Lecture 1

Introduction

Reading: CT1 Core Reading Unit 1
Further reading: http://www.actuaries.org.uk

After some general information about relevant history and about the work of an actuary, we introduce cash-flow models as the basis of this course and as a suitable framework to describe and look beyond the contents of this course.

1.1 The actuarial profession

Actuarial Science is a discipline with its own history. The Institute of Actuaries was formed in 1848, (the Faculty of Actuaries in Scotland in 1856, the two merged in 2010), but the roots go back further. An important event was the construction of the first life table by Sir Edmund Halley in 1693. However, Actuarial Science is not oldfashioned. The language of probability theory was gradually adopted as it developed in the 20th century; computing power and new communication technologies have changed the work of actuaries. The growing importance and complexity of financial markets continues to fuel actuarial work; current debates and changes in life expentancy, retirement age, viability of pension schemes are core actuarial topics that the profession vigorously embraces.

Essentially, the job of an actuary is risk assessment. Traditionally, this was insurance risk, life insurance, later general insurance (health, home, property etc). As typically large amounts of money, reserves, have to be maintained, this naturally extended to investment strategies including the assessment of risk in financial markets. Today, the Actuarial Profession claims yet more broadly to make "financial sense of the future".

To become a Fellow of the Institute/Faculty of Actuaries in the UK, an actuarial trainee has to pass nine mathematics, statistics, economics and finance examinations (core technical series – CT), examinations on risk management, reporting and communication skills (core applications – CA), and three specialist examinations in the chosen areas of specialisation (specialist technical and specialist applications series – ST and SA) and for a UK fellowship an examination on UK specifics. This programme takes normally at least three or four years after a mathematical university degree and while working for an insurance company under the guidance of a Fellow of the Institute/Faculty of Actuaries.

This lecture course is an introductory course where important foundations are laid and an overview of further actuarial education and practice is given. An upper second mark in the examination following the full OBS4/BS4 unit normally entitles to an exemption from the CT1 paper. The CT3 paper is covered by the Part A Probability and Statistics courses. A further exemption, from CT4, is available for BS3 Stochastic Modelling.

1.2 The generalised cash-flow model

The cash-flow model systematically captures payments either between different parties or, as we shall focus on, in an inflow/outflow way from the perspective of one party. This can be done at different levels of detail, depending on the purpose of an investigation, the complexity of the situation, the availability of reliable data etc.

Example 1 Look at the transactions on a bank statement for September 2011.

Date	Description	Money out	Money in
01-09-11	Gas-Elec-Bill	£21.37	
04-09-11	Withdrawal	£100.00	
15-09-11	Telephone-Bill	£14.72	
16-09-11	Mortgage Payment	£396.12	
28-09-11	Withdrawal	£150.00	
30-09-11	Salary		£1,022.54

Extracting the mathematical structure of this example we define elementary cash-flows.

Definition 2 A *cash-flow* is a vector $(t_j, c_j)_{1\le j\le m}$ of *times* $t_j \in \mathbb{R}$ and *amounts* $c_j \in \mathbb{R}$. Positive amounts $c_j > 0$ are called *inflows*. If $c_j < 0$, then $|c_j|$ is called an *outflow*.

Example 3 The cash-flow of Example 1 is mathematically given by

j	t_j	c_j
1	1	−21.37
2	4	−100.00
3	15	−14.72

j	t_j	c_j
4	16	−396.12
5	28	−150.00
6	30	1,022.54

Often, the situation is not as clear as this, and there may be uncertainty about the time/amount of a payment. This can be modelled stochastically.

Definition 4 A *generalised cash-flow* is a *random* vector $(T_j, C_j)_{1\le j\le M}$ of *times* $T_j \in \mathbb{R}$ and *amounts* $C_j \in \mathbb{R}$ with a possibly random length $M \in \mathbb{N}$.

Sometimes, in fact always in this course, the random structure is simple and the times or the amounts are deterministic, or even the only randomness is that a well specified payment may fail to happen with a certain probability.

Example 5 Future transactions on a bank account (say for November 2011)

j	T_j	C_j	Description
1	1	−21.37	Gas-Elec-Bill
2	T_2	C_2	Withdrawal?
3	15	C_3	Telephone-Bill

j	T_j	C_j	Description
4	16	−396.12	Mortgage payment
5	T_5	C_5	Withdrawal?
6	30	1,022.54	Salary

Here we assume a fixed Gas-Elec-Bill but a varying telephone bill. Mortgage payment and salary are certain. Any withdrawals may take place. For a full specification of the generalised cash-flow we would have to give the (joint!) laws of the random variables.

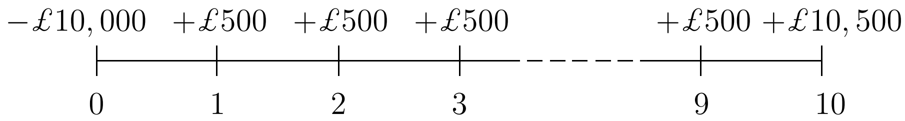
This example shows that simple situations are not always easy to model. It is an important part of an actuary's work to simplify reality into tractable models. Sometimes, it is worth dropping or generalising the time specification and just list approximate or qualitative ('big', 'small', etc.) amounts of income and outgo. cash-flows can be represented in various ways as the following important examples illustrate.

1.3 Examples and course overview

Example 6 (Zero-coupon bond) Usually short-term investments with interest paid at the end of the term, e.g. invest £99 for ninety days for a payoff of £100.

j	t_j	c_j
1	0	−99
2	90	100

Example 7 (Government bonds, fixed-interest securities) Usually long-term investments with annual or semi-annual *coupon payments* (interest), e.g. invest £10,000 for ten years at 5% per annum. [The government borrows money from investors.]



Example 8 (Corporate bonds) The underlying cash-flow looks the same as for government bonds, but they are not as secure. *Credit rating agencies* assess the insolvency risk. If a company goes bankrupt, invested money is often lost. One may therefore wish to add probabilities to the cash-flow in the above figure. Typically, the interest rate in corporate bonds is higher to allow for this extra *risk of default* that the investor takes.

