Lecture 11

Duration, convexity and immunisation

Reading: McCutcheon-Scott Chapter 10, CT1 Unit 13.5

Suppose an institution holds assets of value V_A to meet liabilities of V_L and that at time 0, we have $V_A \geq V_L$. If interest rates applicable for discounting fall (rise), both V_A and V_L will increase (decrease). Under what conditions can we ensure that we still have $V_A \geq V_L$ under modified interest rates?

Obviously, full matching of our assets to our liabilities would achieve this. In practice however, full matching is difficult and it is instructive to ask in what circumstances might a partial match be sufficient?

In this lecture we introduce the notions of "volatility" and "convexity" of a cash-flow that reflect how present values of cash-flows change when interest rates change. Matching volatilities and dominating the convexity of liabilities provides a useful partial match.

11.1 Duration/volatility

For simplicity we assume a constant force/rate of interest. Consider a discrete cash-flow $C = ((t_k, c_k), 1 \leq k \leq n)$. Let

$$V = \sum_{k=1}^{n} c_k \left(\frac{1}{1+i} \right)^{-t_k} = \sum_{k=1}^{n} c_k e^{-t_k \delta}$$

be the present value of the payments in the constant-i model or equivalently in the constant-δ model. We will consider V to be a function of i or δ:

Definition 77 For a cash-flow C with present value V in the constant interest model, we introduce the notions of

(a) *volatility* or *effective duration* $\nu(i) = -\dfrac{d(\ln(V))}{di} = -\dfrac{1}{V}\dfrac{dV}{di}$.

(b) *discounted mean term (DMT)* or *MacAuley duration* $\tau(\delta) = -\dfrac{d(\ln(V))}{d\delta} = -\dfrac{1}{V}\dfrac{dV}{d\delta}$.

Note that with $1+i=e^\delta$ we have $di/d\delta = e^\delta$, or with $\delta = \log(1+i)$ we have $d\delta/di = (1+i)^{-1}$. Therefore

$$\tau(\delta) = -\frac{1}{V}\frac{dV}{d\delta} = -\frac{1}{V}\frac{dV}{di}\frac{di}{d\delta} = (e^\delta - 1)\nu(e^\delta - 1) = (1+i)\nu(i).$$

Furthermore,

$$\nu(i) = -\frac{1}{V}\frac{dV}{di} = \frac{1}{V}\sum_{k=1}^n c_k t_k \left(\frac{1}{1+i}\right)^{t_k+1}, \quad \tau(\delta) = \sum_{k=1}^n t_k \frac{c_k e^{-\delta t_k}}{\sum_{j=1}^n c_j e^{-\delta t_j}},$$

where the last formula explains the name "discounted mean term", since $\tau(\delta)$ is indeed the mean term of the cash-flow, weighted by present value, provided that all weights have the same sign, i.e. provided that the cash-flow has either only inflows or only outflows.

Example 78 DMT of an n-year coupon-paying bond, annual coupons of D, redemption proceeds of R, is

$$\tau = \frac{D(Ia)_{\overline{n}|} + Rnv^n}{Da_{\overline{n}|} + Rv^n}, \qquad \text{where } v = (1+i)^{-1} = e^{-\delta}.$$

To see this, note that

$$V = D\sum_{k=1}^n v^k + Rv^n \quad \Rightarrow \quad \frac{dV}{d\delta} = D\sum_{k=1}^n (-k)v^k - Rnv^n = -D(Ia)_{\overline{n}|} - Rnv^n.$$

Note that for $D=0$ and $R=1$, we obtain the special case of a zero-coupon bond *of duration* n, which has DMT n. Note however, that it is the MacAuley duration, not the effective duration which equals n. The effective duration is $n(1+i)^{-1}$.

11.2 Convexity

Definition 79 For a cash-flow C with present value A in the constant interest model, we introduce the notion of *convexity* $c(i) = \frac{1}{V}\frac{d^2V}{di^2}$.

For a series of cash-flows $C = (t_k, c_k)$, $1 \le k \le n$, this is

$$c(i) = \frac{1}{V}\sum_{k=1}^n c_k t_k (t_k+1)\left(\frac{1}{1+i}\right)^{t_k+2}.$$

Convexity gives a measure of the change in duration of a bond when the interest rate changes.

Example 80 (a) For a zero-coupon bond of duration n, we obtain

$$V_n = (1+i)^{-n} \quad \Rightarrow \quad \nu_n(i) = n(1+i)^{-1} \quad \text{and} \quad c_n(i) = n(n+1)(1+i)^{-2}$$

.

