# A.2 Yields, $p$thly payments, annuities and loans

1. Under the terms of a savings scheme an investor who makes an initial investment of £4,000 may receive either

   - £2,000 after 2 years and a further £3,000 after 7 years; or
   - £4,400 at the end of 4 years.

   Which of these options corresponds to a higher rate of interest on the investor's money?

2. Find the yield of the following cash-flows:

   (a) an investment of $X$ at time $s$, followed by a single payment of $Y$ received at time $t > s$. How does the yield change as $X$, $Y$, $s$, $t$ vary?

   (b) a bond costing $P$ at time 0, which generates coupon payments of $X$ at times $1, 2, \ldots, n$ and a redemption payment of $P$ at time $n$.

3. For $m \in \mathbb{N}$, the prefix ${}_{m|}$ before an annuity symbol indicates that the sequence of payments concerned is *deferred* by an amount of time $m$. For example, the discounted present value (in the constant interest-rate model) of a *deferred annuity*, with unit payments per unit time payable from $m+1$ to $m+n$, is denoted by ${}_{m|}a_{\overline{n}|}$.

   (a) Express ${}_{m|}a_{\overline{n}|}$ in terms of the ordinary annuity symbols introduced in the lectures.

   (b) The case $m = -1$ corresponds to an annuity-due and is denoted by $\ddot{a}_{\overline{n}|}$. Express $\ddot{a}_{\overline{n}|}$ as simply as you can (i) in terms of $a_{\overline{n}|}$ and (ii) in terms of $a_{\overline{n-1}|}$

   (c) The discounted present value of an *increasing annuity* with payments $j$ at time $j = 1, \ldots, n$ is denoted by $(Ia)_{\overline{n}|}$. Express $(Ia)_{\overline{n}|}$ in terms of ordinary annuity symbols.

   (d) Consider a security redeemable at par, with term $n$ and $p$thly coupon payments at nominal rate $j$. Express the accumulated (time $n$) and discounted (time 0) values in the constant $i$ model in terms of annuity symbols.

4. Find, on the basis of an effective interest rate of 4% per unit time, the values of

$$a^{(4)}_{\overline{67}|}, \quad s^{(12)}_{\overline{18}|}, \quad \ddot{a}^{(4)}_{\overline{16.5}|}, \quad \ddot{s}^{(12)}_{\overline{15.25}|}, \quad {}_{4.25|}a^{(4)}_{\overline{3.75}|}, \quad (Is)_{\overline{4}|}.$$

   Describe the meaning of each of the symbols.

5. Buy now, pay later!! A luxury sofa is sold with the following payment options: either pay immediately and receive a 5% discount, or pay by monthly instalments (in arrears) for 15 months. To calculate the monthly payments, add a 5% finance charge to the purchase price, then divide the total amount into 15 equal payments. What annual effective rate of interest is being charged for paying by instalments?

6. Show that in any constant $i$ interest rate model, the cash-flows $c_\infty(s) = \delta$, $0 \leq s \leq 1$, and $c_p = (k/p, i^{(p)}/p)_{k=1,\ldots,p}$ are equivalent (have the same value) for all $p \in \mathbb{N}$.

7. A loan of £30,000 is to be repaid by a level annuity payable monthly in arrears for 25 years, and calculated on the basis of an (effective) interest rate of 12% pa. Calculate the intitial monthly repayments.

   After ten years of repayments the borrower asks to:

   (a) pay off the loan which is outstanding. Calculate the lump sum which would be required to pay off the outstanding loan.

   (b) extend the loan by a further five years (i.e. to 30 years in total), and with repayments changed from monthly to quarterly in arrears. Calculate the revised level of quarterly repayments.

   (c) reduce the loan period by five years (i.e. to 20 years in total) and for repayments to be biannually (i.e. once every two years) in arrears. Calculate the revised level biannual repayments.

   [*Hint: calculate annual repayment levels first, then combine each two consecutive payments into an equivalent single payment.*]

8. An insurance company issues an annuity of £10,000 p.a. payable monthly in arrears for 25 years. The cost of the annuity is calculated using an effective rate of 10% p.a.

   (a) Calculate the interest component of the first instalment of the sixth year.

   (b) Calculate the total interest paid in the first 5 years.

*Optional questions for further practice:*

9. A ten-year loan for £10,000 has a fixed rate of 5% for the first 3 years. Thereafter the rate is 8% for the rest of the term. Repayments are annual. Find the amounts (i) of the first 3 repayments and (ii) of the remaining 7 payments. Show that the APR of the loan is 6.4%.

10. (a) An annuity-certain is payable annually in advance for $n$ years. The first payment of the annuity is 1. Thereafter the amount of each payment is $(1 + r)$ times that of the preceding payment.

    Show that, on the basis of an interest rate of $i$ per annum, the present value of the annuity is $\ddot{a}_{\overline{n}|j}$ where $j = (i - r)/(1 + r)$.

    (b) Suppose instead that the annuity is payable annually in arrear. Is its present value (at rate $i$) now equal to $a_{\overline{n}|j}$?

    (c) In return for a single premium of £10,000 an investor will receive an annuity payable annually in arrear for 20 years. The annuity payments increase from year to year at the (compound) rate of 5% per annum.

    Given that the initial amount of the annuity is determined on the basis of an interest rate of 9% per annum, find the amount of the first payment.

**Course webpage with assignments and, after the classes, solutions**:
`http://www.stats.ox.ac.uk/~winkel/bs4a.html`