Example 9 (Equities) Shares in the ownership of a company that entitle to regular *dividend payments* of amounts depending on the profit and strategy of the company. Equities can be bought and sold on *stock markets* (via a stock broker) at fluctuating market prices. In the above diagram (including payment probabilities) the inflow amounts are not fixed, the term at the discretion of the shareholder and sales proceeds are not fixed. There are *advanced stochastic models* for stock price evolution. A wealth of derivative products is also available, e.g. forward contracts, options to sell or buy shares. We will discuss forward contracts, but otherwise refer to B10b Mathematical Models for Financial Derivatives.

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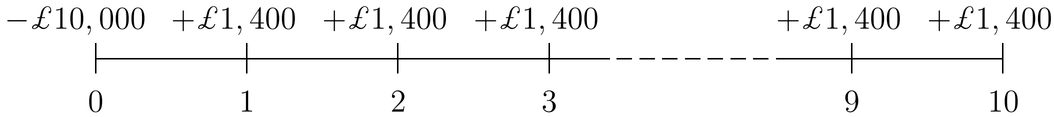
Example 10 (Index-linked securities) Inflation-adjusted securities: coupons and redemption payment increase in line with inflation, by tracking an *inflation index*.

Example 11 (Annuity-certain) Long term investments that provide a series of regular annual (semi-annual or monthly) payments for an initial lump sum, e.g.



Here the term is $n = 10$ years. *Perpetuities* provide regular payment forever ($n = \infty$).

Example 12 (Loans) Formally the negative of a bond cash-flow (interest-only loan) or annuity-certain (repayment loan), but the rights of the parties are not exactly opposite. Whereas the bond investor may be able to redeem or sell the bond early, the lender of a loan often has to obey stricter rules, to protect the borrower.

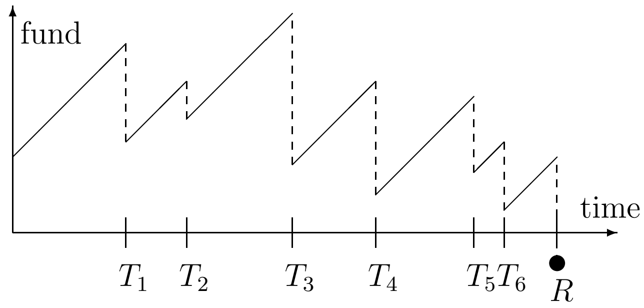
Example 13 (Appraisal of investment projects) Consider a building project. An initial construction period requires certain payments, the following exploitation (e.g. letting) yields income in return for the investment, but maintenance has to be taken into accout as well. Under what circumstances is the project profitable? How reliable are the estimated figures?

These "qualitative" questions, that can be answered qualitatively using specifications as stable, predictable, variable, increasing etc. are just as important as precise estimates.

Example 14 (Life annuity) Life annuities are like annuities-certain, but do not terminate at a fixed time but when the beneficiary dies. Risks due to age, health, profession etc. when entering the annuity contract determine the payment level. They are a basic form of a pension. Several modifications exist (minimal term, maximal term, etc.).

Example 15 (Life assurance) Pays a lump sum on death for monthly or annual premiums that depend on age and health of the policy holder when the policy is underwritten. The sum assured may be decreasing in accordance with an outstanding mortgage.

Example 16 (Property insurance) A class of general insurance (others are health, building, motor etc.). In return for regular premium payments, an insurance company replaces or refunds any stolen or damaged items included on the policy. From the insurer's point of view, all policy holders pay into a common fund to provide for those who claim. The claim history of policy holders affects their premium.



A branch of an insurance company is said to suffer *technical ruin* if the fund runs empty.

Lecture 2

The theory of compound interest

Reading: CT1 Core Reading Units 2-3, McCutcheon-Scott Chapter 1, Sections 2.1-2.4

Quite a few problems that we deal with in this course can be approached in an intuitive way. However, the mathematical and more powerful approach to problem solving is to set up a mathematical model in which the problem can be formalised and generalised. The concept of cash-flows seen in the last lecture is one part of such a model. In this lecture, we shall construct another part, the compound interest model in which interest on capital investments, loans etc. can be computed. This model will play a crucial role throughout the course.

In any mathematical model, reality is only partially represented. An important part of mathematical modelling is the discussion of model assumptions and the interpretation of the results of the model.

2.1 Simple versus compound interest

We are familiar with the concept of interest in everyday banking: the bank pays interest on positive balances on current accounts and savings accounts (not much, but some), and it charges interest on loans and overdrawn current accounts. Reasons for this include that

- people/institutions borrowing money are willing to pay a fee (in the future) for the use of this money now,
- there is price inflation in that £100 lose purchasing power between the beginning and the end of a loan as prices increase,
- there is often a risk that the borrower may not be able to repay the loan.

To develop a mathematical framework, consider an "interest rate h" per unit time, under which an investment of C at time 0 will receive interest Ch by time 1, giving total value $C(1+h)$:

$$C \quad \longrightarrow \quad C(1+h),$$

e.g. for $h = 4\%$ we get $C \longrightarrow 1.04C$.

There are two natural ways to extend this to general times t:

Definition 17 (Simple interest) *Invest C, receive $C(1+th)$ after t years. The* simple *interest on C at rate h for time t is Cth.*

Definition 18 (Compound interest) *Invest C, receive $C(1+i)^t$ after t years. The* compound *interest on C at rate i for time t is $C((1+i)^t-1)$.*

For integer $t=n$, this is as if a bank balance was updated at the end of each year

$$C \longrightarrow C(1+i) \longrightarrow (C(1+i))(1+i) = C(1+i)^2 \longrightarrow (C(1+i)^{n-1})(1+i) = C(1+i)^n.$$

Example 19 Given an interest rate of $i = 6\%$ per annum (p.a.), investing $C = £1,000$ for $t=2$ years yields

$$I_{\text{simp}} = C2i = £120.00 \quad \text{and} \quad I_{\text{comp}} = C\left((1+i)^2-1\right) = C(2i+i^2) = £123.60,$$

where we can interpret Ci^2 as interest on interest, i.e. interest for the second year paid at rate i on the interet Ci for the first year.

Compound interest behaves well under term-splitting: for $t=s+r$

$$C \longrightarrow C(1+i)^s \longrightarrow (C(1+i)^s)\,(1+i)^r = C(1+i)^t,$$

i.e. investing C at rate i first for s years and then the resulting $C(1+i)^s$ for a further r years gives the same as directly investing C for $t=s+r$ years. Under simple interest

$$C \longrightarrow C(1+hs) \longrightarrow C(1+hs)(1+hr) = C(1+ht+srh^2) > C(1+ht),$$

(in the case $C>0$, $r>0$, $s>0$). The difference $Csrh^2 = (Chs)hr$ is interest on the interest Chs that was already paid at time s for the first s years.