(b) For the sum of two zero-coupon bonds with $V = (1+i)^{-(n-m)} + (1+i)^{-(n+m)}$ we can derive simple expressions as averages of the respective quantities for zero-coupon bonds, with notation as in (a):

$$\nu(i) = \nu_{n-m}(i)\frac{(1+i)^m}{(1+i)^m + (1+i)^{-m}} + \nu_{n+m}(i)\frac{(1+i)^{-m}}{(1+i)^m + (1+i)^{-m}}$$

and

$$c(i) = c_{n-m}(i)\frac{(1+i)^m}{(1+i)^m + (1+i)^{-m}} + c_{n+m}(i)\frac{(1+i)^{-m}}{(1+i)^m + (1+i)^{-m}}.$$

Note in particular that in the relevant case of small $i > 0$ and small $n \geq 1$ the largest convexities among $m \in \{0, \ldots, n\}$ are found for the most spread-out cases of m close to n. In the next section, we will see that suitable mixtures of short and long term assets can provide protection against changes in interest rates.

11.3 Immunisation

Definition 81 Consider a fund with asset cash-flow A and liability cash-flow L. Let V_A and V_L be their present values. We say that at interest rate i_0 the fund is *immunised against small movements in the interest rate* if $V_A(i_0) = V_L(i_0)$ and if there is $\varepsilon > 0$ such that $V_A(i) \geq V_L(i)$ for all $i \in (i_0 - \varepsilon, i_0 + \varepsilon)$.

Theorem 82 (Redington) *If $V_A(i_0) = V_L(i_0)$, $\nu_A(i_0) = \nu_L(i_0)$ and $c_A(i_0) > c_L(i_0)$, then at rate i_0, the fund is immunised against small movements in the interest rate.*

Proof: Consider the surplus $S(i) = V_A(i) - V_L(i)$. By Taylor's theorem, we have

$$\begin{aligned} S(i) &= S(i_0) + (i-i_0)S'(i_0) + \tfrac{1}{2}(i-i_0)^2 S''(i_0) + O((i-i_0)^3) \\ &= (V_A(i_0) - V_L(i_0)) - (i-i_0)(V_A(i_0)\nu_A(i_0) - V_L(i_0)\nu_L(i_0)) \\ &\qquad + \tfrac{1}{2}(i-i_0)^2\left(V_A(i_0)c_A(i_0) - V_L(i_0)c_L(i_0)\right) + O((i-i_0)^3) \\ &= 0 - 0 + \tfrac{1}{2}(i-i_0)^2 V_A(i_0)(c_A(i_0) - c_L(i_0)) + O((i-i_0)^3) \geq 0 \end{aligned}$$

for $|i - i_0|$ sufficiently small. □

11.4 Limitations of classical immunisation theory

1. The theory relies on a *small* change in interest rates. The fund may not be protected against large changes. In practice, this is not usually a problem as the theory is fairly robust; only large changes and strange liabilities may lead to problems at this point; rebalancing helps when interest rates change gradually rather than abruptly.

2. The need of constant rebalancing of the portfolio is not unproblematic as this can be costly, in practice.

3. We assumed a constant interest rate now and at future time. This is rather suspect since yield curves are not flat in practice. On the problem sheet, we even point out some arbitrage problems under such model assumptions when a flat yield curve shifts up or down.

4. The theory is aimed at meeting fixed monetary liabilities, whereas in practice many liabilities are real. The theory can be adjusted to include inflation by using index-linked assets, but time lags may be a problem, also when rebalancing.

5. Assets of suitably long term may not exist.

6. There may be uncertainties in timing or amount of liability outgo.

What is done in practice? A broader risk management is based on asset-liability modelling using stochastic models with Monte-Carlo simulation, sensitivity analysis and/or scenario testing.

Lecture 12

No arbitrage and forward contracts

Reading: CT1 Unit 10.4, 12

Arbitrage is a risk-free trading profit. In practice, arbitrage opportunities exist, but they are usually quickly eliminated, since markets are driven by supply and demand and particularly financial markets are highly efficient: whenever there is an arbitrage opportunity, arbitrageurs buy a product at a cheap price in one market, this extra demand meets the cheapest supply thereby increasing the remaining supply price; they sell the product at a higher price typically in another market, this extra supply meets the highest demand thereby decreasing the remaining demand price; arbitrageurs exploit such opportunities until all supply prices exceed all demand prices. With arbitrageurs constantly removing arbitrage opportunities, other market participants act in virtually arbitrage-free markets.

12.1 Arbitrage and the law of one price

Definition 83 We say that an *arbitrage opportunity* exists if either

- an investor can make a deal giving an immediate profit, with no risk of future loss,
- or an investor can make a deal that has zero initial cost, no risk of future loss, and a non-zero probability of a future profit.

We will not make use of the second bullet point. An important consequence (that only relies on the first bullet point) is that no arbitrage implies the Law of One Price. Its proof can be seen as an illustratation of the exploitation of arbitrage opportunities.

