A.6 Term structure, random interest rates/cash-flows

Maths&Stats students with an assessed practical deadline on Tuesday of week 8 may want to leave questions 1, 3 and 5 for further practice after term (or, better, solve these questions between their practical deadline and their class).

For some of the exercises on this sheet, you will need values of the standard normal distribution function. If $Z \sim N(0,1)$, then:

z	-3	-2.3263	-2.4778	-1.8349	-0.039	0.233
$P(Z \le z)$	0.00135	0.01000	0.00661	0.03326	0.48445	0.59211

1. Define what is meant by an n-year spot rate of interest. Suppose that the spot-yield curve is such that the t-year spot rate of interest is $y_t = 0.03e^{-0.05t} + 0.02$. Calculate the following quantities and explain their meaning

   (a) $f_{2,4}$, (b) P_6, (c) $F_{3,5}$, (d) F_8.

2. Consider a fixed-interest bond with coupon 6% p.a., payable annually in arrears, with term to redemption of 3 years, redeemable at par. If the 1-year spot rate is 5% p.a. and the 2-year spot rate is 5.8% p.a., what 3-year spot rate would give a price of the bond at $t = 0$ of £99.07 per £100 nominal?

3. Three bonds paying annual coupons of 6% p.a. annually in arrears and redeemable at par will be redeemed in exactly one year, two years and three years, respectively. The price of each of the bonds is £96 per £100 nominal.

   (a) Determine the gross redemption yield of the 3-year bond.
   (b) Determine the discount factors $v(1)$, $v(2)$ and $v(3)$ that the market is using to discount payments due in 1, 2 and 3 years, respectively.
   (c) Calculate $f_{0,1}$, $f_{1,1}$ and $f_{2,1}$, where $f_{n,k}$ is the forward interest rate from time n to $n + k$.

4. Suppose that the interest rate for the next six months is known to be 5.5% (effective rate per annum), while the rate for the six months after that is unknown and assumed to be uniformly distributed on the interval $(4\%, 6\%)$. Under this assumption, find the expectations of:

   (a) the accumulated value after one year of £100 invested now;
   (b) the discounted present value of a payment of £100 in a year's time.

5. Let $1 + I$ be a lognormal random variable with parameters μ and σ^2, mean $1 + j$ and variance s^2. Show that

$$\sigma^2 = \log\left(1 + \left(\frac{s}{1+j}\right)^2\right) \qquad \text{and} \qquad \mu = \log\left(\frac{1+j}{\sqrt{1 + \left(\frac{s}{1+j}\right)^2}}\right)$$

6. The yield of an investment in a given year is denoted by Y. Suppose $1+Y$ is lognormally distributed. The expected value of Y is 5% and its standard deviation is 11%. Calculate the parameters of the lognormal distribution of $1+Y$ and the probability that the yield for the year lies between 4% and 7%.

7. A company is adopting a particular investment strategy such that the expected annual effective rate of return from investments is 7% and the standard deviation of annual returns is 9%. Annual returns are independent and $(1+I_j)$ is lognormally distributed, where I_j is the return in the jth year.

   (a) Calculate the expected value and standard deviation of an investment of 1 unit over 10 years, deriving all formulae that you use.

   (b) Calculate the probability that the accumulation of such an investment will be less than 50% of its expected value in ten years' time.

   (c) The company has an outstanding debt and must make a payment of £140,000 in 10 years time. Calculate the probability that an investment of £120,000 now will provide sufficient funds to meet this liability.

8. Let I_j denote the effective rate of interest in the year $j-1$ to j. Suppose that, for $j \geq 1$

$$I_{j+1} = \begin{cases} I_j + 0.02 & & 0.25 \\ I_j & \text{with probability} & 0.5 \\ I_j - 0.02 & & 0.25 \end{cases}$$

   Given that $I_1 \equiv 0.06$, calculate the probability that an investment of 1 at time 0 accumulates to more than 1.2 at time 3.