What is the best we can achieve by term-splitting under simple interest?

Denote by $S_t(C) = C(1+th)$ the accumulated value under simple interest at rate h for time t. We have seen that $S_r \circ S_s(C) > S_{r+s}(C)$.

Proposition 20 *Fix $t>0$ and h. Then*

$$\sup_{n\in\mathbb{N},r_1,\ldots,r_n\in\mathbb{R}_+:r_1+\cdots+r_n=t} S_{r_n}\circ S_{r_{n-1}}\circ\cdots\circ S_{r_1}(C) = \lim_{n\to\infty} S_{t/n}\circ\cdots\circ S_{t/n}(C) = e^{th}C.$$

Proof: For the second equality we first note that

$$S_{t/n}\circ\cdots\circ S_{t/n}(C) = \left(1+\frac{t}{n}h\right)^n C \to e^{th}C,$$

because

$$\log((1+th/n)^n) = n\log(1+th/n) = n(th/n + O(1/n^2)) \to th.$$

For the first equality,

$$e^{rh} = 1 + rh + \frac{r^2h^2}{2} + \cdots \geq 1 + rh$$

so if $r_1+\cdots+r_n = t$, then

$$e^{th}C = e^{r_1h}e^{r_2h}\cdots e^{r_nh}C \geq (1+r_1h)(1+r_2h)\cdots(1+r_nh)C = S_{r_n}\circ S_{r_{n-1}}\circ\cdots\circ S_{r_1}(C).$$

□

So, the optimal achievable is $C \longrightarrow Ce^{th}$. If $e^{th} = (1+i)^t$, i.e. $e^h = 1+i$ or $h = \log(1+i)$, we recover the compound interest case.

From now on, we will always consider compound interest.

Definition 21 Given an *effective interest rate* i *per unit time* and an *initial capital* C at time 0, the *accumulated value at time* t under the compound interest model (with constant rate) is given by

$$C(1+i)^t = Ce^{\delta t},$$

where

$$\delta = \log(1+i) = \frac{\partial}{\partial t}(1+i)^t \bigg|_{t=0},$$

is called the *force of interest.*

The second expression for the force of interest means that it is the "instantaneous rate of growth per unit capital per unit time".

2.2 Nominal and effective rates

The effective annual rate is i such that $C \longrightarrow C(1+i)$ after one year. We have already seen the force of interest $\delta = \log(1+i)$ as a way to describe the same interest rate model. In practice, rates are often quoted in other ways still.

Definition 22 A *nominal rate* h *convertible pthly* (or compounded p times per year) means that an accumulated value $C(1+h/p)$ is achieved after time $1/p$.

By compounding, the accumulated value at time 1 is $C(1+h/p)^p$, and at time t is $C(1+h/p)^{pt}$. This again describes the same model of accumulated values if $(1+h/p)^p = 1+i$, i.e. if $h = p((1+i)^{1/p} - 1$. Actuarial notation for the nominal rate convertible $p$thly associated with effective rate i is $i^{(p)} = p((1+i)^{1/p} - 1)$.

Example 23 An annual rate of 8% convertible quarterly, i.e. $i^{(4)} = 8\%$ means that $i^{(4)}/4 = 2\%$ is credited each 3 months (and compounded) giving an annual effecive rate $i = (1+i^{(4)}/4)^4 - 1 \approx 8.24\%$.

The most common frequencies are for $p = 2$ (half-yearly, semi-annually), $p = 4$ (quarterly), $p = 12$ (monthly), $p = 52$ (weekly), although the latter used to be approximated using

$$\lim_{p\to\infty} i^{(p)} = \lim_{p\to\infty} \frac{(1+i)^{1/p} - 1}{1/p} = \frac{\partial}{\partial t}(1+i)^t \bigg|_{t=0} = \log(1+i) = \delta;$$

the force of interest δ can be called the "nominal rate of interest convertible continuously".

Example 24 Here are two genuine and one artificial options for a savings account.

(1) 3.25% p.a. effective ($i_1 = 3.25\%$)

(2) 3.20% p.a. nominal convertible monthly ($i_2^{(12)} = 3.20\%$)

(3) 3.20% p.a. nominal "convertible continuously" ($\delta_3 = 3.20\%$)

After one year, an initial capital of £10,000 accumulates to

(1) $10,000 \times (1 + 3.25\%) = 10,325.00 = R_1$,

(2) $10,000 \times (1 + 3.20\%/12)^{12} = 10,324.74 = R_2$,

(3) $10,000 \times e^{3.20\%} = 10,325.18 = R_3$.

Although interest may be credited to the account differently, an investment into (j) just consists of deposit and withdrawal, so the associated cash-flow is $((0, -10000), (1, R_j))$, and we can use R_j to decide between the options. We can also compare $i_2 \approx 3.2474\%$ and $i_3 \approx 3.2518\%$ or calculate δ_1 and δ_2 to compare with δ_3 etc.

Interest rates always refer to some time unit. The standard choice is one year, but it sometimes eases calculations to choose six months, one month or one day. All definitions we have made reflect the assumption that the interest rate does not vary with the initial capital C nor with the term t. We refer to this model of accumulated values as the constant-i model, or the constant-δ model.

2.3 Discount factors and discount rates

Before we more fully apply the constant-i model to cash-flows in Lecture 3, let us discuss the notion of discount. We are used to discounts when shopping, usually a percentage reduction in price, time being implicit. Actuaries use the notion of an *effective rate of discount d per time unit* to represent a reduction of C to $C(1-d)$ if payment takes place a time unit early.

This is consistent with the constant-i model, if the payment of $C(1-d)$ accumulates to $C = (C(1-d))(1+i)$ after one time unit, i.e. if

$$(1-d)(1+i) = 1 \iff d = 1 - \frac{1}{1+i}.$$

A more prominent role will be played by the *discount factor* $v = 1 - d$, which answers the question

How much will we have to invest now to have 1 at time 1?