Definition 84 The *Law of One Price* (LOOP) stipulates that any two assets with identical cash-flows in all scenarios must trade at the same price on all markets.

Proposition 85 *An assumption of no arbitrage implies the Law of One Price.*

Proof: Assume that an asset trades at different prices on different markets. Then an arbitrageur will buy the asset on a market with a lower price and sell it on a market with a higher price. Then an immediate profit is made and all future cash-flow is zero leaving no risk of future loss – this is arbitrage. □

We have applied the Law of One Price implicitly all the time. We have now shown that failure of the Law of One Price implies arbitrage. And in a mathematical model, arbitrage essentially means $0 = 1$, from where we could deduce anything.

12.2 Standard form of no arbitrage pricing argument

Pricing a derivative security under an assumption of no arbitrage usually makes use of replicating portfolio arguments. We consider two portfolios involving

A: the derivative security (plus cash),

B: a portfolio of underlying securities (plus cash),

where B is constructed to provide exactly the same payoffs as A in all possible scenarios.

Under the no-arbitrage assumption, which implies the Law of one Price, A and B must have the same price at any time. But we know the price of portfolio B as a weighted sum of assets we know the prices of. This gives the price of the derivative security.

In practice, there are transaction costs to buy derivative securities, fulfil the derivative security contract, buy or sell the underlying security. We will ignore transaction costs to focus on the key part of the no-arbitrage argument.

12.3 No-arbitrage computation of forward prices

Example 86 (Forward contract to buy a security with no income) Let

- S_t be the market price of the underlying security at time $0 \le t \le T$; consider as known the present market price S_0, but not future market prices S_t, $0 < t \le T$,
- δ a known constant force of interest on risk-free investments,
- K the forward price to be determined, i.e. the price agreed at time $t = 0$ to be paid at time $t = T$ to purchase the underlying security at time $t = T$,

Under the forward contract, no money changes hands until time $t = T$, i.e. the forward contract does not cost anything, it's just setting a price at time $t = 0$ for a purchase at time $t = T$. To compute the (unique arbitrage-free) forward price K, consider

A: enter into the forward contract to buy asset S with forward price K maturing at time T; buy $Ke^{-\delta T}$ units of the risk-free asset at time $t = 0$;

B: buy one unit of the asset S at the current market price S_0 at $t = 0$.

The only payoff is at time T; under A, the forward contract is worth $S_T - K$, the cash K; under B the asset is worth S_T, so the payoff is the same. By LOOP, A and B have the same price at time $t = 0$, hence $0 + Ke^{-\delta T} = S_0$, i.e. $K = S_0 e^{\delta T}$. We have not made any assumptions on the distribution of S_T other than no-arbitrage. In particular, K does not depend on such distributional assumptions on S_T, which is surprising at first sight.

What if the actual forward price K_{actual} exceeds $K = S_0 e^{\delta T}$? Arbitrageurs would buy B and sell A (until $K_{\text{actual}} = K$).

More generally, it is not necessary to assume a constant force of interest. All we needed to work out $K = S_0 e^{\delta T}$ was a discount factor from time $t = T$ to time $t = 0$. In practice, this discount factor is reflected in the time-0 price P_0 of a zero-coupon bond maturing at time T paying £1. This is our risk-free asset with which portfolio A is set up by buying K units at price KP_0. Reasoning as above, we obtain $K = S_0/P_0$. Since $1/P_0$ is the associated accumulation factor from time $t = 0$ to time $t = T$, we can read $K = S_0/P_0$ as the current price S_0 of the asset accumulated risk-free.

Example 87 (Forward contract to buy a security with fixed cash income) Suppose that the security underlying the forward contract provides a fixed amount c_1 at time $t_1 \in (0, T)$ to the holder. With notation as in the previous example, consider

A: enter into the forward contract to buy asset S with forward price K maturing at time T; invest $Ke^{-\delta T} + c_1 e^{-\delta t_1}$ into the risk-free asset at time $t = 0$;

B: buy one unit of the asset S at the current market price S_0 at $t = 0$. At time t_1, invest the income of c_1 in the risk-free asset.

At time T,

A: Forward contract: $S_T - K$; risk-free holding: $K + c_1 e^{\delta(T-t_1)}$;

B: Asset S: S_T; risk-free holding from coupon: $c_1 e^{\delta(T-t_1)}$.

By LOOP, A and B have the same price at $t = 0$:

$$0 + Ke^{-\delta T} + ce^{-\delta t_1} = S_0 \quad \Rightarrow \quad K = S_0 e^{\delta T} - c_1 e^{\delta(T-t_1)}.$$

More generally, $K = (S_0 - I)e^{\delta T}$, where I is the present value at time $t = 0$ of the fixed income payments due during the term of the forward contract.

