsilon) = \mu + \sigma/\sqrt{\varepsilon}$ is the premium to be charged to achieve a loss probability below ε.

Given independent and identically distributed $X_1, \ldots, X_n$ from n independent policies, we obtain

$$\mathbb{P}\left(\sum_{j=1}^{n} X_j \geq n\left(\mu + \frac{\sigma}{\sqrt{n\varepsilon}}\right)\right) \leq \varepsilon,$$

i.e. $A_n(\varepsilon) = \mu + \sigma/\sqrt{n\varepsilon}$ suffices if the risk of n policies is pooled.

Proof: The statements follow as consequences of Tchebychev's inequality:

$$\begin{aligned}
\mathbb{P}\left(\sum_{j=1}^{n} X_j \geq n\left(\mu + \frac{\sigma}{\sqrt{n\varepsilon}}\right)\right) &\leq \mathbb{P}\left(\left|\frac{1}{n}\sum_{j=1}^{n} X_j - \mu\right| \geq \frac{\sigma}{\sqrt{n\varepsilon}}\right) \\
&\leq \frac{\mathrm{Var}\left(\frac{1}{n}\sum_{j=1}^{n} X_j\right)}{\left(\frac{\sigma}{\sqrt{n\varepsilon}}\right)^2} = \frac{\sigma^2}{n\frac{\sigma^2}{n\varepsilon}} = \varepsilon.
\end{aligned}$$

□

The estimates used in this proposition are rather weak, and the premiums suggested require some modifications in practice, but adding a multiple of the standard deviation is one important method, also since often the variance, and hence the standard deviation, can be easily calculated. However, for large n, so-called safety loadings $A_n(\varepsilon) - \mu$ proportional to $n^{-1/2}$ are of the right order, e.g. for normally distributed risks, or in general by the Central Limit Theorem for large n, when

$$\mathbb{P}\left(\frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n\sigma^2}} \geq c\right) \approx \mathbb{P}(Z > c), \qquad \text{with } Z \text{ standard normally distributed.}$$

The important observation in these result is that the premiums $A_n(\varepsilon)$ decrease with n. This means, that the more policies an insurer can sell the smaller gets the (relative) risk, allowing him to reduce the premium. The proposition indicates this for identical policies, but in fact, this is a general rule about risks with sufficient independence and no unduly large risks.