Definition 25 In the constant-i model, we refer to $v = 1/(1+i)$ as the associated *discount factor* and to $d = 1 - v$ as the associated effective annual rate of discount.

Example 26 How much do we have to invest now to have 1 at time t? If we invest C, this accumulates to $C(1+i)^t$ after t years, hence we have to invest $C = 1/(1+i)^t = v^t$.

Lecture 3

Valuing cash-flows

Reading: CT1 Core Reading Units 3 and 5, McCutcheon-Scott Chapter 2

In Lecture 2 we set up the constant-i interest rate model and saw how a past deposit accumulates and a future payment can be discounted. In this lecture, we combine these concepts with the cash-flow model of Lecture 1 by assigning time-t values to cash-flows. We also introduce general (deterministic) time-dependent interest models, and continuous cash-flows that model many small payments as infinitesimal payment streams.

3.1 Accumulating and discounting in the constant-i model

Given a cash-flow $c = (c_j, t_j)_{1 \leq j \leq m}$ of payments c_j at time t_j and a time t with $t \geq t_j$ for all j, we can write the joint accumulated value of all payments by time t according to the constant-i model as

$$\mathrm{AVal}_t(c) = \sum_{j=1}^{m} c_j (1+i)^{t-t_j} = \sum_{j=1}^{m} c_j e^{\delta(t-t_j)},$$

because each payment c_j at time t_j earns compound interest for $t - t_j$ time units. Note that some c_j may be negative, so the accumulated value could become negative. We assume implicitly that the same interest rate applies to positive and negative balances.

Similarly, given a cash-flow $c = (c_j, t_j)_{1 \leq j \leq m}$ of payments c_j at time t_j and a time $t > t_j$ for all j, we can write the joint *discounted value* at time t of all payments as

$$\mathrm{DVal}_t(c) = \sum_{j=1}^{m} c_j v^{t_j - t} = \sum_{j=1}^{m} c_j (1+i)^{-(t_j - t)} = \sum_{j=1}^{m} c_j e^{-\delta(t_j - t)}.$$

This discounted value is the amount we invest at time t to be able to spend c_j at time t_j for all j.

3.2 Time-dependent interest rates

So far, we have assumed that interest rates are constant over time. Suppose, we now let $i = i(k)$ vary with time $k \in \mathbb{N}$. We define the accumulated value at time n for an investment of C at time 0 as

$$C(1 + i(1))(1 + i(2)) \cdots (1 + i(n-1)) \cdots (1 + i(n)).$$

Example 27 A savings account pays interest at $i(1) = 2\%$ in the first year and $i(2) = 5\%$ in the second year, with interest from the first year reinvested. Then the account balance evolves as $1,000 \longrightarrow 1,000(1 + i(1)) = 1,020 \longrightarrow 1,000(1 + i(1))(1 + i(2)) = 1,071$.

When varying interest rates between non-integer times, it is often nicer to specify the force of interest $\delta(t)$ which we saw to have a local meaning as the infinitesimal rate of capital growth under compound interest:

$$C \longrightarrow C \exp\left(\int_0^t \delta(s)ds\right) = R(t).$$

Note that now (under some right-continuity assumptions)

$$\left.\frac{\partial}{\partial t}R(t)\right|_{t=0} = \delta(0) \qquad \text{and more generally} \qquad \frac{\partial}{\partial t}R(t) = \delta(t)R(t),$$

so that the interpretation of $\delta(t)$ as local rate of capital growth at time t still applies.

Example 28 If $\delta(\cdot)$ is piecewise constant, say constant δ_j on $(t_{j-1}, t_j]$, $j = 1, \ldots, n$, then

$$C \longrightarrow Ce^{\delta_1 r_1} e^{\delta_2 r_2} \cdots e^{\delta_n r_n}, \qquad \text{where } r_j = t_j - t_{j-1}.$$

Definition 29 Given a time-dependent force of interest $\delta(t)$, $t \in \mathbb{R}_+$, we define the *accumulated value* at time $t \geq 0$ of an initial capital $C \in \mathbb{R}$ *under a force of interest* $\delta(\cdot)$ as

$$R(t) = C \exp\left(\int_0^t \delta(s)ds\right).$$

Also, we may refer to $I(t) = R(t) - C$ as the *interest from time* 0 *to time* t *under* $\delta(\cdot)$.

3.3 Accumulation factors

Given a time-dependent interest model $\delta(\cdot)$, let us define *accumulation factors from* s *to* t

$$A(s,t) = \exp\left(\int_s^t \delta(r)dr\right), \qquad s < t. \tag{1}$$

Just as $C \longrightarrow R(t) = CA(0,t)$ for an investment of C at time 0 for a term t, we use $A(s,t)$ as a factor to turn an investment of C at time s into its accumulated value $CA(s,t)$ at time t. This behaves well under term-splitting, since

$$C \longrightarrow CA(0,s) \longrightarrow (CA(0,s))A(s,t) = C \exp\left(\int_0^s \delta(r)dr\right) \exp\left(\int_s^t \delta(r)dr\right) = CA(0,t).$$

More generally, note the *consistency property* $A(r,s)A(s,t) = A(r,t)$, and conversely:

Proposition 30 *Suppose,* $A\colon \{(s,t) : s \le t\} \to (0,\infty)$ *satisfies the consistency property and* $t \mapsto A(s,t)$ *is differentiable for all* s*, then there is a function* $\delta(\cdot)$ *such that (1) holds.*

Proof: Since consistency for $r = s = t$ implies $A(t,t) = 1$, we can define as (right-hand) derivative

$$\delta(t) = \lim_{h\downarrow 0} \frac{A(t,t+h) - A(t,t)}{h} = \lim_{h\downarrow 0} \frac{A(0,t+h) - A(0,t)}{hA(0,t)},$$

where we also applied consistency. With $g(t) = A(0,t)$ and $f(t) = \log(A(0,t))$

$$\delta(t) = \frac{g'(t)}{g(t)} = f'(t) \quad \Rightarrow \quad \log(A(0,t)) = f(t) = \int_0^t \delta(s)ds.$$

Since consistency implies $A(s,t) = A(0,t)/A(0,s)$, we obtain (1). □

We included the apparently unrealistic $A(s,t) < 1$ (accumulated value less than the initial capital) that leads to negative $\delta(\cdot)$. This can be useful for some applications where $\delta(\cdot)$ is not pre-specified, but connected to investment performance where prices can go down as well as up, or to inflation/deflation. Similarly, we allow any $i \in (-1,\infty)$, so that the associated 1-year accumulation factor $1 + i$ is positive, but possibly less than 1.