Example 88 (Forward contract to buy a security with known dividend yield) Suppose that the security underlying the forward contract pays dividend continuously at rate D. Such income is not fixed since the dividend rate is applied to the market price S_t that varies with t and is unknown for $t \in (0, T)$. If we set up portfolios as before, but now reinvesting dividend in the security, the accumulated holding at time T would be e^{DT} units of the security, since the *number of units* of the security held as t varies behaves like a bank account accumulating interest continuously at rate D. Instead, consider

A: enter into the forward contract to buy asset S with forward price K maturing at time T; invest $Ke^{-\delta T}$ into the risk-free asset at time $t = 0$;

B: buy e^{-DT} units of the asset S at the current market price S_0 at $t = 0$. Reinvest dividend income in S immediately when it is received.

At time T,

A: Forward contract: $S_T - K$; risk-free holding: K;

B: Asset S: $e^{DT} e^{-DT} S_T = S_T$;

By LOOP, A and B have the same price at $t = 0$:

$$0 + Ke^{-\delta T} = S_0 e^{-DT} \quad \Rightarrow \quad K = S_0 e^{(\delta - D)T}.$$

Note, we can work out K if fixed income is reinvested in the risk-free asset and income proportional to S is reinvested in S.

12.4 Values of forward contracts

The standard form of the no-arbitrage pricing argument can also be applied to assign a no-arbitrage value to forward contracts at intermediate times.

Example 89 (Forward contract to buy a security with no income) The forward contract initially changes hands at no cost to either party. The sole purpose has been to fix the forward price $K = S_0 e^{\delta T}$. However, at maturity T, the contract is worth $S_T - K$ to the buyer (and $K - S_T$ to the seller). What about intermediate times? Let V_r denote the value of the forward contract at time $r < T$. Consider

A: at time r, pay V_r to take over the forward contract to buy asset S at forward price K at time T; buy $Ke^{-\delta(T-r)}$ units of the risk-free asset at time $t = r$;

B: buy one unit of the asset S at the current market price S_r at $t = r$.

Then the payoffs of A and B (all at time T) are equal, and by LOOP the values of A and B at $t = r$ must be equal, i.e.

$$V_r + Ke^{-\delta(T-r)} = S_r \quad \Rightarrow \quad V_r = S_r - Ke^{-(T-r)\delta} = S_r - S_0 e^{r\delta}.$$

Similar reasoning can be applied with fixed or dividend income.

Terminology and warning. A forward contract legally binds two parties to act, respectively, as seller and buyer of the underlying asset at a given future time for a given price. Since the forward contract has no money value at issue, the parties "enter" the contract, they do not "buy" the contract. We adopt the usual jargon that the party committing to buy (resp. sell) enters a "long" (resp. "short") forward contract. In fact, the buyer holds the stock in the long term, after the agreed sale, the seller in the short term, before the agreed sale. The similar term "short-selling" refers to selling stock not actually owned but borrowed. It is mathematically convenient to allow short-selling. In practice, short-selling is also possible, but there are some legal restrictions.

We have assigned arbitrage-free values to the contract at intermediate times. It is instructive (but neglecting some legal issues) to think of the long (or the short) contract as a piece of paper that someone else can "buy", but the value may well be negative so that the term "buying" can be misleading. It is more appropriate to "take over" the long (or the short) position of the contract.

Lecture 13

Term structure of interest rates

Reading: CT1 Unit 13, McCutcheon-Scott Section 10.2

In practice, the interest rate offered on investments usually varies according to the term of the investment. Thinking particularly of government bonds, we find prices for a wide range of maturities. In this lecture, we use the method of arbitrage-free pricing to deduce from such bond prices an implied time-varying interest-rate model, called *term structure of interest rates.*

13.1 Spot rates as yields of zero-coupon bonds

Definition 90 Given a zero-coupon bond maturing at time t and trading at a price P_t at time 0, we call its yield y_t the *t-year spot rate of interest.* We refer to $Y_t = \log(1 + y_t)$ as the *t-year spot force of interest.*

The equation of value for this zero-coupon bond is $P_t = (1 + y_t)^{-t} = e^{-tY_t}$, so we obtain $y_t = (P_t)^{-1/t} - 1$ and $Y_t = -t^{-1}\log(P_t)$. Also, By no-arbitrage arguments, every riskless fixed-interest investment (discrete cash-flow) can be regarded as a combination of (perhaps notional) zero-coupon bonds.

Example 91 Consider a bond of nominal amount N paying annual coupons at rate j per unit nominal and redeemable at R per unit nominal at time n.

A: Buy the coupon bond at time 0 for a price P;

B: At time 0, buy n zero-coupon bonds with maturity values $jN, \dots, jN, jN + RN$ at times $1, \dots, n-1, n$.

