A.5 No-arbitrage, term structure, immunisation

1. (a) Suppose that there exists a risk-free asset with constant force of interest r. Using the no arbitrage assumption, show from first principles that the forward price F, agreed at time 0 and to be paid at time T, for an asset S, with no income and with value S_0 at time 0, is given by $F = S_0 e^{rT}$. Assume that there are no transaction costs.

   (b) Extend the argument in (a) to derive the forward price where the asset provides a fixed known cash-flow $c = ((t_1, X_1), \ldots, (t_n, X_n))$ of income.

   (c) In the setting of (a) and (b), calculate the value V_t of the contract to buy asset S at time T at intermediate times $t \in (0, T)$.

2. An asset has a current price of $100p$. It will pay an income of $5p$ in 20 days' time. Given a risk-free rate of interest of 6% per annum convertible half-yearly and assuming no arbitrage, calculate the forward price to be paid in 40 days.

3. Consider an interest rate model with annual rates i_n from time $n-1$ to time n, where

$$i_1 = 4.0\%, \quad i_2 = 4.5\%, \quad i_3 = 4.8\%.$$

   (a) Calculate

      i. the price per £100 nominal of a 3-year bond paying an annual coupon in arrears of 5%, redeemed at par in exactly three years, and

      ii. the gross redemption yield from the bond.

   (b) Explain why a bond with a higher coupon would have a lower gross redemption yield, for the same term to redemption.

4. An investment project involves payments of £1,000,000 now and £500,000 two years from now. The proceeds from the project are £2,000,000, payable in exactly four years' time.

   Calculate the volatility and the convexity of this project, as a function of the underlying interest rate i (assumed constant).

   Calculate the yield and compare volatility and convexity at $i = 4\%$ and $i = 8\%$.

5. Consider a portfolio of n fixed income bonds (any positive cash flows) with present values $A_1, \ldots, A_n$. For $j = 1, \ldots, n$, let the discounted mean term and the convexity of the jth bond be denoted by τ_j and c_j. Let $p_j = A_j/A$ be the proportion of the jth security in the portfolio, $A = A_1 + \cdots + A_n$.

   Show that the discounted mean term and the convexity of the portfolio are

$$\sum_{j=1}^{n} p_j \tau_j \qquad \text{and} \qquad \sum_{j=1}^{n} p_j c_j.$$

6. A fund has to provide an annuity of £50,000 p.a. payable yearly in arrears for the next 9 years followed by a final payment of £625,000 in 10 years' time.

   The fund has earmarked cash assets equal to the present value of the payments and the fund manager wants to invest these in two zero coupon bonds, A, repayable after 5 years, and B, repayable after 20 years.

   How much should the manager invest in A and B to have the same volatility in the assets and liabilities, assuming an effective rate of interest of 7% p.a.?

7. A fund will need to make payments of £10,000 at the end of each of the next five years. It wishes to immunise using two zero coupon bonds, one maturing in 5 years and one in 1 year. The rate of interest is 5% p.a.

   (a) Calculate the present value of the liabilities. Find the discounted mean term.

   (b) Calculate the nominal amounts of the zero-coupon bonds needed to equate the present value and duration of assets and liabilities.

   (c) Calculate the convexity of the assets. Comment on Redington immunisation.

8. A government bond pays a coupon half-yearly in arrears of 10% per annum. It is to be redeemed at par in exactly ten years. The gross redemption yield from the bond is 6% per annum convertible half-yearly. Calculate the discounted mean term of the bond in years.

   Explain why the duration of the bond would be longer if the coupon rate were 8% per annum instead of 10% per annum.

Optional questions for further practice (for Question 10, terminology is as in Lecture 13):

9. Redington's immunisation assumes a flat yield curve that may shift up or down over time. Show that such yield curves are inconsistent with an assumption of no arbitrage. You may assume that at any future time instant the flat yield is different to the initial flat yield with some positive probability.

   [Hint: Consider a flat yield curve with flat yield at time 0 of i_0 and at time 1 of i_1. Consider two portfolios A and B at times 0 and 1. Portfolio A comprises one unit of zero coupon bond with term 1, and $(1+i_0)^2$ units of zero coupon bond with term 3. Portfolio B comprises $2 \times (1+i_0)$ units of zero coupon bond with term 2.]

10. Denote by $f_{t,r}$ the forward rate applicable over the period t to $t+r$ and by i_t the spot rate over the period 0 to t. The gross redemption yield from a one-year bond with a 6% annual coupon is 6% p.a. effective; the gross redemption yield from a two-year bond with a 6% annual coupon is 6.3%; and the gross redemption yield from a three year bond with a 6% annual coupon is 6.6% p.a. effective. All the bonds are redeemed at par and are exactly one year from the next coupon payment.

    (a) i. Calculate i_1, i_2 and i_3 assuming no arbitrage.

    ii. Calculate $f_{0,1}$, $f_{1,1}$ and $f_{2,1}$ assuming no arbitrage.

    (b) Explain why the forward rates increase more rapidly with term than the spot rates.