3.4 Time value of money

We have discussed accumulated and discounted values in the constant-i model. In the time-varying $\delta(\cdot)$ model with accumulation factors $A(s,t) = \exp(\int_s^t \delta(r)dr)$, we obtain

$$\mathrm{AVal}_t(c) = \sum_{j=1}^m c_j A(t_j,t) \quad \text{if all } t_j \le t, \qquad \mathrm{DVal}_t(c) = \sum_{j=1}^m c_j V(t,t_j) \quad \text{if all } t_j > t,$$

where $V(s,t) = 1/A(s,t) = \exp(-\int_s^t \delta(r)dr)$ is the discount factor from time t back to time $s \le t$. With $v(t) = V(0,t)$, we get $V(s,t) = v(t)/v(s)$. Notation $v(t)$ is useful, as it is often the *present value*, i.e. the discounted value at time 0, that is of interest, and we then have

$$\mathrm{DVal}_0(c) = \sum_{j=1}^m c_j v(t_j), \quad \text{if all } t_j > 0,$$

where each payment is discounted by $v(t_j)$. Each future payment has a different present value. Note that the formulas for AVal_t and DVal_t are identical, if we express $A(s,t)$ and $V(s,t)$ in terms of $\delta(\cdot)$.

Definition 31 The *time-t value* of a cash-flow c is defined as

$$\mathrm{Val}_t(c) = \mathrm{AVal}_t(c_{\le t}) + \mathrm{DVal}_t(c_{>t}),$$

where $c_{\le t}$ and $c_{>t}$ denote restrictions of c to payments at times $t_j \le t$ resp. $t_j > t$.

Proposition 32 *For all* $s \le t$ *we have* $\mathrm{Val}_t(c) = \mathrm{Val}_s(c)A(s,t) = \mathrm{Val}_s(c)\dfrac{v(s)}{v(t)}$.

The proof is straightforward and left as an exercise. Note in particular, that if $\mathrm{Val}_t(c) = 0$ for some t, then $\mathrm{Val}_t(c) = 0$ for all t.

Remark 33 1. A sum of money without time specification is meaningless.

2. Do not add or directly compare values at different times.

3. If values of two cash-flows are equal at one time, they are equal at all times.

3.5 Continuous cash-flows

If many small payments are spread evenly over time, it is natural to model them by a continuous stream of payment.

Definition 34 A *continuous cash-flow is a function* $c\colon \mathbb{R} \to \mathbb{R}$. *The total* net *inflow between times* s *and* t *is*

$$\int_s^t c(r)dr,$$

and this may combine periods of inflow and outflow.

As before, we can consider random c. We can also mix continuous and discrete parts. Note that the net inflow "adds" values at different times ignoring the time-value of money. More useful than the net inflow are accumulated and discounted values

$$\mathrm{AVal}_t(c) = \int_0^t c(s)A(s,t)ds \quad \text{and} \quad \mathrm{DVal}_t(c) = \int_t^\infty c(s)V(t,s)ds = \frac{1}{v(t)}\int_t^\infty c(s)v(s)ds.$$

Everything said in the previous section applies in an analogous way.

3.6 Example: withdrawal of interest as a cash-flow

Consider a savings account that does not credit interest to the savings account itself (where it is further compounded), but triggers a cash-flow of interest payments.

1. In a $\delta(\cdot)$-model, $1 \longrightarrow 1 + I = \exp(\int_0^1 \delta(ds))$. Consider the interest cash-flow $(1, -I)$. Then $\mathrm{Val}_1((0,1),(1,-I)) = 1$ is again the capital, at time 1.

2. In the constant-i model, recall the nominal rate $i^{(p)} = p((1+i)^{1/p} - 1)$. Interest on an initial capital 1 up to time $1/p$ is $i^{(p)}/p$. After one or indeed k such $p$thly interest payments of $i^{(p)}/p$, we have $\mathrm{Val}_{k/p}((0,1),(1/p,-i^{(p)}/p),\ldots,(k/p,-i^{(p)}/p)) = 1$.

3. In a $\delta(\cdot)$-model, continuous cash-flow $c(s) = \delta(s)$ has $\mathrm{Val}_t((0,1),-c_{\le t})=1$ for all t.

We leave as an exercise to check these directly from the definitions.

Note, in particular, that accumulation of interest itself does not correspond to events in the cash-flow. Cash-flows describe external influences on the account. Although interest is not credited continuously or at every withdrawal in practice, our mathematical model does assign a balance=value that changes continuously between instances of external cash-flow. We include the effect of interest in a cash-flow by withdrawal.

Lecture 4

The yield of a cash-flow

Reading: CT1 Core Reading Unit 7, McCutcheon-Scott Section 3.2

Given a cash-flow representing an investment, its yield is the constant interest rate that makes the cash-flow a fair deal. Yields allow to assess and compare the performance of possibly quite different investment opportunities as well as mortgages and loans.

4.1 Definition of the yield of a cash-flow

In that follows, it does not make much difference whether a cash-flow c is discrete, continuous or mixed, whether the time horizon of c is finite or infinite (like e.g. for perpetuities). However, to keep statements and technical arguments simple, we assume:

The time horizon of c is finite and payment rates of c are bounded. (H)

Since we will compare values of cash-flows under different interest rates, we need to adapt our notation to reflect this:

$$\mathrm{NPV}(i) = i\text{-}\mathrm{Val}_0(c)$$

denotes the Net Present Value of c discounted in the constant-i interest model, i.e. the value of the cash-flow c at time 0, discounted using discount factors $v(t) = v^t = (1+i)^{-t}$.