The cash-flows in portfolios A and B are equal. By LOOP, they have the same price

$$P = jN(P_1 + \cdots + P_{n-1}) + (jN + RN)P_n = jN(v_{y_1} + v_{y_2}^2 + \cdots + v_{y_n}^n) + RNv_{y_n}^n,$$

where $v_{y_k} = (1 + y_k)^{-1}$.

13.2 Forward rates

A forward interest rate is an interest rate implied by current zero-coupon rates for a specified future time period. We denote by $f_{t,r}$ an annual effective interest rate agreed at time zero for an investment made at time $t > 0$ for r years.

Proposition 92 *Under the assumption of no-arbitrage*

$$(1 + f_{t,r})^r = (1 + y_{t+r})^{t+r}/(1 + y_t)^t = P_t/P_{t+r}.$$

Proof: Consider two portfolios set up at time $t = 0$:

A: Spend £1 on a term-t zero-coupon bond and enter into a forward contract to invest £$(1 + y_t)^t$ at time t for r years at rate $f_{t,r}$ p.a.;

B: Spend £1 on a term-$(t + r)$ zero-coupon bond.

Both portfolios have price £1 at time 0 and provide riskless payoff at time $t + r$. By no-arbitrage, the payoffs must be the same, so

$$(1 + y_t)^t(1 + f_{t,r})^r = (1 + y_{t+r})^{t+r}.$$

Since $P_t = (1 + y_t)^{-t}$, we also obtain

$$(1 + f_{t,r})^r = \frac{(1 + y_{t+r})^{t+r}}{(1 + y_t)^t} = \frac{P_t}{P_{t+r}}.$$

□

To a forward rate $f_{t,r}$, we associate the equivalent force of interest $F_{t,r} = \log(1 + f_{t,r})$. Then

$$e^{rF_{t,r}} = (1 + f_{t,r})^r = P_t/P_{t+r} \quad \Rightarrow \quad F_{t,r} = \frac{1}{r}\log(P_t/P_{t+r}) = \frac{\log(P_t) - \log(P_{t+r})}{r}.$$

Note also that geometric averages of rates turn into arithmetic averages of forces:

$$e^{(t+r)Y_{t+r}} = (1+y_{t+r})^{t+r} = (1+y_t)^t(1+f_{t,r})^r = e^{tY_t}e^{rF_{t,r}} \quad \Rightarrow \quad Y_{t+r} = \frac{t}{t+r}Y_t + \frac{r}{t+r}F_{t,r}.$$

13.3 Instantaneous forward rates

Spot rates and implied forward rates describe a market interest rate model. As a time-varying interest rate model, it is natural to express all interest rates and discount factors in terms of an "implied time-varying force of interest".

Recall that for discount factors $v(t) = \exp(-\int_0^t \delta(s)ds)$, we have $\delta(t) = -(\log(v))'(t) = -v'(t)/v(t)$. Here, our discount factors are P_t, since P_t is the price of a time-t zero-coupon bond and hence precisely the amount to invest so as to dispose of 1 at time t. Hence, we want

$$F_t = -\frac{\partial}{\partial t}(\log(P_t)) = \lim_{r\downarrow 0}\frac{\log(P_t) - \log(P_{t+r})}{r} = \lim_{r\downarrow 0} F_{t,r},$$

and in practice, when working from market prices of real-world assets, such a limit will only ever stand for an approximation "for small r". The force F_t is referred to as the *instantaneous forward rate*. In theory, we can now write $P_t = \exp(-\int_0^t F_s ds)$.

13.4 Yield curves

For market spot rates y_t, the function $t \mapsto y_t$ is called *yield curve*. There are three main shapes of yield curves: increasing, decreasing, and humped, where the latter means that medium-term interest rates are higher than both short- and long-term rates. A constant yield curve is also called "flat".

In Example 91, we expressed coupon bond prices in terms of spot rates, based on the coupon rate. We now turn this equation of value around:

Definition 93 The *n-year par yield* is defined as the coupon rate py_n for which a coupon bond trades at par.

Since $1 = py_n(v_{y_1} + v_{y_2}^2 + \cdots + v_{y_n}^n) + v_{y_n}^n$, we get

$$py_n = \frac{1 - v_{y_n}^n}{v_{y_1} + v_{y_2}^2 + \cdots + v_{y_n}^n} = \frac{1 - v_{y_n}}{v_{y_n}} \frac{v_{y_n} + v_{y_n}^2 + \cdots + v_{y_n}^n}{v_{y_1} + v_{y_2}^2 + \cdots + v_{y_n}^n} = y_n \frac{v_{y_n} + v_{y_n}^2 + \cdots + v_{y_n}^n}{v_{y_1} + v_{y_2}^2 + \cdots + v_{y_n}^n}.$$

Clearly, if the yield curve is flat at $y_t = i$, $t \geq 0$, then the par yield curve $n \mapsto py_n$ is also flat at $py_n = i$. If $n \mapsto y_n$ increases, then $n \mapsto py_n$ also increases, but does so more slowly.