The following questions relate to week 8 lecture material, optional for the week 8 classes.

9. Suppose the force of interest Δ_j during the year from $j-1$ to j is given by

$$\Delta_j = \mu + \frac{1}{\sqrt{2}}\epsilon_{j-1} + \frac{1}{\sqrt{2}}\epsilon_j,$$

   where $\epsilon_0, \epsilon_1, \epsilon_2, \ldots$ are i.i.d. random variables with distribution $N(0, \sigma^2)$.

   (a) Show that $\Delta_j \sim N(\mu, \sigma^2)$ for all j.

   (b) Write an expression for $\Delta_1 + \Delta_2 + \cdots + \Delta_n$ in terms of the random variables ϵ_j. Hence show that the accumulated value at time n of 1 unit invested at time 0 has a lognormal distribution, and find its parameters.

10. (a) Consider a discounted dividend model for equity valuation. Assume that dividends are payable annually with the first dividend of D_1 payable in exactly one year and that dividends grow at constant annual rate g. Show that the value of the equity in an interest-rate model of constant rate i is $D_1/(i-g)$.

    (b) Under the above set of assumptions, what would be the value of the equity immediately after the fifth dividend?

11. Mr Strauss dies and leaves an estate valued at £50,000 in a bank account earning interest at rate $i^{(12)} = 9\%$. She has three children: Jim, aged 7, Fred, aged 5, and Sandra, just turned 4. The estate will be divided among the surviving children 14 years from now, on Sandra's 18th birthday. Find the expected value of the inheritance for each child given the following, and assuming independence.

Age today	Probability of Survival for 14 years
7	0.95
5	0.97
4	0.98

12. The *expected yield* of a random cash-flow is used to mean the rate i such that $\mathbb{E}NPV(i) = 0$, that is, the yield of the mean cash-flow is zero (or, put otherwise, the rate under which the expected value of the cash-flow is zero — this is not in general the same as the expectation of the yield of the cash-flow).

    A £1000 20-year bond has coupons at $j^{(2)} = 12\%$ redeemable at par. Find the purchase price which provides an expected annual yield of $i^{(2)} = 14\%$, under the assumption of a semi-annual default probability of 2%. After default, no more payments take place. (You may find it convenient to work with 6 months as the basic time unit).

13. An insurer issues n identical policies. Let Y_j be the claim amount from the jth policy, and suppose that the random variables Y_j, $j = 1, \ldots, n$ are i.i.d. with mean $\mu > 0$ and variance σ^2. The insurer charges a premium of A for each policy.

    (a) Show that if $A = \mu + 10\sigma n^{-1/2}$, then the probability that total claims exceed total premiums is no more than 1%, for any value of n.

    (b) Use the Central Limit Theorem to show that if instead $A = \mu + 3\sigma n^{-1/2}$, then this probability is still less than 1%, provided n is large enough.

And a last question, on earlier material, for further practice.

14. Makeham's formula. Consider a security of nominal amount N which is to be repaid after n years at a price of R per unit nominal, and let $C = NR$. Thus C is the cash payable on redemption. Let the coupon rate be j (per unit nominal), and assume that coupon payments are made $p$thly in arrear.

    Fix a yield i to be obtained (i.e. constant i model). Denote by g the annual coupon rate *per unit of redemption price* and K_n the present value of the capital repayment.

    (a) Find expressions for g and K_n in terms of j, N, C, R, i, n.

    (b) Prove Makeham's formula for the purchase price $A_n = K_n + \frac{g}{i^{(p)}}(C - K_n)$ from first principles and compare with formulas for A_n seen previously.

    (c) What can you say about the monotonicity of A_n as a function of n? Argue both mathematically and by general reasoning.

    (d) Let $n_1 < n_2$ be the terms of two otherwise equal securities that you are offered for a price B each. Under what conditions do you choose the first, when the second?