Lemma 35 *Given a cash-flow c satifying hypothesis* (H)*, the function $i \mapsto \mathrm{NPV}(i)$ is continuous on $(-1, \infty)$.*

Proof: In the discrete case $c = ((t_1, c_1), \ldots, (t_n, c_n))$, we have $\mathrm{NPV}(i) = \sum_{k=1}^{n} c_k(1+i)^{-t_k}$, and this is clearly continuous in i for all $i > -1$. For a continuous-time cash-flow $c(s)$, $0 \le s \le t$ (and mixed cash-flows) we use the uniform continuity of $i \mapsto (1+i)^{-s}$ on compact intervals $s \in [0, t]$ for continuity to be maintained after integration

$$\mathrm{NPV}(i) = \int_0^t c(s)(1+i)^{-s} ds.$$

□

Corollary 36 *Under hypothesis* (H)*, $i \mapsto i\text{-}\mathrm{Val}_t(i)$ is continuous on $(-1, \infty)$ for any t.*

Often the situation is such that an investment deal is profitable ($\text{NPV}(i) > 0$) if the interest rate i is below a certain level, but not above, or vice versa. By the intermediate value theorem, this threshold is a zero of $i \mapsto \text{NPV}(i)$, and we define

Definition 37 Given a cash-flow c, if $i \mapsto \text{NPV}(i)$ has a unique root on $(-1, \infty)$, we define the *yield* $y(c)$ to be this root. If $i \mapsto \text{NPV}(i)$ does not have a root in $(-1, \infty)$ or the root is not unique, we say that the yield is not well-defined.

The yield is also known as the "internal rate of return" or also just "rate of return". We can say that the yield is the fixed interest rate at which c is a "fair deal". The equation $\text{NPV}(i) = 0$ is called *yield equation.*

Example 38 Suppose that for an initial investment of £1,000 you obtain a payment of £400 after one year and 770 after two years. What is the yield of this deal? Clearly $c = ((0, -1000), (1, 400), (2, 770)$. By definition, we are looking for roots $i \in (-1, \infty)$ of

$$\begin{aligned} \text{NPV}(i) = -1,000 + 400(1+i)^{-1} + 800(1+i)^{-2} = 0 \\ \iff \qquad 1,000(i+1)^2 - 400(i+1) - 770 = 0 \end{aligned}$$

The solutions to this quadratic equation are $i_1 = -1.7$ and $i_2 = 0.1$. Since only the second zero lies in $(-1, \infty)$, the yield is $y(c) = 0.1$, i.e. 10%.

Sometimes, it is onvenient to solve for $v = (1+i)^{-1}$, here $1,000 - 400v - 770v^2 = 0$ etc. Note that $i \in (-1, \infty) \iff v \in (0, \infty)$.

Example 39 Consider the security of Example 7 in Lecture 1. The yield equation $\text{NPV}(i) = 0$ can be written as

$$10,000 = 500 \sum_{k=1}^{10} (1+i)^{-k} + 10,000(1+i)^{-10}.$$

We will introduce some short-hand actuarial notation in Lecture 5. Note, however, that we already know a root of this equation, because the cash-flow is the same as for a bank account with capital £10,000 and a cash-flow of annual interest payments of £500, i.e. at 5%, so $i = 5\%$ solves the yield equation. We will now see in much higher generality that there is usually only one solution to the yield equation for investment opportunities.

4.2 General results ensuring the existence of yields

Since the yield does not always exist, it is useful to have sufficient existence criteria.

Proposition 40 *If c has in- and outflows and all inflows of c precede all outflows of c (or vice versa), then the yield $y(c)$ exists.*

Remark 41 This includes the vast majority of projects that we will meet in this course. Essentially, investment projects have outflows first, and inflows afterwards, while loan schemes (from the borrower's perspective) have inflows first and outflows afterwards.

Proof: By assumption, there is T such that all inflows are strictly before T and all outflows are strictly after T. Then the accumulated value

$$p_i = i\text{-Val}_T(c_{<T})$$

is positive strictly increasing in i with $p_{-1} = 0$ and the discounted value $p_\infty = \infty$ (by assumption there are inflows) and

$$n_i = i\text{-Val}_T(c_{>T})$$

is negative strictly increasing with $n_{-1} = -\infty$ (by assumption there are outflows) and $n_\infty = 0$. Therefore

$$b_i = p_i + n_i = i\text{-Val}_T(c)$$

is strictly increasing from $-\infty$ to ∞, continuous by Corollary 36; its unique root i_0 is also the unique root of $i \mapsto \text{NPV}(i) = (1+i)^{-T}\,(i\text{-Val}_T(c))$ by Corollary 32.

For the "vice versa" part, replace c by $-c$ and use $\text{Val}_0(c) = -\text{Val}_0(-c)$ etc. □

Corollary 42 *If all inflows precede all outflows, then*

$$y(c) > i \iff \text{NPV}(i) < 0.$$

If all outflows precede all inflows, then

$$y(c) > i \iff \text{NPV}(i) > 0.$$

Proof: In the first setting assume $y(c) > 0$, we know $b_{y(c)} = 0$ and $i \mapsto b_i$ increases with i, so $i < y(c) \iff b_i < 0$, but $b_i = i\text{-Val}_T(c) = (1+i)^T\text{NPV}(i)$.

The second setting is analogous (substitute $-c$ for c). □

As a useful example, consider $i = 0$, when $\text{NPV}(0)$ is the sum of undiscounted payments.

Example 39 (continued) By Proposition 40, the yield exists and equals $y(c) = 5\%$.

4.3 Example: APR of a loan

A yield that is widely quoted in practice, is the Annual Percentage Rate (APR) of a loan. This is straightforward if the loan agreement is based on a constant interest rate i. Particularly for mortgages (loans to buy a house), it is common, however, to have an initial period of lower interest rates and lower monthly payments followed by a period of higher interest rates and higher payments. The APR then gives a useful summary value:

Definition 43 Given a cash-flow c representing a loan agreement (with inflows preceding outflows), the yield $y(c)$ rounded down to next lower 0.1% is called the *Annual Percentage Rate (APR)* of the loan.

Example 44 Consider a mortgage of £85,000 with interest rates of 2.99% in year 1, 4.19% in year 2 and 5.95% for the remainder of a 20-year term. A Product Fee of £100 is added to the loan amount, and a Funds Transfer Fee is deducted from the Net Amount provided to the borrower. We will discuss in Lecture 6 how this leads to a cash flow of

$$c = ((0, 84975), (1, -5715), (2, -6339), (3, -7271), (4, -7271), \ldots, (20, -7271)),$$

and how the annual payments are further transformed into equivalent monthly payments. Let us here calculate the APR, which exists by Proposition 40. Consider

$$f(i) = 84,975 - 5,715(1+i)^{-1} - 6,339(1+i)^{-2} - 7,271\sum_{k=3}^{20}(1+i)^{-k}.$$

Solving the geometric progression, or otherwise, we find the root iteratively by evaluation

$$f(5\%) = -3,310, \quad f(5.5\%) = 396, \quad f(5.4\%) = -326, \quad f(5.45\%) = 36.$$

From the last two, we see that $y(c) \approx 5.4\%$, actually $y(c) = 5.44503\ldots\%$. We can see that APR=5.4% already from the middle two evaluations, since we always round down, by definition of the APR.