Why do interest rates vary over time?

1. Supply and Demand: different institutions demand and supply bonds of different terms (e.g. pension fund demand may drive long term yield, while bank demand may drive short-term yields.)

2. Base rate: the rate of interest which the central bank (e.g. Bank of England) charges to commercial banks in its role as lender of last resort. One may argue that forward rates reflect expected changes in the base rate.

3. Liquidity preference: longer dated bonds are more sensitive to interest rate movements, so a risk averse investor may require a higher yield to compensate.

13.5 Example

The following n-year spot rates y_n were observed at time $t = 0$.

n	1	2	3	4	5	6
y_n	4%	5%	6%	7%	7.5%	8%

We can calculate the implied forward rates, e.g. $f_{3,2}$ satisfies $(1+y_5)^5 = (1+y_3)^3(1+f_{3,2})^2$ and so $f_{3,2} = 9.79\%$. This is higher than the spot rates since the spot rates are geometric averages of forward rates and so the later forward rates have to make up for low short-term spot rates. Indeed, if the yield curve is increasing, the forward rates increase more rapidly. Here,

n	1	2	3	4	5	6
$f_{n-1,1}$	4.00%	6.01%	8.03%	10.06%	9.52%	10.54%

We can also calculate par yields, e.g. the six-year par yield py_6 at time $t = 0$ satisfies the equation of value

$$1 = py_6 \left(\frac{1}{1.04} + \frac{1}{1.05^2} + \frac{1}{1.06^3} + \frac{1}{1.07^4} + \frac{1}{1.075^5} + \frac{1}{1.08^6} \right) + \frac{1}{1.08^6}$$

which is solved by $py_6 = 7.71\%$.

n	1	2	3	4	5	6
py_n	4.00%	4.98%	5.92%	6.83%	7.28%	7.71%

Note that the par yields are lower than the spot yields, because, in the interest rate model given by the forward rates, the equal coupon payments are higher than the interest due in earlier years, so if the coupon was paid at the spot rate, some payment would happen too early leading to a discounted value above par.

Lecture 14

Stochastic interest-rate models

Reading: McCutcheon-Scott Chapter 12, CT1 Unit 14

So far, we usually assumed that we knew all interest rates, or we compared investments under different interest rate assumptions. We have indicated how to model uncertainty of investment proceeds by random variables, as part of generalised cash-flows. In practice, interest rates themselves are not always fixed in advance and are therefore uncertain. In this lecture, we model interest rates by random variables.

14.1 Basic model: the constant-I model

Just as the constant-i model is the proto-type of an interest rate model, we now start from the corresponding model, where i is replaced by a random variable I (which does not vary with time). Examples also include yields of generalised cash-flows, but let us keep cash-flows non-random in this lecture and look at a more elementary example.

Example 94 Suppose, you invest £100 for 1 year at an interest rate I not known in advance. The interest rate for the previous year was 3% and you expect one of three possibilities, equally likely: a rise by 1%, no change or a fall by 1%, i.e.

$$\mathbb{P}(I = 2\%) = \mathbb{P}(I = 3\%) = \mathbb{P}(I = 4\%) = 1/3.$$

Let $R = 100(1 + I)$ be the accumulated value at time 1. Then

$$\mathbb{P}(R = 102) = \mathbb{P}(R = 103) = \mathbb{P}(R = 104) = 1/3$$

and hence $\mathbb{E}(R) = \dfrac{102 + 103 + 104}{3} = 103 \qquad \left[= 100(1 + \mathbb{E}(I)). \right]$

However, for a different term t and $R_t = 100(1 + I)^t$, we have $\mathbb{P}(R_t = 100(1.02)^t) = \mathbb{P}(R_t = 100(1.03)^t) = \mathbb{P}(R_t = 100(1.04)^t) = 1/3$ and

$$\mathbb{E}(R_t) = 100\frac{(1.02)^t + (1.03)^t + (1.04)^t}{3} \neq 100(1.03)^t \qquad \text{unless } t = 1.$$

Assuming that I can only take 3 values is of course an unnecessary restriction, and we can take as stochastic interest rate any random variable I that ranges $(-1, \infty)$, discrete or continuous.