4.4 Numerical calculation of yields

Suppose we know the yield exists, e.g. by Proposition 40. Remember that $f(i) = \mathrm{NPV}(i)$ is continuous and (usually) takes values of different signs at the boundaries of $(-1, \infty)$.

Interval splitting allows to trace the root of f: $(l_0, r_0) = (-1, \infty)$, make successive guesses $i_n \in (l_n, r_n)$, calculate $f(i_n)$ and define

$$(l_{n+1}, r_{n+1}) := (i_n, r_n) \quad \text{or} \quad (l_{n+1}, r_{n+1}) = (l_n, i_n)$$

such that the values at the boundaries $f(l_{n+1})$ and $f(r_{n+1})$ are still of different signs. Stop when the desired accuracy is reached.

The challenge is to make good guesses. Bisection

$$i_n = (l_n + r_n)/2$$

(once $r_n < \infty$) is the ad hoc way, *linear interpolation*

$$i_n = l_n \frac{f(r_n)}{f(r_n) - f(l_n)} + r_n \frac{-f(l_n)}{f(r_n) - f(l_n)}$$

an efficient improvement. There are more efficient variations of this method using some kind of convexity property of f, but that is beyond the scope of this course.

Actually, the iterations are for computers to carry out. For assignment and examination questions, you should make good guesses of l_0 and r_0 and carry out one linear interpolation, then claiming an *approximate* yield.

Example 44 (continued) Good guesses are $r_0 = 6\%$ and $l_0 = 5\%$, since $i = 5.95\%$ is mostly used. [Better, but a priori less obvious guess would be $r_0 = 5.5\%$.] Then

$$\left.\begin{array}{l} f(5\%) = -3,310.48 \\ f(6\%) = 3874.60 \end{array}\right\} \Rightarrow y(c) \approx 5\% \frac{f(6\%)}{f(6\%) - f(5\%)} + 6\% \frac{-f(5\%)}{f(6\%) - f(5\%)} = 5.46\%.$$

Lecture 5

Annuities and fixed-interest securities

Reading: CT1 Core Reading Units 6, 10.1, McCutcheon-Scott 3.3-3.6, 4, 7.2

In this chapter we introduce actuarial notation for discounted and accumulated values of regular payment streams, so-called *annuity symbols*. These are useful not only in the pricing of annuity products, but wherever regular payment streams occur. Our main example here will be fixed-interest securities.

5.1 Annuity symbols

Annuity-certain. An annuity-certain of term n entitles the holder to a cash-flow

$$c = ((1, X), (2, X), \ldots, (n-1, X), (n, X)).$$

Take $X = 1$ for convenience. In the constant-i model, its Net Present Value is

$$a_{\overline{n}|} = a_{\overline{n}|i} = \mathrm{NPV}(i) = \mathrm{Val}_0(c) = \sum_{k=1}^{n} v^k = v\frac{1-v^n}{1-v} = \frac{1-v^n}{i}.$$

The symbols $a_{\overline{n}|}$ and $a_{\overline{n}|i}$ are annuity symbols, pronounced "a angle n (at i)".

The accumulated value at end of term is

$$s_{\overline{n}|} = s_{\overline{n}|i} = \mathrm{Val}_n(c) = v^{-n}\mathrm{Val}_0(c) = \frac{(1+i)^n - 1}{i}.$$

$p$thly payable annuities. A $p$thly payable annuity spreads (nominal) payment of 1 per unit time equally into p payments of $1/p$, leading to a cash-flow

$$c_p = ((1/p, 1/p), (2/p, 1/p), \ldots, (n - 1/p, 1/p), (n, 1/p))$$

with

$$a^{(p)}_{\overline{n}|} = \mathrm{Val}_0(c_p) = \frac{1}{p}\sum_{k=1}^{np} v^{k/p} = v^{1/p}\frac{1-v^n}{1-v^{1/p}} = \frac{1-v^n}{i^{(p)}},$$

where $i^p = p((1+i)^{1/p} - 1)$ is the nominal rate of interest convertible $p$thly associated with i. This calculation hence the symbol is meaningful for n any integer multiple of $1/p$.

We saw in Section 3.6 that, (now expressed in our new notation)

$$ia_{\overline{n}|} = \mathrm{Val}_0((1,i),(2,i),\ldots,(n,i)) = \mathrm{Val}_0((1/p, i^{(p)}/p),(2/p, i^{(p)}/p),\ldots,(n, i^{(p)}/p)) = i^{(p)} a^{(p)}_{\overline{n}|},$$

since both cash-flows correspond to the income up to time n on 1 unit invested at time 0, at effective rate i.

The accumulated value of c_p at the end of term is

$$s^{(p)}_{\overline{n}|} = \mathrm{Val}_n(c_p) = v^{-n}\mathrm{Val}_0(c_p) = \frac{(1+i)^n - 1}{i^{(p)}}.$$

Perpetuities. As $n \to \infty$, we obtain perpetuities that pay forever

$$a_{\overline{\infty}|} = \mathrm{Val}_0((1,1),(2,1),(3,1),\ldots) = \sum_{k=1}^{\infty} v^k = \frac{v}{1-v} = \frac{1}{i}.$$

Continuously payable annuities. As $p \to \infty$, the cash-flow c_p "tends to" the continuous cash-flow $c(s) = 1$, $0 \le s \le n$, with

$$\overline{a}_{\overline{n}|} = \mathrm{Val}_0(c) = \int_0^n c(s) v^s ds = \int_0^n v^s ds = \int_0^n e^{-\delta s} ds = \frac{1 - e^{-\delta n}}{\delta} = \frac{1 - v^n}{\delta}.$$

Or $\overline{a}_{\overline{n}|} = \mathrm{Val}_0(c) = \lim_{p\to\infty} a^{(p)}_{\overline{n}|} = \lim_{p\to\infty} \frac{1-v^n}{i^{(p)}} = \frac{1-v^n}{\delta}$. Similarly $\overline{s}_{\overline{n}|} = \mathrm{Val}_n(c) = v^{-n}\overline{a}_{\overline{n}|}$.