Proposition 95 *Given a stochastic interest rate I. Invest c at time 0. Then its expected accumulated value at time t is given by*

$$\mathbb{E}(R_t) = \mathbb{E}(I\text{-Val}_t((0,c))) = \mathbb{E}(c(1+I)^t).$$

Let $\lambda = \mathbb{E}(I)$ and $\mathbb{P}(I \neq \lambda) > 0$. Then

- *if $t < 1$, then $\mathbb{E}(R_t) < \lambda\text{-Val}_t((0,c))$,*
- *if $t > 1$, then $\mathbb{E}(R_t) > \lambda\text{-Val}_t((0,c))$.*

Proof: The proof of the inequalities is essentially an application of Jensen's inequality (see following lemma) for the function $f(x) = (1+x)^t$ which is *convex* (increasing derivative, $f'' > 0$) if $t > 1$ and *concave* (decreasing derivative, $f'' < 0$) if $t < 1$, so that then $-f$ is convex. □

Lemma 96 (Jensen's inequality) *For any convex function $f : \mathbb{R} \to \mathbb{R}$ and any random variable X with $\mathbb{E}(X) \in \mathbb{R}$, we have*

$$f(\mathbb{E}(X)) \leq \mathbb{E}(f(X)).$$

If f is strictly convex and $\mathbb{P}(X = \mathbb{E}(X)) < 1$, then the inequality is strict.

The result is still true if f is only defined on an interval $J \subset \mathbb{R}$ with $\mathbb{P}(X \in J) = 1$.

Proof: By Taylor's formula

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + (x - x_0)^2 f''(\xi)/2 \geq f(x_0) + (x - x_0)f'(x_0)$$

for all $x, x_0 \in \mathbb{R}$ (strict inequality for $x \neq x_0$). For $x = X$ and $x_0 = \mathbb{E}(X)$, we obtain

$$f(X) \geq f(\mathbb{E}(X)) + (X - \mathbb{E}(X))f'(\mathbb{E}(X)),$$

and since $\mathbb{E}$ is linear and order-preserving, we conclude that

$$\mathbb{E}(f(X)) \geq \mathbb{E}\left(f(\mathbb{E}(X)) + (X - \mathbb{E}(X))f'(\mathbb{E}(X))\right) = f(\mathbb{E}(X)).$$

□

Obviously, instead of modelling the interest rate I, we could model the force of interest $\Delta = \log(1 + I)$. This is useful since calculating expected discounted/accumulated values is then based on Laplace transforms/moment generating function $\mathbb{E}(e^{-t\Delta})$ and $\mathbb{E}(e^{t\Delta})$ of Δ.

14.2 Independent interest rates

The model in the previous section is often artificial (particularly for long terms). It is natural to allow the interest rate to change. Interest rates change. Another simple model is to take independent identically distributed (i.i.d.) annual interest rates. Let $I_j \in (-1, \infty)$ be the interest rate in year j (i.e. between times $j-1$ and j). Let us concentrate on the accumulation factor S_n from 0 to n:

$$S_n = (1 + I_1) \cdots (1 + I_n).$$

Proposition 97 *Suppose I_j, $j \geq 1$ are i.i.d. Consider a payment of c at time 0. Its expected accumulated value at $n \in \mathbb{N}$ is $\mathbb{E}(cS_n) = c(1 + \mathbb{E}(I_1))^n$.*

Proof: $\mathbb{E}(cS_n) = \mathbb{E}\left(c\prod_{j=1}^{n}(1 + I_j)\right) = c\prod_{j=1}^{n}\mathbb{E}(1 + I_j) = c\prod_{j=1}^{n}(1 + \mathbb{E}(I_j)) = c(1 + \mathbb{E}(I_1))^n$, where the second equality uses independence and the last identical distribution of the I_j. □

Similarly, there are formulas for discounted values (but note that $\mathbb{E}((1 + I_1)^{-1}) > (1+\mathbb{E}(I_1))^{-1}$, by Jensen's inequality), and for higher moments (involving higher moments of I_1). In particular

$$\text{Var}(cS_n) = c^2\left(\mathbb{E}(S_n^2) - (\mathbb{E}(S_n))^2\right) = c^2\left(1 + 2\mathbb{E}(I_1) + (\mathbb{E}(I_1))^2 + \text{Var}(I_1))^n - (1 + \mathbb{E}(I_1))^{2n}\right).$$

We leave the details as an exercise.

14.3 The log-normal distribution

If $\Delta_j = \log(1 + I_j) \sim \text{Normal}(\mu, \sigma^2)$, then the distribution of $1 + I_j \sim \text{logN}(\mu, \sigma^2)$ is called the *log-normal distribution with parameters $\mu \in \mathbb{R}$ and $\sigma^2 > 0$*. Note that

$$\mu = \mathbb{E}(\Delta_j) = \mathbb{E}(\log(1 + I_j)) \quad \text{and} \quad \sigma^2 = \text{Var}(\Delta_j) = \mathbb{E}((\log(1 + I_j))^2) - (\mathbb{E}(\log(1 + I_j)))^2,$$

so the parameters of the log-normal distribution are *not* mean and variance.