Annuity-due. This simply means that the first payment is *now*

$$((0,1),\ldots,(n-1,1))$$

with

$$\ddot{a}_{\overline{n}|} = \mathrm{Val}_0((0,1),(1,1),\ldots,(n-1,1)) = \sum_{k=0}^{n-1} v^k = \frac{1-v^n}{1-v} = \frac{1-v^n}{d}$$

and

$$\ddot{s}_{\overline{n}|} = \mathrm{Val}_n((0,1),(1,1),\ldots,(n-1,1)) = v^{-n}\ddot{a}_{\overline{n}|} = \frac{(1+i)^n - 1}{d}.$$

Similarly

$$\ddot{a}^{(p)}_{\overline{n}|} = \mathrm{Val}_0((0,1/p),(1/p,1/p),\ldots,(n-1/p,1/p))$$

and

$$\ddot{s}_{\overline{n}|} = \mathrm{Val}_n((0,1/p),(1/p,1/p),\ldots,(n-1/p,1/p)).$$

Also $\ddot{a}_{\overline{\infty}|}$, $\ddot{a}^{(p)}_{\overline{\infty}|}$, etc.

Deferred and increasing annuities. Further important annuity symbols dealing with regular cash-flows starting some time in the future, and with cash-flows with regular increasing payment streams, are introduced on Assignment 2. The corresponding symbols are

$$_{m|}a_{\overline{n}|}, {}_{m|}a^{(p)}_{\overline{n}|}, {}_{m|}\ddot{a}_{\overline{n}|}, {}_{m|}\ddot{a}^{(p)}_{\overline{n}|}, {}_{m|}\overline{a}_{\overline{n}|}, \qquad (Ia)_{\overline{n}|}, (I\ddot{a})_{\overline{n}|}, (I\overline{a})_{\overline{n}|}, (\overline{I}\overline{a})_{\overline{n}|}, \qquad {}_{m|}(Ia)_{\overline{n}|} \text{ etc.}$$

5.2 Fixed-interest securities

Simple fixed-interest securities. A simple fixed-interest security entitles the holder to a cash-flow

$$c = ((1, Nj), (2, Nj), \ldots, (n-1, Nj), (n, Nj + N)),$$

where j is the *coupon rate*, N is the *nominal amount* and n is the *term*. The value in the constant-j model is

$$\text{NPV}(j) = Nja_{\overline{n}|j} + N(1+j)^{-n} = Nj\frac{1-(1+j)^{-n}}{j} + N(1+j)^{-n} = N.$$

This is not a surprise: compare with point 2. of Section 3.6. The value in the constant-i model is

$$\text{NPV}(i) = Nja_{\overline{n}|i} + N(1+i)^{-n} = Nj\frac{1-(1+i)^{-n}}{i} + N(1+i)^{-n} = Nj/i + Nv^n(1-j/i).$$

More general fixed-interest securities. There are fixed-interest securities with $p$thly payable coupons at a *nominal* coupon rate j and with a redemption price of R per unit nominal

$$c = ((1/p, Nj/p), (2/p, Nj/p), \ldots, (n-1/p, Nj/p), (n, Nj/p + NR)),$$

where we say that the security is redeemable *at (resp. above or below) par* if $R = 1$ (resp. $R > 1$ or $R < 1$). We compute

$$\text{NPV}(i) = Nja^{(p)}_{\overline{n}|} + NR(1+i)^{-n}.$$

If $\text{NPV}(i) = N$ (resp. $> N$ or $< N$), we say that the security is valued or traded at (resp. above or below) par. Redemption at par is standard. If redemption is not at par, this is usually expressed as e.g. "redemption at 120%" meaning $R = 1.2$. If redemption is not at par, we can calculate the coupon rate per unit redemption money as $j' = j/R$; with $N' = NR$, the cash-flow of a $p$thly payable security of nominal amount N' with coupon rate j' redeemable at par is identical.

Interest payments are always calculated from the nominal amount. Redemption at par is the standard. In practice, a security is a piece of paper (with coupon strips to cash in the interest) that can change owner (sometimes under some restrictions).

Fixed-interest securities as investments. Fixed-interest securities are issued by Governments and are also called Government bonds as opposed to corporate bonds, which are issued by companies. Corporate bonds are less secure than Government bonds since (in either case, actually) bankruptcy can stop the payment stream. Since Government typically issues large quantities of bonds, they form a very liquid/marketable form of investment that is actively traded on bond markets.

Government bonds are either issued at a fixed price or *by tender*, in which case the highest bidders get the bonds at a set issue date. Government bonds usually have a term of several years. There are also shorter-term Government bills which have no coupons, so they are just offered at a discount on their nominal value.

Example 45 Consider a fixed-interest security of $N = 100$ nominal, coupon rate $j = 3\%$ payable annually and redeemable at par after a term of $n = 2$. If the security is currently trading below par, with a purchase price of $P = £97$, the investment has a cash-flow

$$c = ((0, -97), (1, 3), (2, 103))$$

and we can calculate the yield by solving the equation of value

$$-97 + 3(1+i)^{-1} + 103(1+i)^{-2} = 0 \iff 97(1+i)^2 - 3(1+i) - 103 = 0$$

to obtain $1 + i = 1.04604$, i.e. $i = 4.604\%$ (the second solution of the quadradic is $1+i = -1.01512$, i.e. $i = -2.01512$, which is not in $(-1, \infty)$; note that we knew already by Proposition 40 that there can only be one admissible solution).

Here, we could solve the quadratic equation explicitly; for fixed-interest securities of longer term, it is useful to note that the yield is composed of two effects, first the coupons payable at rate j/R per unit redemption money (or at rate jN/P per unit purchasing price) and then any capital gain/loss $RN - P$ spread over n years. Here, these rough considerations give

$$j/R + (R - P/N)/n = 4.5\%, \quad \text{or more precisely } (j/R + jN/P)/2 + (R - P/N)/n = 4.546\%,$$

and often even rougher considerations give us an idea of the order of magnitute of a yield that we can then use as good initial guesses for a numerical approximation.