Proposition 98 *If $1 + I \sim \text{logN}(\mu, \sigma^2)$, then*

$$\lambda := \mathbb{E}(I) = \exp(\mu + \sigma^2/2) - 1 \qquad \textit{and} \qquad s^2 = \text{Var}(I) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1).$$

Proof: We need the moment generating function of $\Delta \sim \text{Normal}(\mu, \sigma^2)$:

$$\begin{aligned}
\mathbb{E}(e^{t\Delta}) &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} e^{tx} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} \exp\left(-\frac{(x-\mu+\sigma^2 t)^2}{2\sigma^2}\right) \exp\left(\frac{(\mu+\sigma^2 t)^2 - \mu^2}{2\sigma^2}\right) dx \\
&= \exp\left(\mu t + \frac{\sigma^2}{2}t^2\right).
\end{aligned}$$

So $\mathbb{E}(1+I) = \mathbb{E}(e^{\Delta}) = \exp(\mu + \sigma^2/2)$ and

$$\begin{aligned}\mathrm{Var}(I) = \mathrm{Var}(1+I) = \mathrm{Var}(e^{\Delta}) &= \mathbb{E}(e^{2\Delta}) - \big(\mathbb{E}(e^{\Delta})\big)^2 \\ &= \exp(2\mu + 2\sigma^2) - \exp(2\mu + \sigma^2) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1).\end{aligned}$$

□

The log-normal distribution is useful to calculate probabilities that an accumulation falls into an interval, or exceeds a threshold, because for $Y \sim \mathrm{logN}(\mu, \sigma^2)$, we have

$$\mathbb{P}(Y \in [a,b]) = \mathbb{P}(\log(Y) \in [\log(a), \log(b)]) = \int_{\log(a)}^{\log(b)} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-(x-\mu)^2/2\sigma^2)dx.$$

By a simple change of variables $x = \log(y)$, we can also identify the probability density function $f_Y(y)$, but it will not be useful for us. Instead, what is useful is the property of the normal distribution that

$$X \sim \mathrm{Normal}(\mu, \sigma^2) \quad \Rightarrow \quad Z = \frac{X-\mu}{\sigma} \sim \mathrm{Normal}(0,1),$$

which follows from a simple moment generating function calculation

$$\mathbb{E}(e^{tZ}) = \mathbb{E}(e^{t(X-\mu)/\sigma}) = e^{-\mu t/\sigma}\mathbb{E}(e^{(t/\sigma)X}) = e^{-\mu t/\sigma}e^{(t/\sigma)^2\sigma^2/2+(t/\sigma)\mu} = e^{t^2/2}.$$

This is the moment generating function of Normal(0, 1).

Example 99 Let $1 + I_1, \ldots, 1 + I_n$ be independent log-normal random variables with common parameters μ and σ^2. We can calculate the distribution of the accumulated value at time n of a unit investment at time 0.

$$S_n = \prod_{j=1}^{n}(1+I_j) = \exp\left\{\sum_{j=1}^{n}\Delta_j\right\} \sim \mathrm{logN}(n\mu, n\sigma^2)$$

since sums of independent normal random variables are normal with as parameters the sums of the individual parameters (check this using moment generating functions).

Assume that $\mu = 0.04$, $\sigma = 0.02$ and $n = 5$. If we want to accumulate at least £600,000 with probability 99%, we have to invest A where

$$\begin{aligned} 0.99 &= \mathbb{P}(AS_n > 600,000) = \mathbb{P}(\log\{S_n\} > \log\{600,000/A\}) \\ &= \mathbb{P}\left(Z > \frac{\log\{600,000/A\} - n\mu}{\sqrt{n\sigma^2}}\right) \\ \Rightarrow -2.33 &= \frac{\log\{600,000/A\} - n\mu}{\sqrt{n}\sigma} \Rightarrow A = 600,000\exp\{2.33\sqrt{n}\sigma - n\mu\} = 545,187.90 \end{aligned}$$

Here we used that $\mathbb{P}(Z > -2.33) = 0.99$ for a standard normal random variable Z.

If $1 + I_j$ are i.i.d. but not log-normal, we can approximate by the Central Limit Theorem. We have

$$S_n = (1+I_1)(1+I_2)\cdots(1+I_n) = \exp(\Delta_1 + \cdots + \Delta_n).$$

So, let $\mu = \mathbb{E}(\Delta_1) = \mathbb{E}(\log(1+I_1))$ and $\sigma^2 = \mathrm{Var}(\Delta_1)$. Then $\Delta_1+\cdots+\Delta_n$ is approximately $\mathrm{N}(n\mu, n\sigma^2)$, so S_n is approximately $\mathrm{logN}(n\mu, n\sigma^2)$.